Contour Integration:

Use the residue theorem to evaluate complex integrals over closed paths, obtaining results thereby for real integrals evaluated over real paths. Recalling
\[ \oint_C f(z)\,dz = 2\pi i \sum \text{residues} \]

where \( \sum \text{residues} \) is the sum of the \( n \) residues at all the poles and singularities of \( f(z) \) inside \( C \). For a simple pole the residue is \( C_1 = \lim_{z \to z_0} (z - z_0)f(z) \). For \( f(z) = \phi(z)/\chi(z) \) where \( \chi(z) = (z - z_0)^m \chi'(z_0) + (z - z_0)^{m-1} \chi''(z_0) + \ldots \) one finds \( C_1 = \phi(z_0)/\chi'(z_0) \). If \( z_0 \) is a pole of order \( m \) the residue is
\[ C_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[ \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right] \]

Example: \( I = \int_0^\infty \frac{dx}{1 + x^2} = \lim_{R \to \infty} \int_0^R \frac{dx}{1 + x^2} = \frac{1}{2} \lim_{R \to \infty} \int_{-R}^R \frac{dx}{1 + x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \)

Let us evaluate the complex integral \( J = \frac{1}{2} \oint_C \frac{dz}{1 + z^2} \) along the path shown:

Along \( 1 \): \( z = x \), \( dz = dx \) from \( x = -R \) to \( x = R \)

Along \( 2 \): \( z = e^{i\theta} \), \( dz = iRe^{i\theta} d\theta \) from \( \theta = 0 \) to \( \theta = 2\pi \)

\[ J = \frac{1}{2} \int_{-R}^{R} \frac{dx}{1 + x^2} + \frac{1}{2} \int_{0}^{2\pi} \frac{iRe^{i\theta}}{1 + R^2} d\theta \]

Let \( R \to \infty \) so that \( J = \frac{1}{2} \oint_C \frac{dz}{1 + z^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = I \). Thus, we can evaluate \( I \) by evaluating \( J \). We have \( J = \frac{1}{2} \oint_C \frac{dz}{1 + z^2} = \frac{1}{2} \oint_C \frac{dz}{(z + i)(z - i)} \)

\[ = \frac{1}{2} \left[ \oint_{C_1} \frac{dz}{z - i} - \oint_{C_2} \frac{dz}{z + i} \right] = \frac{1}{2} \left[ 2\pi i (\frac{i}{2}) \right] = \frac{\pi}{2} \]

Thus, \( I = \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2} \)
Reiteration: If the pole or singularity is not enclosed by the contour its contribution is zero.

\[ \oint f(z) \, dz = 2\pi i B_1 + 2\pi i B_2 = 2\pi i (B_1 + B_2) \]

Example: \[ I = \int_0^\pi \ln(\sin x) \, dx \]

Let's evaluate the integral \[ J = \oint_{\text{unit circle} \, |z| = 1} \ln(\epsilon + \sin \theta) \, d\theta \] and let \( \epsilon \to 0 \) at end.

How is \( I \) related to \( J \)?

\[ \text{Re}(J) = \text{Re} \oint_{|z|=1} \ln(-i\epsilon + \sin \theta) \, d\theta \]

As \( \epsilon \to 0 \), \( \text{Re}(J) = \oint_0^\pi \ln(\sin \theta) \, d\theta = 2 \oint_0^\pi \ln(\sin \theta) \, d\theta = 2I \)

Thus, \[ I = \frac{1}{2} \text{Re}(J) \]

and we evaluate \( I \) by calculating \( J \) and then finding its real part.

