Mappings in the Complex Plane:

\[ w = f(z) = u + iv \]

\[ u + iv = f(x + iy) = g(x, y) + i h(x, y) \]

\[ u = g(x, y) \text{ and } v = h(x, y) \] (forward map)

\[ \Rightarrow x = X(u, v), \ y = Y(u, v) \] (inverse map)

Example: \( w = f(z) = az + b \) \( a, b \) complex, \( a \neq 0 \).

1. 1-1: \( f(z) \) analytic, \( a = f'(z) \neq 0 \implies \exists \) region near \( z \) in \( z \)-plane and around \( w = az + b \) in \( w \)-plane s.t. \( w = f(z) \) is 1-1. Thus \( w = az + b \) and \( dw/dz = f'(z) = a \neq 0 \). So 1-1 everywhere.

2. Shape of map: \( a = |a|e^{i\theta} \) and \( \beta = r\theta \) so \( w = |a|r e^{i(\theta + \beta)} \) rotates \( i(\theta + \beta) \) stretches \( z \) by \(|a|r\) displaces \( az \) by \( \beta \).

3. Can show that circles map to circles, ovals to ovals, lines to lines.

Example: \( w = f(z) = az + b \) \( a = 1 + i, \ b = 1 \)

\[ w = az + b = (1+i)z + 1 = \bar{z} + 1; \quad \bar{z} = (1+i)z = \sqrt{2}e^{\pi i/4} \]

\[ \text{displacement.} \quad \text{rotation.} \quad \text{stretch.} \]

\[ z = x + iy \]

\[ \bar{z} = \bar{x} - i\bar{y} \]

\[ \bar{z} = (x - y) + i(x + y) = \bar{z} + i\bar{z} \]

\[ w = u + iv \]

\[ w = u + iv = 3 + 1 = (3 + 1) + i(3 + 1) \]

\[ w = u + iv = 3 + i \]

\[ w = 1 + \bar{z} \]

\[ w = 1 + \bar{z} = 1 + (x - iy) \]

\[ w = 1 + (x - iy) \]

\[ w = 1 + (x - iy) \]
Example: \( w = f(z) = \frac{1}{z} \)

1) 1-1: \( \frac{df}{dz} = -\frac{1}{z^2} \) so 1-1 everywhere except \( z = 0 \). But \( z = 0 \) maps to "the point at infinity" so 1-1 everywhere.

2) polar form: \( w = r e^{i\theta} \) \( \frac{r}{r} e^{i\theta} \Rightarrow r = \frac{1}{r}, \ \theta = -\theta \)

3) straight lines in \( z \)-plane:

Sub-Example: (special case)

Generalize: Straight lines in \( z \)-plane not passing through origin map to circles in the \( w \)-plane, for which one point is \( w = (0,0) \). Note that the two points \((x,y) = (1,i\infty)\) and \((x,y) = (1,-i\infty)\) map to the origin. Note also the sense (direction) of traversal: toward \( ty \) in \( z \)-plane is clockwise in \( w \)-plane.

Exercise: Map \( rsin\theta = z \) and \( rcos(\pi/4 - \theta) = 3 \) in the \( w \)-plane.
4) Circles in the $z$-plane:

Let $r = \rho|z|$, $\theta = \arg(z)$

$\Rightarrow \rho^2(r^2 - R^2) - 2\rho R r \cos(\theta - \theta_0) + 1 = 0$

If $\rho = 0$ then $r^2 = r^2(R = r)$ gives a circle of radius $R$ in the $z$-plane centered at the origin. Then $\rho = \sqrt{r}$ is a circle of radius $\sqrt{r}$ in $w$-plane also centered at the origin.

If $r_0 = R$ and $\theta_0 = 0$ then $r = 2R \cos \theta$ so $x = (2R \cos \theta) \cos \varphi, y = (2R \cos \theta) \sin \varphi$

and $\rho = \frac{1}{2R \cos \varphi}$

As $\theta$ varies from $\theta = 0$ to $\theta = \pi/2$, $\varphi$ varies from $\varphi = 0$ to $\varphi = -\pi/2$;

As $\theta = \pi/2$, $\varphi$ varies from $\varphi = 0$ to $\varphi = \pi/2$.

