
Euler (circa 1750): \[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \Rightarrow e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
Now let \( z = x + iy \) \( \Rightarrow \) \( e^z = e^{x+iy} = \frac{e^{x}e^{iy}}{e^{i\pi}} = e^{x} \left( \cos y + i \sin y \right) \)
Thus: \( e^{x+iy} = e^x \left( \cos y + i \sin y \right) \). A powerful, extremely useful formula.

Exercise: Use \( \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \) to show \( \tan(x) = \frac{1}{\cos(x)} \).

The algebra of \( \sqrt{-1} \): The quantity \( \sqrt{-1} \) "solves" the quadratic equation \( x^2 = -1 \). In fact, its solutions are \( \pm \sqrt{-1} \). We write \( \sqrt{-1} \) so \( x^2 = 1 \) is "solved" by \( \pm 1 \).

- Addition: \( (a + ib) + (c + id) = (a + c) + i(b + d) \)
- Subtraction: \( (a + ib) - (c + id) = (a - c) + i(b - d) \)
- Multiplication: \( (a + ib)(c + id) = (ac - bd) + i(ad + bc) \)
- Division: \( \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} \)
- Exponentiation: \( e^{i\theta} = \cos \theta + i \sin \theta \) (Euler)
- Fractional power: \( (x + iy)^n = (x + iy)^n \)
- Logarithm: \( \ln(x + iy) = \ln(\sqrt{x^2 + y^2}) + i\arctan(y/x) \)

Definition: Complex number \( z = a_0 + a_1 i + a_2 i^2 + a_3 i^3 + \ldots + a_n i^n \)

\[ = \left( \frac{a_0 - a_2 + a_4 + \ldots}{a} \right) + i \left( \frac{a_1 - a_3 + a_5 + \ldots}{b} \right) = a + ib \]

Uniqueness: \( a + ib = c + id \)

\[ a + c + i(b - d) = 0 \] \( \Rightarrow \) \[ (a + c) + i(b - d) = 0 \] \( \Rightarrow \) \[ (a - c)^2 + (b - d)^2 = 0 \]
This is true if \( a = c \) and \( b = d \) so \( a + ib = c + id \) is unique.

Complex Conjugate (used above) is defined s.t. if \( z = a + ib \), then \( \overline{z} = a - ib \).
It follows that \( a = \text{Re}(z) = (z + \overline{z})/2 \); \( b = \text{Im}(z) = (z - \overline{z})/2i \);
\( \overline{z} = a^2 + b^2 = 0 \) iff \( a = b = 0 \).

Complex numbers:
Product: \( z_1z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc) = x + iy = z \)
\( x = ac - bd \); \( y = ad + bc \)
\( (a, b) \cdot (c, d) = (ac - bd, ad + bc) \)

Quotient: \( z = \frac{z_1}{z_2} \) or \( (a + ib)/(c + id) = (cx - dy) + i(dx + cy) \)
\( a = cx - dy \)
\( b = dx + cy \)
\( \Rightarrow \) \( x = (ac + bd)/(c^2 + d^2) \) and \( z = \frac{z_1}{z_2} \) is solved

Easier: \( z = \frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{(c^2 + d^2)} = x + iy \) (as above).
\( z = \frac{z_1}{z_2} = \frac{z_1}{z_{2\overline{z}_2}} = x + iy \)

Square Root: \( z^2 = (x + iy)^2 = w = \alpha + i\beta \)
\( \pm (x + iy) = \sqrt{\alpha + i\beta} \) \( (\pm z = \sqrt{w}) \).

Exercise: Show that \( x = \frac{\sqrt{\alpha + \beta}}{2} \), \( y = \frac{\sqrt{-\alpha + \beta}}{2} \) where \( \beta = \sqrt{\alpha^2 + \beta^2} \) and \( \beta \neq 0 \).
Why do we require the "+" sign?

The formal mathematics of complex number analysis is linked to planar geometry. Graphical solutions can easily be "seen" before the algebraic details are worked out.

Complex number plane:

\[ r = \sqrt{x^2 + y^2} \equiv |Z| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \] Modulus of \( z = x + iy \) is \( |z| = \sqrt{z\overline{z}} \).

Geometry: \( x = r\cos\theta \); \( y = r\sin\theta \).
Argument of complex number $z$ is $\theta = \tan^{-1} \frac{y}{x}$. Thus $z = r \cos \theta + i r \sin \theta = r (\cos \theta + i \sin \theta) = x + iy$.

