On Bifurcations in Nonlinear Consensus Networks

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Collective motion in animal groups I

How does each fish decides on its direction of motion?

**Proposition:** The direction of motion of each fish is determined by the average of its and its neighbors' direction.

These adjacency based averaging or consensus algorithms have been vastly applied in engineering applications.

DeGroot'74, Tsitsiklis'84, Jadbabaie et al '03, Olfati-Saber et al '07

Collective motion in animal groups II

For small differences in preferences of individuals, the animal group decide via consensus.

For significant difference in preferences, under certain conditions, the decision dynamics bifurcates around the consensus value.

Couzin et al '05, Nabet et al '06, '09

Nonlinear phenomena in engineered networks

**Power network**

Arapostathis et al '81: Global analysis of swing dynamics
- There are multiple equilibria
- There are bifurcations as system parameters vary

**Group coordination and Kuramoto oscillators**

Arcak'07: Passive protocols for consensus.
Chopra et al '09: Exponential synchronization of Kuramoto oscillator.
Nabet et al '09: Dynamics of Decision Making in Animal Group Motion.

Objectives of this talk

- Develop frameworks to define spatially distributed nonlinear protocols in networked systems.
- Determine the asymptotic behaviors obtained through these frameworks.
- Study the normal forms of the bifurcations using these frameworks.
- Elucidate on the dynamic properties of these systems.

Review: The Laplacian flow

The consensus dynamics or the Laplacian flow on $\mathbb{R}^n$ is defined by

$$\dot{x}_i = -\mathcal{L}(G)x, \quad \text{i.e.,} \quad \dot{x}_i = \sum_{j \in \text{adj}(i)} x_j - x_i.$$

The Laplacian flow converges to the set

$$C = \{x \in \mathbb{R}^n \mid x_1 = \cdots = x_n\}.$$

Frameworks for distributed nonlinear protocols on networks

**Absolute nonlinear flow**

$$\dot{x}_i = \sum_{j \in \text{adj}(i)} (f_i(x_i) - f_j(x_j))$$

**Relative nonlinear flow**

$$\dot{x}_i = \sum_{j \in \text{adj}(i)} f(x_i - x_j)$$

**Disagreement nonlinear flow**

$$\dot{x}_i = f \left( \sum_{j \in \text{adj}(i)} (x_i - x_j) \right)$$

**Assumption:** The graph $G$ is undirected, connected, and has no weights.

Absolute nonlinear flow

The absolute nonlinear flow on $\mathbb{R}^n$ is defined by

$$\dot{x} = \sum_{j \in \text{adj}(i)} (f(x_i) - f_j(x_j)) \quad \equiv \quad \dot{x} = -\mathcal{L}(G)f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is some smooth function.

The set of equilibrium points is

$$\{x^* \mid (f(x^*))_1 = \cdots = (f(x^*))_n\},$$

an invariant over the set of balanced directed graphs with a globally reachable node.
A toy example - the pitchfork nonlinearity

Pick \( f_i : \mathbb{R} \to \mathbb{R} \) defined by \( f_i(x) = \gamma x - x^3 \) for each \( i \in \{1, \ldots, n\} \).

Set of equilibrium points

Absolute nonlinear flow with pitchfork nonlinearity on a graph with three nodes.

**Equilibrium points**

For \( \gamma \leq 0 \), the set of equilibrium points is \( \mathcal{E}_c = \text{diag}(\mathbb{R}^n) \).

For \( \gamma > 0 \), the set of equilibrium points is

\[
\mathcal{E}_b = \{f_-(\beta), f_0(\beta), f_+ (\beta)\}^n \mid \beta \in [-\sqrt{\frac{4}{3} \gamma}, \sqrt{\frac{4}{3} \gamma}] \}
\]

where \( f_0, f_\pm : [-\sqrt{\frac{4}{3} \gamma}, \sqrt{\frac{4}{3} \gamma}] \to \mathbb{R} \) is defined by

\[
f_0(\beta) = \beta, \quad \text{and} \quad f_\pm (\beta) = -\frac{\beta}{2} \pm \sqrt{\gamma - \frac{3}{4} \beta^2}.
\]

**Convergence**

For \( \gamma \leq 0 \), the pitchfork Laplacian flow converges to the set \( \mathcal{E}_c \), while for \( \gamma > 0 \), each equilibrium point \( x^* \in \mathcal{E}_b \) is locally stable if and only if \( 3x_i^{*2} > \gamma \), for each \( i \in \{1, \ldots, n\} \).

Relative nonlinear flow

The *relative nonlinear flow* on \( \mathbb{R}^n \) is defined by

\[
\dot{x}_i = \sum_{j \in \text{adj}(i)} (f(x_i - x_j)),
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is some smooth function.

- Given an equilibrium point of the relative nonlinear flow, the passivity based approach of Arcak '07 can be used to ascertain stability.

- Unfortunately, it is very hard to determine equilibrium points of the relative nonlinear flow.
A toy example - the pitchfork nonlinearity

Pick $f_i : \mathbb{R} \to \mathbb{R}$ defined by $f_i(x) = \gamma x - x^3$ for each $i \in \{1, \ldots, n\}$.

Disagreement nonlinear flow

The \textit{disagreement nonlinear flow} on $\mathbb{R}^n$ is defined by

$$\dot{x}_i = f_i \left( \sum_{j \in \text{adj}(i)} (x_i - x_j) \right), \quad \equiv \quad \dot{x} = f(L(G)x),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function.

The set of \textit{equilibrium points} is

$$\{x^* \in \mathbb{R}^n | L(G)x^* \in \{z_1, \ldots, z_r\}^n\},$$

where $z_i$, $i \in \{1, \ldots, r\}$ are real roots of $f_i(z) = 0$.

Complete characterization of the stability can be found in the paper.

Conclusions and Future directions

Conclusions

- Three frameworks to define spatially distributed nonlinear protocols
- Very simple local rules yield a rich set of behaviors

Future directions

- May be useful in engineered systems, e.g., group coordination, bio-inspired systems, etc
- To determine an "optimal dynamics" to achieve "certain desired behavior" in a distributed dynamical system.