When $R$ is decreased the size of the circle in the $z$-plane diminishes and the vertical line in the $w$-plane moves to the right. Thus, the inside of the circle maps to the right of the vertical line $u = \sqrt{2}R$, $-\infty < v < \infty$ (as shown above by the shading). The outside of the $z$-plane circle maps to the left of this line (white region).
If \( r_0 \neq R \) then in the \( z \)-plane the circle does not pass through the origin (let’s say \( r_0 > R \)). The mapping in the \( w \)-plane is also a circle that does not pass through the origin. Thus:

\[
\begin{align*}
    r^2 &= 2r_0 \cos(\theta - \theta_0) - (r_0^2 - R^2) \quad \text{same circles} \\
    \rho^2 &= 2r_0 \rho \cos(\alpha + \theta_0) - \frac{1}{r_0^2 - R^2} \quad \text{mappings} = \text{circles} = \text{circles}.
\end{align*}
\]

**Exercise:** Draw the mappings for \( r_0 = 2, R = 1, \theta_0 = 0 \) giving \( r^2 = 4r \cos \theta - 3 \) and \( \rho^2 = \frac{4}{3} \rho \cos \theta - \frac{1}{3} \).

Let us write the transformation \( w = \sqrt{z} \) in terms of \( u, v, x, y \):

\[
    u + iv = \sqrt{x + iy} = \frac{x - iy}{\sqrt{x^2 + y^2}} \quad \text{so} \quad u = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } v = -\frac{y}{\sqrt{x^2 + y^2}}.
\]

Conversely \( z = \sqrt{u - iv} \) so \( x + iy = \frac{x - iy}{u + iv} = \frac{(u - iv)(u^2 + v^2)}{u^2 + v^2} \) so that \( x = \frac{u}{u^2 + v^2} \) and \( y = -\frac{v}{u^2 + v^2} \). Using these formulas it is sometimes easier to examine the geometrical form of the transformations.

The line \( x = c_1 \):

use \( x = u/(u^2 + v^2) \Rightarrow u^2 + v^2 = u = c_1 \). Let \( u = U + \xi \) so

\[
    U^2 + 2U \xi + \xi^2 + v^2 - (U + \xi) = 0.
\]

Choose \( \xi = \frac{1}{2c_1} \), so \( U^2 + v^2 = \frac{1}{4c_1^2} \). Now let \( U = U/2c_1 \) and \( v = V/2c_1 \), to get the circle \( U^2 + V^2 = 1 \).

Clearly as \( c_1 \) increases (the vertical line in the \( z \)-plane moves to the right) the circles in the \( w \)-plane shrink. The region \( x > c_1 \), thus maps to the inside of the circle.
Note: The circle has radius $\frac{1}{2c}$, and is centered at $u=\frac{1}{2c}$, $v=0$. Analytically the half-plane $x>\frac{1}{2}$ has $x=\frac{u}{u^2+v^2} > \frac{1}{2}$ or $u>\frac{1}{c}(u^2+v^2)$ or $u^2+v^2 > \left(u^2-2(u\frac{1}{2c})+\frac{1}{4c^2}\right)+v^2 \Rightarrow \left(\frac{1}{2c}\right)^2 > \left(u-\frac{1}{2c}\right)^2+v^2$. Thus the right half-plane maps to the inside of the circle with radius $\frac{1}{2c}$. Let $c=\frac{1}{2}$ so $1 > (u-1)^2 + v^2$ (circle of radius less than 1).

The line $y=c_2$: Use $y=-v/(u^2+v^2)$ \Rightarrow $u^2+v^2+v/y = 0 = u^2+v^2+v/c_2$. The circle in the $u-v$ plane is tangent to the $u$-axis at the origin. We have $u = x/(x^2+c_2^2)$ and $v = -c_2/(x^2+c_2^2)$. So $v$ is negative.