On the unit circle \( z = 1 + \epsilon e^{i\theta} \)

\[ z = \frac{1}{2} = \frac{1}{2} e^{i\theta} - i \theta \]

\[ = \left( e^{i\theta} + e^{-i\theta} \right) \]

\[ \Rightarrow \sin \theta = \frac{1}{2} \left( e^{i\theta} - e^{-i\theta} \right) = \frac{1}{2i} (z - \frac{1}{2}) \]

\[ \Rightarrow \frac{\epsilon}{i} + \sin \theta = \frac{\epsilon}{i} \left( \frac{z - \frac{1}{2}}{2i} \right) = \frac{z^2 + 2\epsilon z - 1}{2i\epsilon} = \frac{(z - z_1)(z - z_2)}{2i\epsilon} \]

where \( z_1 = -\epsilon + \sqrt{1 + \epsilon^2} \) and \( z_2 = -\epsilon - \sqrt{1 + \epsilon^2} \). As \( \epsilon \to 0 \) get \( z_1 \approx 1 - \epsilon + \ldots \) and \( z_2 \approx -(1 + \epsilon) + \ldots \) so \( z_1 \) is inside \( |z| = 1 \) and \( z_2 \) is outside \( |z| = 1 \). See the figure on next page.
Thus: \( \ln \left( \frac{z-z_1(z-z_2)}{2iz} \right) = \ln \left( \frac{z-z_1}{z} \right) + \ln \left( \frac{z-z_2}{2i} \right) \)

analytic outside the unit circle.

Write \( z = e^{i\theta} \), \( dz = i2\pi e^{i\theta} \) so \( d\theta = dz/iz \) on the unit circle.

Then:

\[
\mathcal{J} = \oint_{|z|=1} \ln \left( \frac{z-z_1(z-z_2)}{2iz} \right) \, dz
\]

\[
= \oint_{|z|=1} \frac{\ln(z-z_1)/z}{iz} \, dz + \oint_{|z|=1} \frac{\ln(z-z_2)/2i}{iz} \, dz
\]

We put a negative sign in front of the first integral and reverse the contour so

\[
\oint_{|z|=1} \frac{\ln(z-z_2)/2i}{iz} \, dz = -\oint_{|z|=1} \frac{\ln(z-z_2)/z}{iz} \, dz.
\]

The function is analytic outside the circle but there are no poles there (the pole \( z=0 \) is inside the circle) so the residue is zero. The second integral is essentially \( f(z) = \phi(z)/\psi(z) \), \( \phi(z) = \ln(z-z_2)/2i \) (analytic inside circle), \( \psi(z) = iz \) so the residue is \( c_1 = \frac{1}{i} \ln \left( \frac{z_2}{2i} \right) = \frac{1}{z} \ln \left( \frac{e+1+i6}{2i} \right) \). Thus,

\[
\mathcal{J} = 2\pi i \left( \frac{1}{i} \ln \left( \frac{e+1+i6^2}{2i} \right) \right) = 2\pi \ln \left( \frac{e+1+i6^2}{2i} \right) = 2\pi \left[ \ln(e+1+i6^2) - \text{Im} \ln 2 - \frac{i\pi}{2} \right]
\]
\[ \text{Re}(J) = 2\pi \ln(e + i\sqrt{1 + e^2}) - 2\pi \ln 2 \]

As \( \varepsilon \to 0 \) get \( \text{Re}(J) = -2\pi \ln 2 \)

Then \( I = \frac{1}{2} \text{Re}(J) \bigg|_{\varepsilon=0} = -\pi \ln 2 \)

It is worthwhile studying this outstanding exercise (taken from Carrier, Krook and Pearson, Functions of a Complex Variable, McGraw-Hill, 1966).

**Exercise:** Evaluate \( I = \int_0^\infty \frac{dx}{1 + x^2} \) by calculating \( J = \oint \frac{dz}{1 + z^2} \) along the contour shown: (Mathews & Walker, pp. 68-69)

**Exercise:** Evaluate \( I = \int_0^\infty \frac{dx}{a + bx^2} \) noting that the integrand is even and then integrating along the unit circle with \( z = e^{i\theta}, \) \( dz = ie^{i\theta}d\theta \) and \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 = (1 + i\varepsilon)/2 \) (Mathews & Walker, pp. 69-70).

**Exercise:** Evaluate \( I = \int_{-\infty}^\infty \frac{e^{ax}}{e^x + 1} \) dx, \( 0 < a < 1 \) using \( J = \oint \frac{e^{az}}{e^z + 1} \) along a suitably-chosen contour (Mathews & Walker, pp. 71-72).