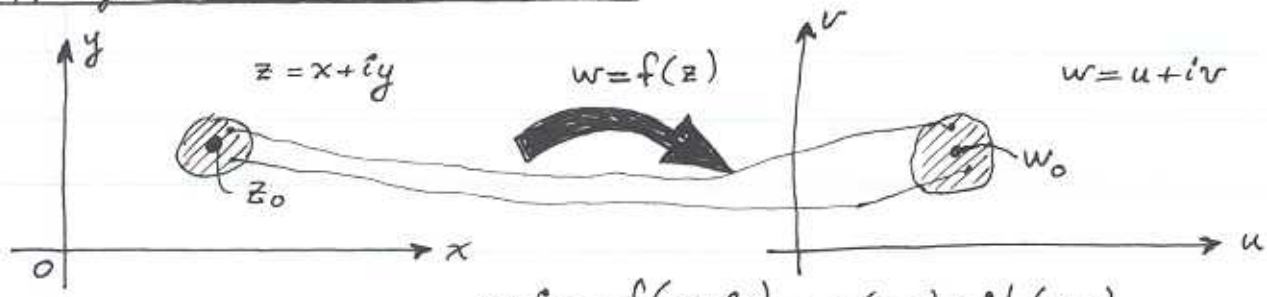
Mappings in the Complex Plane:



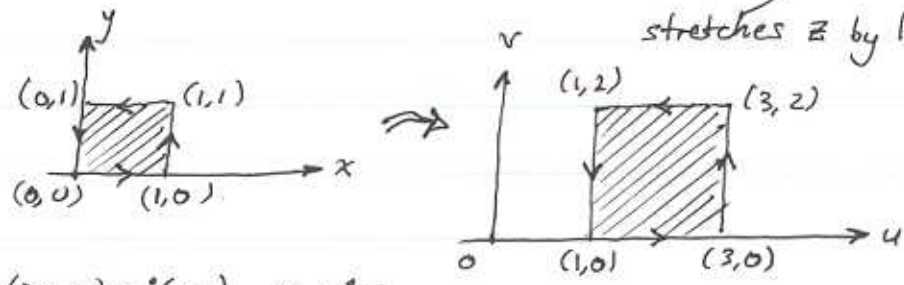
$$u+iv = f(x+iy) = g(x,y) + i\,h(x,y)$$
$$u = g(x,y) \text{ and } v = h(x,y) \quad \text{(forward map)}$$
$$\Rightarrow x = X(u,v),\ y = Y(u,v) \quad \text{(inverse map)}$$

Example: $w = f(z) = az + b$ $\quad a, b$ complex, $a \neq 0$.

1) 1-1: $f(z)$ analytic, $a = f'(z) \neq 0 \Rightarrow \exists$ region near z in z-plane and around $w = az+b$ in w-plane s.t. $w = f(z)$ is 1-1. Thus $w = az+b$ and $dw/dz = f'(z) = a \neq 0$. So 1-1 everywhere.

2) Shape of map: $a = |a|e^{i\theta_a}$ and $z = re^{i\theta}$ so $w = \underbrace{|a|}_{\text{stretches } z \text{ by } |a|} r e^{i(\theta+\theta_a)} + \underbrace{b}_{\text{displaces } az \text{ by "}b\text{"}}$ (rotates)

let $b=1$, $a=2$



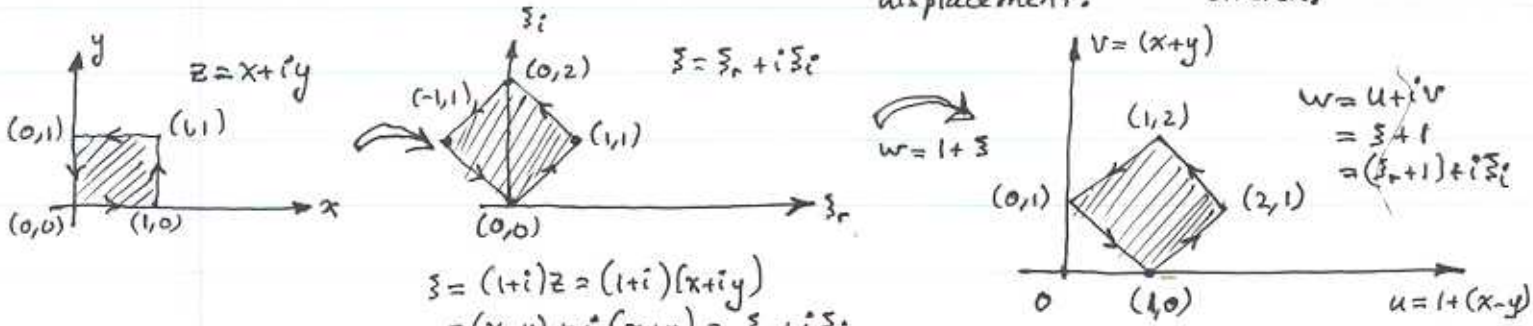
$u+iv = 2z+1 = (2x+1) + i(2y) = u+iv$

3) Can show that circles map to circles, ovals to ovals, lines to lines.

Example: $w = f(z) = az + b$ $\quad a = 1+i,\ b = 1$

$$w = az+b = (1+i)z + 1 = \xi + 1; \quad \xi = (1+i)z = \underbrace{\sqrt{2}}_{\text{stretch.}}\underbrace{e^{i\pi/4}}_{\text{rotation.}} z$$

(displacement.)