The complex conjugate is a reflection about the $x$-axis. $\bar{z} = re^{-i\theta}$.

Product (geometrically):

$z_1 z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) r_2 (\cos \theta_2 + i \sin \theta_2) = r_1 r_2 (\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2))$

Thus: $|z_1 z_2| = |r_1 r_2| = |z_1|/|z_2|$

Using Euler's formula:

$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r e^{i\theta}$

Quotient and powers:

$z = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2}$

$z^n = r^n e^{i\theta} = r^n (\cos n\theta + i \sin n\theta)$

Consider the special case $r = 1$: $e^{i\theta} = \cos \theta + i \sin \theta = (e^{i\pi})^n = (\cos \theta + i \sin \theta)^n$

This is known as de Moivre's formula. Special case $n = 2$ gives $\cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + i(2 \sin \theta \cos \theta)$ so $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$.

Easily extended to $n = 3, 4, \ldots$

Logarithm:

$\ln z = \ln (re^{i\theta}) = \ln r + \ln (e^{i\theta}) = \ln r + i\theta$. This result, superficially so simple and obvious, leads to serious multiplicity issues, and requires careful consideration. Note that $e^{i\theta} = e^{i(\theta + 2\pi n)}$ since $e^{i(2\pi n)} = 1$, so that $\ln (e^{i\theta}) = \ln (e^{i(\theta + 2\pi n)})$ so $\ln r + i\theta = \ln r + i\theta + \frac{i(2\pi n)}{n}$. Either $n = 0$ or there is more to this formulation.

Power law multiplicity:

$z = re^{i\theta} \Rightarrow z^n = r^n e^{i\theta} = re^{i\theta}$
Written out: \( \rho \left( \cos \theta + i \sin \theta \right) = \rho \left( \cos \phi + i \sin \phi \right) \) so \( \rho = 2 \)

Now \( z = \rho \left( \cos \theta + i \sin \theta \right) = \left( e^{i \theta} \right)^{1/n} = (\rho e^{i \theta})^{1/n} = \rho^{1/n} e^{i \theta/n} = \rho^{1/n} e^{i \theta} \)

But this expression applies both for \( \theta = \omega/n \) and \( \theta = \omega/n + 2\pi m \) since \( e^{i \theta} = e^{i (\omega/n + 2\pi m)} = e^{i \omega/n} (\cos 2\pi m + i \sin 2\pi m) \). We can go further, however, and write the integer \( m \) as the quotient \( m = k/n \). Then \( e^{i \omega/n} = e^{i (\omega/n + 2\pi k/n)} = e^{i \omega/n} \left( \cos 2\pi k/n + i \sin 2\pi k/n \right) \). Are there limits on \( k \)? Write \( z = \rho^{1/n} e^{i \omega/n} = \rho^{1/n} e^{i (\omega/n + 2\pi k/n)} \). Distinct (unique) values (roots) for \( z \) are obtained for \( k = 0, 1, 2, \ldots, n-1 \). If \( k = n \) we return to the \( k = 1 \) case, since \( k = 1 \) gives \( z = \rho^{1/n} e^{i (\omega/n + 2\pi/n)} \) and \( k = n \) gives \( z = \rho^{1/n} e^{i (\omega/n + 2\pi n/n)} \) with \( k = 0 \) these are identical "roots" since they land geometrically on the same point.

**Example:** Let \( z = e^{i \pi/4} \) (so \( \rho = 1, \theta = \pi/4 \))

Let \( n = 2 \)

So \( z = (e^{i \pi/4})^{1/2} = e^{i \pi/8} = e^{i \phi} \) (so \( \rho = 1 \) and \( \phi = \pi/8 \)).

Write \( \theta = \frac{\pi}{4} + \frac{2\pi k}{n} = \frac{\pi}{4} + \frac{2\pi k}{2} = \frac{\pi}{2} + \frac{2\pi k}{2} \) with \( k = 0, 1 \).

\[
\begin{align*}
\text{Geometrically:} \\
\rho = 1 (n = 1) \\
\theta = \frac{3\pi}{4} \\
\phi = \frac{\pi}{8} \\
\end{align*}
\]

So \( z^2 = e^{i \pi/2} \):

\[
\begin{align*}
\left( e^{i \pi/4} \right)^2 &= e^{i \pi/2} \\
e^{i \pi/2} &= e \\
e^{i \pi/2} &= -i \\
\end{align*}
\]

**Example:**

\( z^3 = 1 = (\rho e^{i \phi})^3 \) (\( \rho = 1, \phi = 0 \))