The point $(-\infty, c_2)$ maps to the "left-origin" while $(+\infty, c_2)$ maps to the "right origin."

The upper half of the $y$-plane (above $y=c_2$) maps to the inside of the circle (see shading).

Note: If the line $y=c_2$ is traversed oppositely ($c\rightarrow b \rightarrow a$ or right to left) then the $w$-plane circle is traversed oppositely, or clockwise.

Exercise: For the transformation $w=\frac{1}{2z}$ map the sequence of straight lines making a square $h\times h$ as shown below in the $z$-plane to the corresponding shape in the $w$-plane. Draw the shapes as $h\rightarrow 0$ and $h\rightarrow \infty$. 

\begin{align*}
& (-h/2, h/2) \rightarrow (h/2, h/2) \\
& (-h/2, -h/2) \rightarrow (h/2, -h/2)
\end{align*}
The infinite strip: \( 0 < y < \frac{1}{2}c_0 \)

As \( c_0 \) increases the strip becomes narrower and the circle radius becomes larger. Thus, the inside of the strip maps to the outside of the circle, as shown by the right-inclined hatchings. The region above the line \( y = \frac{1}{2}c_0 \) maps to the inside of the circle (left-inclined hatchings).

The Bilinear Transformation:

\[ w = f(z) = \frac{az+b}{cz+d} = \frac{a\frac{z}{c} + \frac{b}{c}}{z + \frac{d}{c}} = \frac{A + \frac{B}{c}}{z + \frac{d}{c}} = \frac{a}{c} + \frac{(bc-ad)}{c(cz+d)} \]

\[ B = \left( \frac{az+b}{c} \right)_{z=d/c} = \frac{bc-ad}{c^2} \]

\[ A = \left[ \frac{az+b}{c} - \frac{(bc-ad)}{c^2} \right]_{z=d/c} = a \]

Note: \( w = \frac{az+b}{cz+d} \Rightarrow w(cz+d) = az+b \Rightarrow z = \frac{wd+b}{wc-a} \)

Let \( d = \alpha, -b = \beta, -c = \gamma, a = \delta \): \( z = \frac{wd+\beta}{wc+\delta} \) (also bilinear).

The examples considered previously are special cases of the bilinear transformation, the first with \( c = 0 \), the second with \( a = d = 0, b = c \).
\[ w = f(z) = \frac{az+b}{cz+d} \]

\[ \frac{df}{dz} = \frac{a}{cz+d} - \frac{c(az+b)}{(cz+d)^2} = \frac{(ad-bc)}{(cz+d)^2} = \frac{\Delta}{(cz+d)^2} \neq 0 \text{ when } \Delta = ad-bc \neq 0. \]

When \( \Delta = 0 \) the mapping is not 1:1 \( \Rightarrow \) not useful.

**Three-Stage Mapping:**

\[ s = f(z) = az + b \quad \text{(stretch and displace)} \]
\[ z = g(s) = \frac{1}{s} \quad \text{(invert)} \]
\[ w = h(z) = yz + \delta \quad \text{(stretch and displace)} \]

\[ w = yz + \delta = \frac{y + \delta}{s} = \frac{y}{s} + \frac{\delta}{s} = \frac{y + \delta(\frac{az+b}{cz+d})}{s} = \frac{y + \delta}{s} + \frac{\delta(\frac{az+b}{cz+d})}{s} \]

\[ = \frac{\alpha z + (y + \delta\beta)}{\alpha z + \beta} = \frac{a z + b}{c z + d} \quad \left( \begin{array}{l}
\alpha = \frac{\alpha z + (y + \delta\beta)}{\alpha z + \beta} \\
b = y + \delta\beta \\
c = \alpha \\
d = \beta
\end{array} \right) \]

\[ z \xrightarrow{f(z)} s \xrightarrow{g(s)} z \xrightarrow{h(z)} w = \frac{az+b}{cz+d} \]

\[ w = h(z) = h(g(s)) = h(g(f(z))) = \frac{az+b}{cz+d}. \]
Other mappings:

\[ w = e^{\pi z / b} \]

\[ w = u + iv = e^{\pi i / \sqrt{b}} \left( \cos \left( \frac{\pi y}{b} \right) + i \sin \left( \frac{\pi y}{b} \right) \right) \]