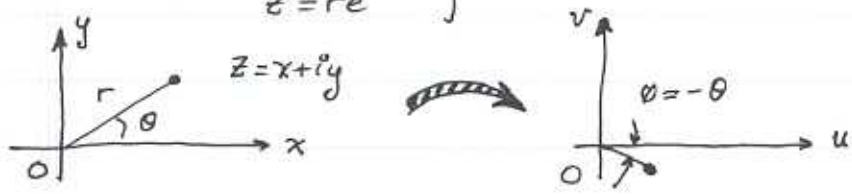


$$\xi = (1+i)z = (1+i)(x+iy)$$
$$= (x-y) + i(x+y) = \xi_r + i\xi_i$$

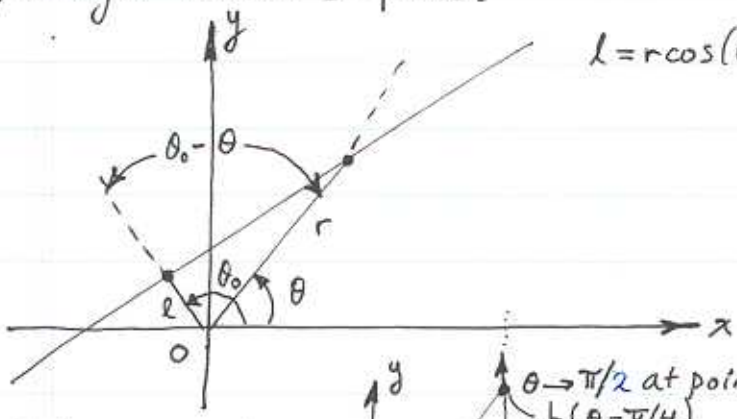
Example: $w = f(z) = 1/z$

1) 1-1: $df/dz = -1/z^2$ so 1-1 everywhere except $z=0$. But $z=0$ maps to "the point at infinity" so 1-1 everywhere.

2) polar form: $\left.\begin{array}{l} w = \rho e^{i\phi} \\ z = re^{i\theta} \end{array}\right\}$ $\rho e^{i\phi} = \dfrac{1}{re^{i\theta}} \Rightarrow \rho = \dfrac{1}{r},\ \phi = -\theta$



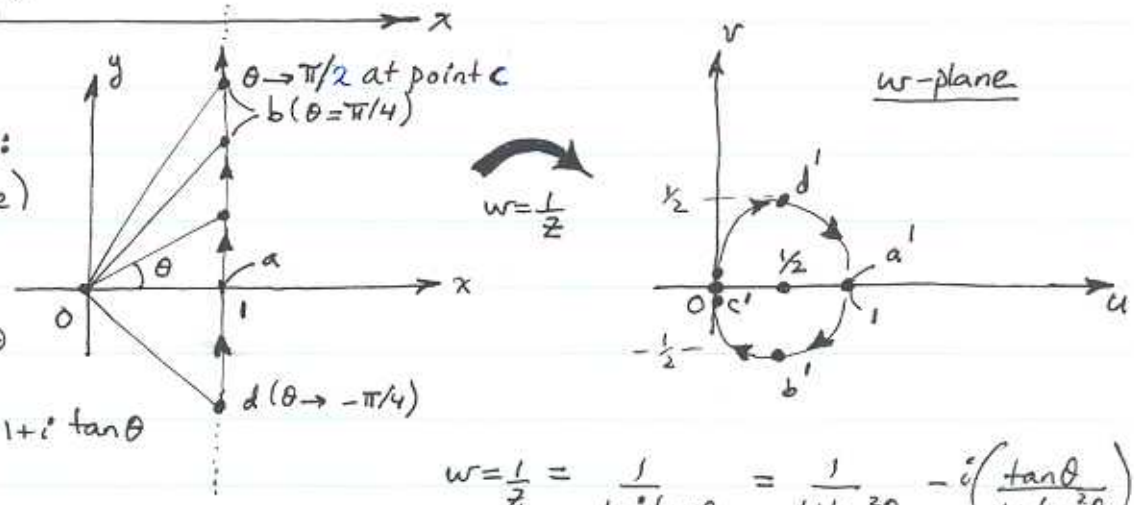
3) straight lines in z-plane:



$\ell = r\cos(\theta_0 - \theta) = \dfrac{1}{\rho}\cos(\theta_0 + \phi)$

$\ell = \dfrac{1}{\rho}\cos(\theta_0 + \phi)$ is a circle.

