Collective Decision Making in Ideal Networks: The Speed-Accuracy Trade-off

Vaibhav Srivastava
Department of Mechanical & Aerospace Engineering
Princeton University

Joint work with: Naomi Leonard

Oct 28, 2014
Symposium on Control of Network Systems
Boston, Massachusetts, USA

Incomplete Literature Review

Human Decision-making

Information assimilation in social networks

Collective decision-making in animal and human groups

Collective Decision Making in Socio-Cognitive Networks

Social information assimilation + decision-making = Socio-Cognitive Networks

Ideal group versus Condorcet group

Role of network structure in performance


V. Srivastava & N. E. Leonard (Princeton University)
Drift Diffusion Model and the Free Response Paradigm

- Models human decision making in two alternative choice tasks
- Evidence evolution in a two alternative choice task is modeled by
  \[ dx(t) = \beta dt + dW(t), \quad x(0) = x_0 \]
- Decision process at time \( \tau \) is
  \[ \begin{cases} x(\tau) > \eta, & \text{choose alternative 1,} \\ x(\tau) < -\eta, & \text{choose alternative 2,} \\ \text{else}, & \text{collect more evidence.} \end{cases} \]
- Continuous time version of Sequential Probability Ratio Test

Choice of threshold dictates the speed-accuracy trade-off

Social Interaction and the DeGroot Model

- \( p \): vector of opinions in a network
- \( A \): row stochastic matrix
- models consensus seeking in a social network by
  \[ p(t + 1) = Ap(t). \]
- same as the celebrated consensus dynamics in multi-agent systems
- Continuous time consensus seeking in a social network modeled by
  \[ \dot{p}(t) = -Lp(t), \quad p(0) = p_0 \]
- \( L = \text{Laplacian Matrix} \)

Coupled Drift Diffusion Model

- \( n \) decision-makers collect noisy signals and interact with each other
- the evidence aggregation process well modeled by
  \[ dx(t) = -Lx(t)dt + \beta_1 dt + \sigma dW(t), \quad x(0) = 0_n, \quad (1) \]

Quantities of interest:

- Expected decision times
- Error rates (probability of wrong decision)

Coupled Drift Diffusion Model

- \( n \) decision-makers collect noisy signals and interact with each other
- the evidence aggregation process well modeled by
  \[ dx(t) = -Lx(t)dt + \beta_1 dt + \sigma dW(t), \quad x(0) = 0_n, \quad (1) \]

Quantities of interest:

- Expected decision times
- Error rates (probability of wrong decision)

Standard approach:

- solve first passage time associated with the FP equation for (1)
- requires the solution of a second order PDE with \( n \) variables

Asymptotic Optimality of the Coupled DDM

- Evidence vector: \( x(t) = x_{\text{cen}}(t)1_n + \epsilon(t) \)

\[
dx_{\text{cen}}(t) = \beta dt + \frac{1}{n} \sum_{i} \epsilon_i(t) dW(t), \quad x_{\text{cen}}(0) = 0
\]

\[
de(t) = -Le(t)dt + (L_n - \frac{1}{n} \sum_{i} \epsilon_i(t)) dW_n(t), \quad \epsilon(0) = 0_n.
\]

- \( \epsilon_k(t) \to N(0, 1/\mu_k), \quad \frac{1}{\mu_k} = \sum_{p=2}^{n} \frac{1}{2 \pi \rho^2} u_k(\rho)^2 \)
- \( \mu_k \) is a certainty index determined purely by the interaction graph
Decoupled Approximation to the Coupled DDM

- Decoupled approximation to $\epsilon(t)$
  
  \[ \frac{d\epsilon(t)}{dt} = -L\epsilon(t)dt + (L_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T)dw(t), \epsilon(0) = 0_n \]

- $\epsilon_k(t)$ is a continuous Gaussian process and converges to $\mathcal{N}(0, 1/\mu_k)$
- $\mu_k$ is a measure of node certainty