The inside of the strip in the \( z \)-plane maps into the upper half of the \( w \)-plane (and vice-versa).

\[ w = -\cosh(\pi z / a) \]

\[ w = u + iv = \cosh(\pi x / a) \left( \cos \left( \frac{\pi y}{a} \right) + i \sin \left( \frac{\pi y}{a} \right) \right) \]

Now \( \cosh \theta = \frac{e^\theta + e^{-\theta}}{2} \) and \( \sinh \theta = \frac{e^\theta - e^{-\theta}}{2} \).

Let \( \theta = \pi x / a \).

\[ \cosh(\pi x / a) = \frac{e^{\pi x / a} + e^{-\pi x / a}}{2} = \cosh \theta \quad \text{and} \quad \sinh(\pi x / a) = \frac{e^{\pi x / a} - e^{-\pi x / a}}{2} = \frac{1}{\cosh \theta} \sinh \theta. \]

Hence:

\[ w = u + iv = \left( \cosh(\pi x / a) \cos \left( \frac{\pi y}{a} \right) + i \sin \left( \frac{\pi y}{a} \right) \right) \]

This maps the vertical semi-infinite strip in the \( z \)-plane into the upper-half \( w \)-plane.
Sector: $w = \left( \frac{\pi}{R} \right)^{\frac{1}{2}} \sqrt{1 - \left( \frac{x}{R} \right)^2}$

$\left. \begin{array}{c}
0 \leq \theta \leq \alpha \\
\end{array} \right.$

$w = u + iv = \left( \frac{x + iy}{R} \right)^{\frac{1}{2}} e^{i \frac{\theta}{\alpha}}$ since $x + iy = re^{i \theta}$

Write $w = \rho e^{i \varphi} \Rightarrow \rho = \left( \frac{\sqrt{x^2 + y^2}}{R} \right)^{\frac{1}{2}}$, $\varphi = \pi \frac{\theta}{\alpha}$.

Thus $r = 0 \Rightarrow \rho = 0$.
$r = R \Rightarrow \rho = 1$.
$\theta = 0 \Rightarrow \varphi = 0$.
$\theta = \alpha \Rightarrow \varphi = \frac{\pi}{2}$.

$w = f(z) = \left( \frac{z}{R} \right)^{\frac{1}{2}}$ maps the sector $0 \leq r \leq R$, $0 \leq \theta \leq \alpha$ into the upper half of the unit semicircle in the $w$-plane. Interior points of the sector map to interior points of the semicircle.

$w = \frac{z - 1}{z + 1}$ (mapping onto the unit disc)

$u + iv = \frac{z - 1}{z + 1} = \frac{-x + i(1-y)}{x - i(1+y)} = \frac{\left[ x-i(1+y) \right] \left[ x+i(1+y) \right]}{x^2 + (1+y)^2}$

$w = f(z) = \frac{z - 1}{z + 1}$

The upper half of the $z$-plane maps into the interior of the unit circle in the $w$-plane. Note that points $(1)$ and $(5^*)$ approach each other ("the point at infinity").
Exercise: For mapping $z = 2$ take $b = 2$ and draw the exact transformation for specific $(x,y)$ values, say $(x,y) = (-10,2), (0,2), (0,0)$, and so on.

Exercise: For mapping $z = 2$ use $a = 2$ and do the same.

Exercise: Generalize the map $z = 2$ and show what happens for $z > R$.

Exercise: For the mapping $w = \frac{1}{z}$ map the strip shown into a region of the $w$-plane. Draw it clearly.