Sub-Example: (special case)



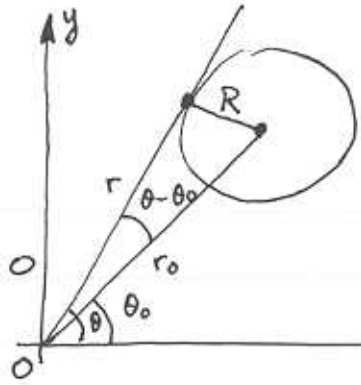
$x = 1 = r\cos\theta$
$y = r\sin\theta$
$z = x + iy = 1 + i\tan\theta$

$$w = \frac{1}{z} = \frac{1}{1+i\tan\theta} = \frac{1}{1+\tan^2\theta} - i\left(\frac{\tan\theta}{1+\tan^2\theta}\right)$$

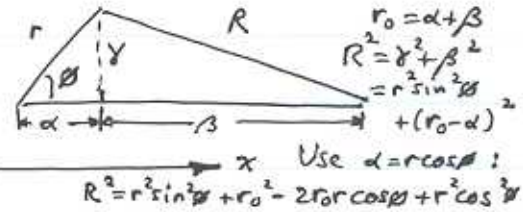
Generalize: Straight lines in z-plane not passing through origin map to circles in the w-plane, for which one point is $w=(0,0)$. Note that the two points $(x,y) = (1, i\infty)$ and $(x,y) = (1, -i\infty)$ map to the origin. Note also the sense (direction) of traversal: toward $+y$ in z-plane is clockwise in w-plane.

Exercise: Map $r\sin\theta = 2$ and $r\cos(\pi/4 - \theta) = 3$ in the w-plane.

4) <u>Circles in the z-plane:</u>



$$R^2 = r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)$$



$r_0 = \alpha+\beta$

$R^2 = \gamma^2+\beta^2$

$= r^2\sin^2\phi + (r_0-\alpha)^2$

Use $\alpha = r\cos\phi$:

$R^2 = r^2\sin^2\phi + r_0^2 - 2r_0 r\cos\phi + r^2\cos^2\phi$

$\Rightarrow R^2 = r^2 - 2r_0 r\cos\phi + r_0^2$ ✓

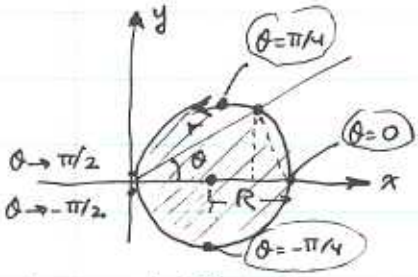
Let $r = 1/\rho$, $\theta = -\phi$

$\Rightarrow \rho^2(r_0^2 - R^2) - 2\rho r_0\cos(\theta_0+\phi) + 1 = 0$

<u>If $r_0 = 0$</u> then $R^2 = r^2$ $(R = r)$ gives a circle of radius R in the z-plane centered at the origin. Then $\rho = 1/R$ is a circle of radius $1/R$ in w-plane also centered at the origin.

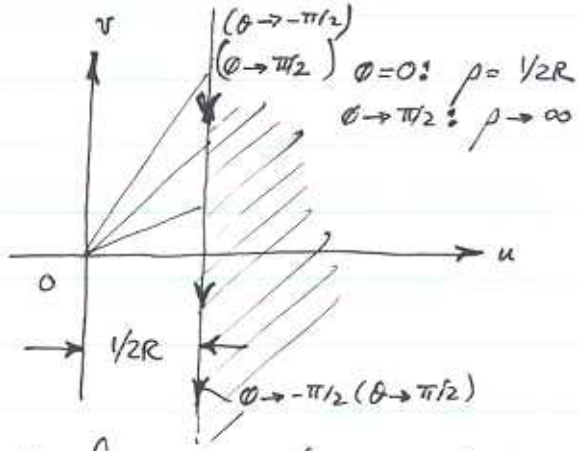
<u>If $r_0 = R$ and $\theta_0 = 0$</u> then $r = 2R\cos\theta$ so $x = (2R\cos\theta)\cos\theta$, $y = (2R\cos\theta)\sin\theta$

and $\rho = \dfrac{1}{2R\cos\phi}$



$\theta = 0$: $x = 2R$, $y = 0$

$\theta = \pi/4$: $x = y = R$



As θ varies from $\theta = 0$ to $\theta = \pi/2$, ϕ varies from $\phi = 0$ to $\phi = -\pi/2$;

As θ " " " " $\theta = -\pi/2$, ϕ " " " " $\phi = \pi/2$.

When R is decreased the size of the circle in the z-plane diminishes and the vertical line in the w-plane moves to the right. Thus, the inside of the circle maps to the right of the vertical line $u = 1/2R$, $-\infty < v < \infty$ (as shown above by the shading). The outside of the z-plane circle maps to the left of this line (white region).

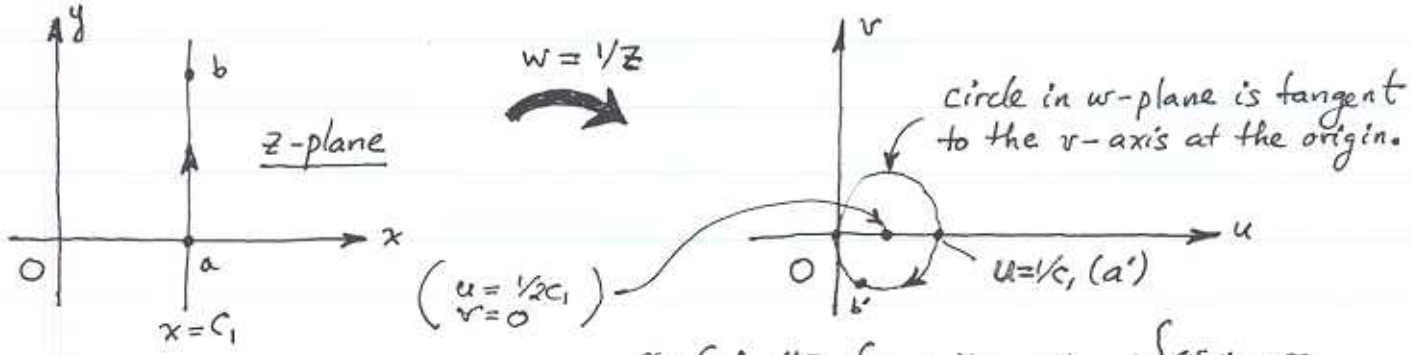
If $r_0 \neq R$ then in the z-plane the circle does <u>not</u> pass through the origin (let's say $r_0 > R$). The mapping in the w-plane is also a circle that does not pass through the origin. Thus:

$$\left.\begin{aligned} r^2 &= 2r_0 r\cos(\theta-\theta_0) - (r_0^2-R^2) \\ \rho^2 &= \frac{2r_0\rho\cos(\phi+\theta_0)}{r_0^2-R^2} - \frac{1}{r_0^2-R^2} \end{aligned}\right\} \text{same mappings} \Rightarrow \text{circles} \to \text{circles.}$$

<u>Exercise:</u> Draw the mappings for $r_0=2$, $R=1$, $\theta_0=0$ giving $r^2 = 4r\cos\theta - 3$ and $\rho^2 = \frac{4}{3}\rho\cos\phi - \frac{1}{3}$.

Let us write the transformation $w = 1/z$ in terms of u, v, x, y: $u+iv = 1/(x+iy) = (x-iy)/(x^2+y^2)$ so $u = x/(x^2+y^2)$ and $v = -y/(x^2+y^2)$. Conversely $z = 1/w$ so $x+iy = 1/(u+iv) = (u-iv)/(u^2+v^2)$ so that $x = u/(u^2+v^2)$ and $y = -v/(u^2+v^2)$. Using these formulas it is sometimes easier to examine the geometrical form of the transformations.

<u>The line $x=C_1$:</u> use $x = u/(u^2+v^2) \Rightarrow u^2+v^2 - \frac{u}{C_1} = 0$. Let $u = U+\alpha$ so $U^2 + 2\alpha U + \alpha^2 + v^2 - \frac{(U+\alpha)}{C_1} = 0$. Choose $\alpha = 1/C_1$ so $U^2+v^2 = \frac{1}{4C_1^2}$. Now let $U = \mathbb{U}/2C_1$ and $v = \mathbb{V}/2C_1$ to get the circle $\mathbb{U}^2 + \mathbb{V}^2 = 1$.

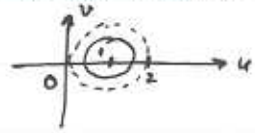


$x = C_1$: $u = \frac{C_1}{C_1^2+y^2}$, $v = \frac{-y}{C_1^2+y^2}$: $\begin{cases} \text{as } y\to\infty \\ u\to 0^+,\ v\to 0^- \\ \text{as } y\to-\infty \\ u\to 0^+,\ v\to 0^+ \end{cases}$

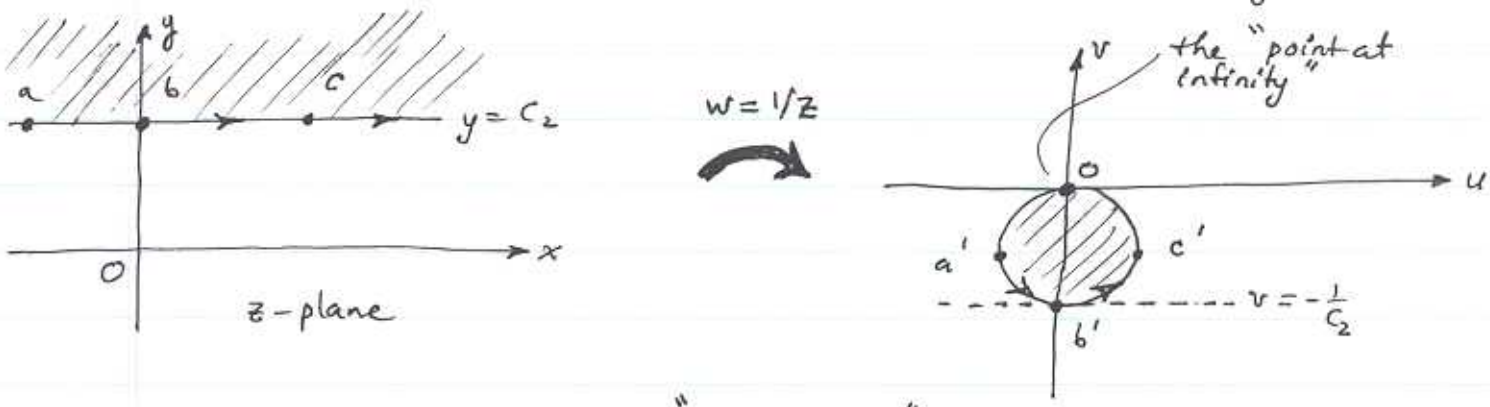
At $y=0$, $v=0$ and $u = 1/C_1$

Clearly as C_1 increases (the vertical line in the z-plane moves to the right) the circles in the w-plane shrink. The region $x > C_1$ thus maps to the inside of the circle.