\( z^3 = \rho^3 (\cos 3\phi + i \sin 3\phi) = \rho^3 e^{i 3\phi} = e^{i (\omega + 2\pi k/3)} \)

So \( \rho = 1, \phi = 0 \) or \( \phi = \omega/3 \)

Also \( \omega = \frac{2\pi}{3} + \frac{2\pi k}{3} \) with \( k = 0, 1, 2 \).
So \( k = 0 : \theta = 0 \)
\( k = 1 \): \( \theta = 2\pi/3 \)
\( k = 2 \): \( \theta = 4\pi/3 \)

(Note: \( k = 3 \) would give \( \theta = 6\pi/3 = 2\pi \) so we return to the \( k=0 \) case)

The three roots of \( z^3 = 1 \) are:
\[
\begin{align*}
Z_1 &= \cos(0) + i\sin(0) = 1 \\
Z_2 &= \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
Z_3 &= \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{align*}
\]

\[
\begin{align*}
Z_1 &= e^{i0} = 1 \\
Z_2 &= e^{i2\pi/3} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
Z_3 &= e^{i4\pi/3} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{align*}
\]

Geometrically:

Unit circle: \( r = |z| = 1 \). Defines the unit circle.

Triangle inequality: \( |z_1 + z_2| \geq |z_1| + |z_2| \geq |z_1 - z_2| \).

Proof: \( |z_1 + z_2| \geq |z_1| + |z_2| \). Let \( z_1 = a_1 + i b_1 \), \( z_2 = a_2 + i b_2 \). Then
\[
|a_1^2 + b_1^2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}.
\]

Squaring both sides:
\[
\left|a_1^2 + b_1^2\right| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}.
\]

Squaring both sides:
\[
(a_1 + a_2)^2 + (b_1 + b_2)^2 = a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2.
\]

This is known as the Schwarz inequality and can be proved to be rigorously true. The Schwarz inequality thus proves the triangle inequality.

Exercise: By appropriate redefinitions and normalizations write the Schwarz inequality as \( 2x\beta \leq x^2 + \beta^2 \).

Exercise: Write \( x^2 + \beta^2 \geq 2x\beta \) as \( (x - \beta)^2 \geq 0 \), which is always true. This proves the Schwarz inequality.

Exercise: Show that \( |z_1 + z_2| \geq |z_1| - |z_2| \) by writing \( |(z_1 + z_2) - z_2| \leq |z_1 + z_2| + |z_2| \).
The number \(i = \sqrt{-1}\) was not at first accepted by many mathematicians. The use of complex number grew and their capacity to explain and clarify many problems grew until their use became expedient and convenient. In physics and engineering usually only the modulus or the real part of a complex function (a "solution") is taken to possess "physical meaning." If we write the pressure field oscillation as \(A = e^{i(kx + wt)}\) then either of \(\text{Re}(A) = \beta \cos (kx + wt)\) or \(|A| = \varepsilon\) can be said "to exist." The imaginary part \(\xi \sin (kx + cw)\) is "not real" or unrealistic or unphysical.

Example: If only real roots are allowed then the polynomial of third degree \(x^3 = 1\) has only one root \((x = 1)\). If complex numbers are allowed it has three roots. Two are complex conjugates, one is real. Thus the general result: An \(n\)-th degree polynomial has \(n\) roots some (possibly all) complex.

Example: \(x^2 + 2bx + c = 0 = (x-x_1)(x-x_2)\). What are \(x_1\) and \(x_2\)?

\[x_{1,2} = -b \pm \sqrt{b^2 - c}\]

If \(b^2 < c\) then \(x_{1,2} = -b \pm i \sqrt{c-b^2} = \left\{ \begin{array}{ll} -b + i\alpha \nonumber \\
-b - i\alpha \nonumber \end{array} \right.\]

The two roots are complex conjugates. It is easily checked that

\[
(x-x_1)(x-x_2) = x^2 - (x_1+x_2)x + x_1x_2
\]

\[= x^2 - (-2b)x + (b^2 - c^2) = x^2 + 2bx + c = 0 \checkmark\]

\[x_1 = -b + i\alpha \]

there are no real roots when \(b^2 < c\).

\[x_2 = -b - i\alpha \]
Exercise: For $b=\{-1,0,1\}$ let $-\infty < c < \infty$ and plot in the $x-y$ plane the evolution of $x_0, x_2$. Identify $x_1$ and $x_2$ on each branch. Identify the $c$-value at which the solution(s) become "physical." Do the "unphysical" solutions have any mathematical relevance?