- Approximate $\epsilon_k(t)$ by the O-U process
  
  \[ \frac{d\epsilon_k(t)}{dt} = \frac{\mu_k}{2} \epsilon_k(t)dt + dw(t), \epsilon_k(0) = 0 \]

Efficiency of approximation

\[ \lim_{t \to +\infty} \text{corr}(\epsilon_k(t), \epsilon_k(t)) = \mu_k \sum_{p=1}^{n} \frac{1}{2 \text{eig}_p(L + \text{diag}(\mu/2))} (\text{eig}_p)^2 - \frac{2}{n} \]

- Approximate evidence at node $k$: $x_{\text{cen}}(t) + \epsilon_k(t)$
- Decision time and Error Rate: need to solve $n$ elliptic PDEs with two variables opposed to a PDE with $n$ variables earlier

Numerical Illustration: Decoupled Approximation

The reduced DDM approximates the coupled DDM well.
Further Approximations

- bound the contribution by the O-U process $\varepsilon_k(t)$
- for sufficiently large $K$, with high probability
  \[ \max_{t \in [0,T]} |\varepsilon_k(t)| \leq \frac{K}{\sqrt{4K}} \]
- effective threshold for the centralized DDM belongs to the set $(\eta - K/\sqrt{4K}, \eta + K/\sqrt{4K})$

Expected decision time

Bounds on Decision Time and Error Rates

\[ \frac{\eta_k - K}{\beta \sqrt{\gamma}} \tanh \left( \beta n (\eta_k - K) \right) \leq ET_k \leq \frac{\eta_k + K}{\beta \sqrt{\gamma}} \tanh \left( \beta n (\eta_k + K) \right) \]
\[ \frac{1}{1 + \exp \left( 2\beta n (\eta_k + \frac{K}{\sqrt{4K}}) \right)} \leq ER_k \leq \frac{1}{1 + \exp \left( 2\beta n (\eta_k - \frac{K}{\sqrt{4K}}) \right)} \]

Empirical Estimates for Threshold Correction

- coupled DDM approximated well by centralized DDM with a modified threshold
- effective threshold at node $k = \eta - \frac{K(\beta)}{\sqrt{4K}}$, $K(\beta) = \frac{e^{\frac{-1}{\sqrt{\gamma}}}}{\sqrt{\pi(1+\beta/3)}}$

Numerical Illustration: Threshold Corrected Centralized DDM

Centralized DDM with corrected thresholds approximates the coupled DDM well.
**Information Centrality and Node Certainty**

**Information Centrality**

The inverse of the mean of the effective path lengths from the given node to every other node in the interaction graph.

- It is known that
  \[
  \frac{1}{\mu_k} = \frac{\sigma^2}{2} \left( \frac{1}{\text{Kirchoff Index}(k)} - \frac{K_f}{\sigma^2} \right)
  \]

- Node centrality is equivalent to information centrality


**Speed-Accuracy Trade-off**

**Wald Criterion:**
- Choose threshold to achieve a desired accuracy (error rate)
- More centrally located node has smaller threshold
- More centrally located node has smaller expected decision time

**Bayes Risk Criterion:**
- Choose threshold to minimize Bayes risk
  \[
  \text{BR}_k = c\text{ER}_k + ET_k
  \]
- More centrally located node has smaller threshold
- Each node has the same expected decision time and error rate

Similar story for reward-rate criterion

**Conclusions and Future Directions**

**Conclusions:**
- towards rigorous modeling and analysis of socio-cognitive networks
- coupled DDM as model for social decision-making in 2-AC tasks
- a computationally tractable decoupled approximation to coupled DDM
- further approximation by the threshold corrected centralized DDM
- ideas extend to multi-alternative choice tasks and 2-AC tasks with recency effect

**Future Directions:**
- relaxing the continuous communication assumption
- heterogeneous individuals
- general decision-making tasks, e.g., multi-armed bandits
- experiments with fish schools and humans