<u>Note:</u> The circle has radius $1/2C_1$ and is centered at $u=1/2C_1$, $v=0$.
Analytically the half-plane $x>C_1$ has $x=\frac{u}{u^2+v^2}>C_1$ or $u>C_1(u^2+v^2)$ or $u/C_1>u^2+v^2$ or $0>\left(u^2-2\left(\frac{u}{2C_1}\right)+\frac{1}{4C_1^2}-\frac{1}{4C_1^2}\right)+v^2 \Rightarrow \left(\frac{1}{2C_1}\right)^2>\left(u-\frac{1}{2C_1}\right)^2+v^2$.
Thus the right-half plane maps to the inside of the circle with radius $1/2C_1$. Let $C_1=1/2$ so $1>(u-1)^2+v^2$ (circle of radius less than 1 centered at $u=1, v=0$).

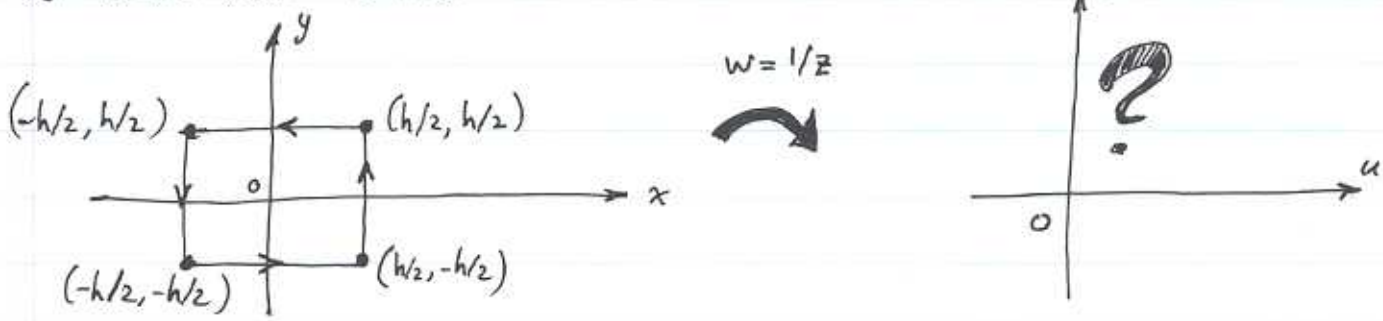


<u>The line $y=C_2$:</u> Use $y=-v/(u^2+v^2) \Rightarrow u^2+v^2+v/y=0=u^2+v^2+v/C_2$.
The circle in the u-v plane is tangent to the u-axis at the origin.
We have $u=x/(x^2+C_2^2)$ and $v=-C_2/(x^2+C_2^2)$. So v is negative.

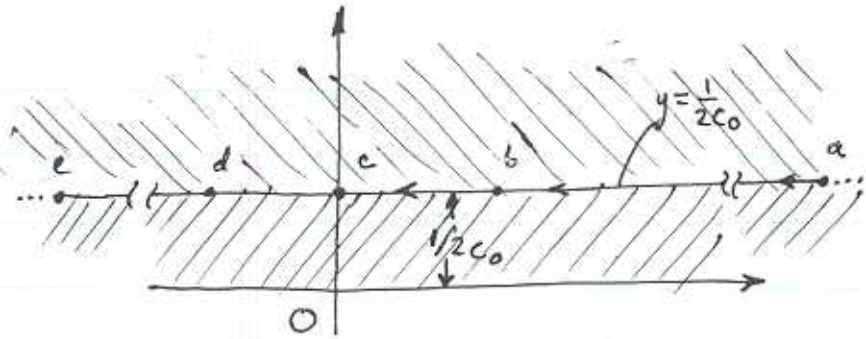


The point $(-\infty, C_2)$ maps to the "left-origin" while $(+\infty, C_2)$ maps to the "right origin."
The upper half of the y-plane (above $y=C_2$) maps to the inside of the circle (see shading)
<u>Note:</u> If the line $y=C_2$ is traversed oppositely ($c\to b\to a$ or right$\to$left) then the w-plane circle is traversed oppositely, or <u>clockwise</u>.

<u>Exercise:</u> For the transformation $w=1/z$ map the sequence of straight lines making a square $h\times h$ as shown below in the z-plane to the corresponding shape in the w-plane. Draw the shapes as $h\to 0$ and $h\to\infty$

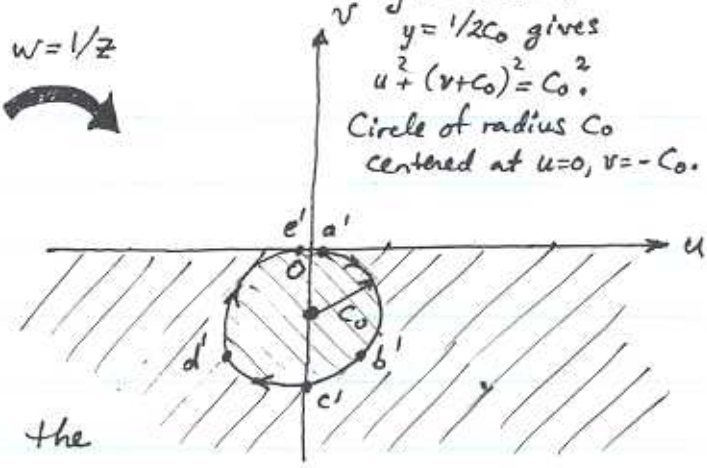


The infinite strip: $0 < y < 1/2c_0$



$w = 1/z$

$x = u/(u^2+v^2)$
$y = -v/(u^2+v^2)$
$y = 1/2c_0$ gives
$u^2 + (v+c_0)^2 = c_0^2$.
Circle of radius c_0
centered at $u=0$, $v=-c_0$.



As c_0 increases the strip becomes narrower and the circle radius becomes larger. Thus, the inside of the strip maps to the outside of the circle, as shown by the right-inclined hatchings. The region <u>above</u> the line $y = 1/2c_0$ maps to the inside of the circle (left-inclined hatchings).

The Bilinear Transformation:

$$w = f(z) = \frac{az+b}{cz+d} = \frac{az+b}{c(z+d/c)} = \frac{A}{c} + \frac{B}{z+d/c} = \frac{a}{c} + \frac{(bc-ad)}{c(cz+d)}$$

$$\left(\begin{aligned} B &= \left(\frac{az+b}{c}\right)_{z=-d/c} = \frac{bc-ad}{c^2} \\ \Rightarrow A &= \left[\frac{az+b}{c(z+d/c)} - \frac{(bc-ad)}{c^2(z+d/c)}\right]c = a \end{aligned} \right)$$

Note: $w = \dfrac{az+b}{cz+d} \Rightarrow w(cz+d) = az+b \Rightarrow z = \dfrac{wd+(-b)}{w(-c)+a}$

Let $d=\alpha$, $-b=\beta$, $-c=\gamma$, $a=\delta$: $z = \dfrac{w\alpha+\beta}{w\gamma+\delta}$ (also bilinear).

The examples considered previously are special cases of the bilinear transformation, the first with $c=0$, the second with $a=d=0$, $b=c$.

<u>Note:</u> $w = f(z) = \dfrac{az+b}{cz+d}$

$$\frac{df}{dz} = \frac{a}{cz+d} - \frac{c(az+b)}{(cz+d)^2} = \frac{(ad-bc)}{(cz+d)^2} \equiv \frac{\Delta}{(cz+d)^2} \neq 0 \quad \text{when } \Delta = ad-bc \neq 0.$$

When $\Delta = 0$ the mapping is not 1:1 $\Rightarrow$ not useful.

<u>Three-Stage Mapping:</u>

$\zeta = f(z) = \alpha z + \beta$ (stretch and displace)

$\eta = g(\zeta) = 1/\zeta$ (invert)

$w = h(\eta) = \gamma\eta + \delta$ (stretch and displace)

$$w = \gamma\eta + \delta = \frac{\gamma}{\zeta} + \delta = \frac{\gamma}{\alpha z+\beta} + \delta = \frac{\gamma + \delta(\alpha z + \beta)}{\alpha z + \beta}$$

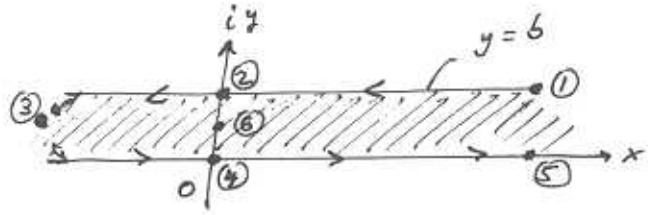
$$= \frac{\alpha\delta z + (\gamma + \delta\beta)}{\alpha z + \beta} = \frac{az+b}{cz+d} \quad \begin{pmatrix} a = \alpha\delta \\ b = \gamma + \delta\beta \\ c = \alpha \\ d = \beta \end{pmatrix}$$

$$z \xrightarrow{f(z)} \zeta \xrightarrow{g(\zeta)} \eta \xrightarrow{h(\eta)} w = \frac{az+b}{cz+d}$$

$$w = h(\eta) = h(g(\zeta)) = h(g(f(z))) = \frac{az+b}{cz+d}.$$

Other mappings:

I $\boxed{w = e^{\pi z/b}}$ $\quad -\infty < x < \infty$, $0 \le y \le b$

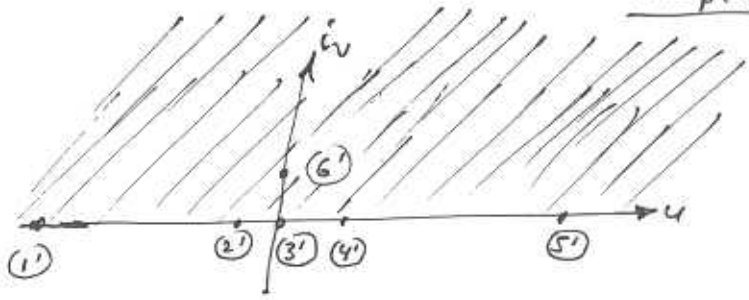


z-plane

$$w = u + iv = e^{\pi x/b}\left(\cos\frac{\pi y}{b} + i\sin\frac{\pi y}{b}\right)$$

	(x,y)	(u,v)	
1	(∞, b)	$(-\infty, 0)$	1'
2	$(0, b)$	$(-1, 0)$	2'
3	$(-\infty, y)$	$(0, 0)$	3'
4	$(0, 0)$	$(1, 0)$	4'
5	$(\infty, 0)$	$(\infty, 0)$	5'
6	$(0, b/2)$	$(0, i)$	6'

w-plane



The inside of the strip in the z-plane maps into the upper half of the w-plane (and vice-versa)

II $\boxed{w = -\cos\left(\frac{\pi z}{a}\right)}$

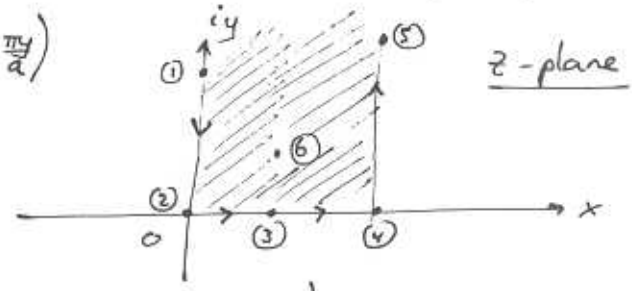
$$w = u + iv = -\cos\frac{\pi}{a}(x+iy) = -\left[\cos\frac{\pi x}{a}\cos\left(\frac{i\pi y}{a}\right) - \sin\frac{\pi x}{a}\sin\left(\frac{i\pi y}{a}\right)\right]$$

Now $\cos\phi = \dfrac{e^{i\phi} + e^{-i\phi}}{2}$ and $\sin\phi = \dfrac{e^{i\phi} - e^{-i\phi}}{2i}$

Let $\phi = i\theta$

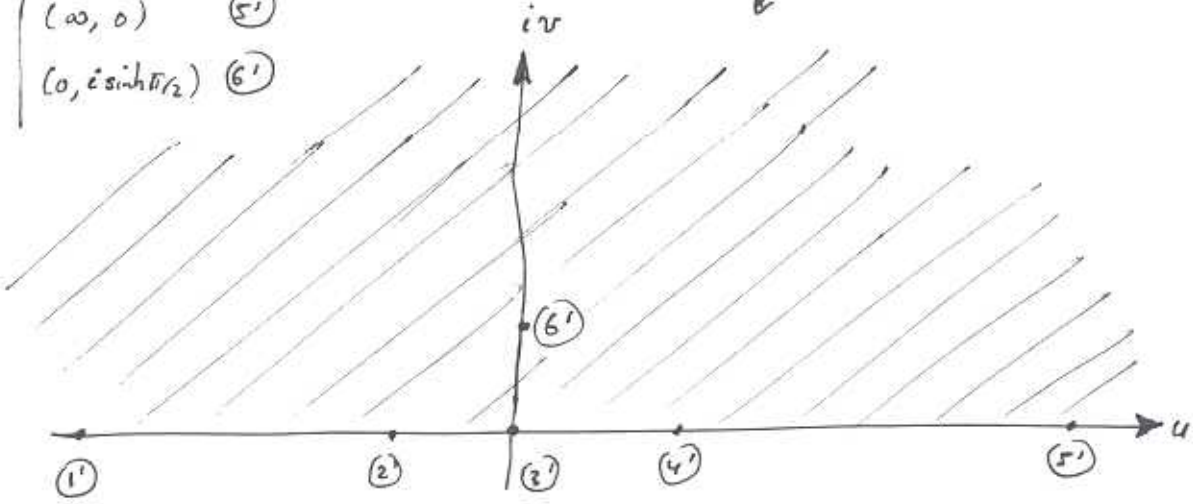
$$\cos(i\theta) = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta \quad \text{and} \quad \sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2i} = -\frac{1}{i}\left(\frac{e^{\theta} - e^{-\theta}}{2}\right) = -\frac{1}{i}\sinh\theta.$$

Hence: $u + iv = \left(-\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a}\right) + i\left(\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}\right)$



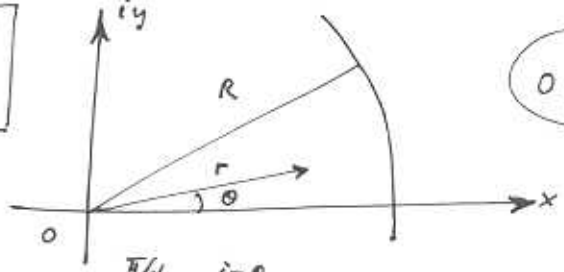
z-plane

	(x,y)	(u,v)	
1	$(0, \infty)$	$(-\infty, 0)$	1'
2	$(0, 0)$	$(-1, 0)$	2'
3	$(a/2, 0)$	$(0, 0)$	3'
4	$(a, 0)$	$(1, 0)$	4'
5	(a, ∞)	$(\infty, 0)$	5'
6	$(a/2, a/2)$	$(0, i\sinh\pi/2)$	6'



This maps the vertical semi-infinite strip in the z-plane into the upper-half w-plane

III Sector: $\boxed{w = \left(\frac{z}{R}\right)^{\pi/\alpha}}$

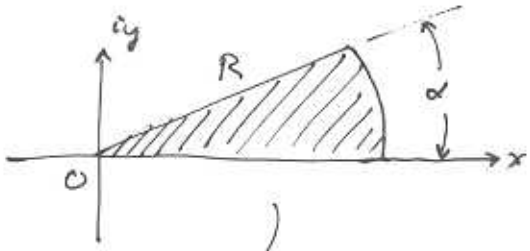


$0 \le \theta \le \alpha$

$$w = u + iv = \left(\frac{x+iy}{R}\right)^{\pi/\alpha} = \left(\frac{r}{R}\right)^{\pi/\alpha} e^{\frac{i\pi\theta}{\alpha}} \quad \text{since} \quad x+iy = re^{i\theta}$$

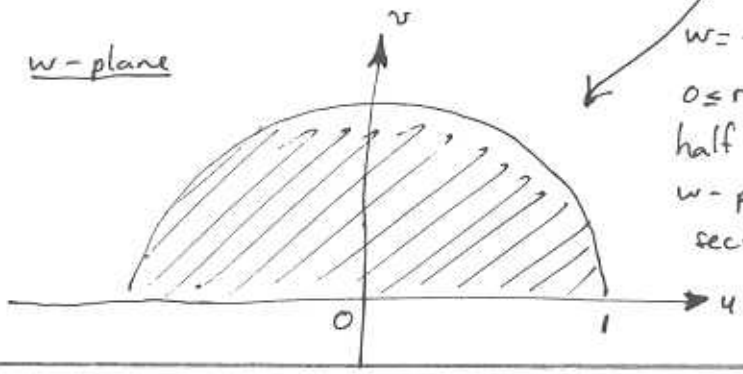
Write $w = \rho e^{i\phi} \Rightarrow \rho = \left(\frac{r}{R}\right)^{\pi/\alpha}, \quad \phi = \pi\theta/\alpha$.

Thus
$r = 0 \Rightarrow \rho = 0$
$r = R \Rightarrow \rho = 1.$
$\theta = 0 \Rightarrow \phi = 0$
$\theta = \alpha \Rightarrow \phi = \pi$



z-plane

w-plane



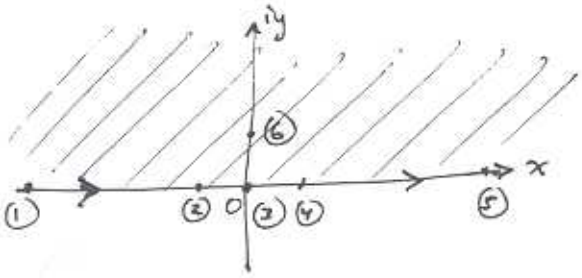
$w = f(z) = \left(\frac{z}{R}\right)^{\pi/\alpha}$ maps the sector $0 \le r \le R$, $0 \le \theta \le \alpha$ into the upper half of the unit semicircle in the w-plane. Interior points of the sector map to interior points of the semicircle.

IV $w = \frac{i-z}{i+z}$ (mapping onto the unit disk)

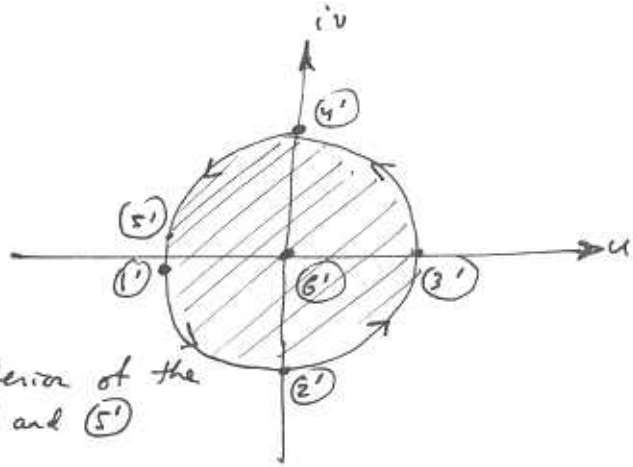
$$u + iv = \frac{i-(x+iy)}{i+(x+iy)} = \frac{-x+i(1-y)}{x+i(1+y)} = \frac{[-x+i(1-y)][x-i(1+y)]}{x^2+(1+y)^2} = \frac{-x^2+(1+y)(1-y)+i[x(1-y)+x(1+y)]}{x^2+(1+y)^2}$$

$$= \frac{1-(x^2+y^2)}{x^2+(1+y)^2} + i\left(\frac{2x}{x^2+(1+y)^2}\right)$$

	①	②	③	④	⑤	⑥
(x,y)	$(-\infty,0)$	$(-1,0)$	$(0,0)$	$(1,0)$	$(\infty,0)$	$(0,1)$
(u,v)	$(-1,0^-)$	$(0,-1)$	$(1,0)$	$(0,1)$	$(-1,0^+)$	$(0,0)$
	①'	②'	③'	④'	⑤'	⑥'



$w = f(z) = \frac{i-z}{i+z}$



The upper half of the z-plane maps into the interior of the unit circle in the w-plane. Note that points ①' and ⑤' approach one another ("the point at infinity")

Exercise: For mapping (I) take $b = 2$ and draw the exact transformation for specific (x,y) values say $(x,y) = (-10,2), (0,2), (0,0)$ and so on.

Exercise: For mapping (II) use $a = 2$ and do the same.

Exercise: Generalize the map (III) and show what happens for $z > R$.

Exercise: For the mapping $w = 1/z$ map the strip shown into a region of the w-plane. Draw it clearly