Abstract—First-passage time (FPT) of an Ornstein-Uhlenbeck (OU) process is of interest in various contexts. This paper investigates tuning of FPT moments of an OU process. The region of interest is defined by two boundaries, out of which at least one is absorbing. We find that the FPT distribution of an OU process is scale invariant with respect to the drift parameter, i.e., the drift parameter just controls the mean FPT and does not affect the shape of the distribution. This facilitates independent control of the mean and the coefficient of variation (CV) of the FPT. We also explore the effect of control parameters on the FPT distribution, and find parameters that minimize the distance between the FPT distribution and a desired distribution.

I. INTRODUCTION

The first passage time (FPT) is the earliest time at which a trajectory of a stochastic process initially inside a bounded region leaves the region. The FPTs are extensively used across disciplines, including neuroscience [1], biology [2]–[4], finance [5], ecology [6], engineering [7], statistical physics [8], finance [9], and health science [10] to model several interesting phenomena. For example, the FPT of diffusion processes is used to model human decision-making [1], animal foraging [6], financial markets [9], and clock synchronization [7].

In this paper, we study control of the FPT statistics of an Ornstein-Uhlenbeck (OU) process. An OU process belongs to the class of diffusion processes and is a generalization of drift-diffusion process. The OU process is also a continuum approximation to several discrete time Markov models. For biological phenomena modeled by FPTs, the analysis in this paper can provide insights into the mechanisms these systems employ to cope with uncertainty and ensure resilient performance. For example, how attention and memory is modulated in human decision-making, or how a gene’s expression is regulated to control timing of its response, or how animals regulate their foraging activity. For engineered systems, these analysis may provide insights into optimal control laws that delay an undesired event such as epidemic outbreak, or optimal control laws that achieve a desired distribution for time to certain event such as adoption of a product by certain fraction of population.

A related problem of steering a linear stochastic system to a desired final distribution has been previously studied [11]. However, computing and controlling FPT distribution is significantly more complicated than controlling the evolution of trajectories without boundaries. Indeed, the Fokker-Plank equation for the OU process is nonlinear and has limited tractability [12]. Also related is the work in [13], wherein the boundary of the region is controlled to steer the FPT distribution to a Gamma distribution. Loosely speaking, this problem can be thought of as a boundary control of a PDE [14], where underlying PDE is the Fokker-Planck equation.

Investigation of optimal feedback strategy that provides desired FPT statistics for a bursty birth-death process process has also been in our recent work [15]. It was found that the best strategy to minimize the coefficient of variation (CV) of FPT for a fixed mean FPT is to have zero death rate and a constant rate of birth. These results interpreted in continuum limit would mean that the optimal stochastic process (within OU processes) for minimizing the CV of FPT for a given mean is the drift-diffusion process. In other words, the optimal control does not require state feedback. In this paper, we explore this in more detail.

Although FPT properties of OU processes have been extensively studied in the literature [16], analysis of how the process can be steered to some desired FPT statistics is lacking. This paper utilizes characteristic function to study how FPT moments as well as FPT distribution can be steered towards their respectively desired values/functions. The approach is analytically and numerically tractable and provides important insights into the FPT behavior of the OU process. We also show that the FPT distribution for OU process is scale-invariant with respect to drift parameter, which facilitates independent tuning of the mean and CV of the FPT.

The paper is organized as follows. Section II introduces the problem. Section III presents background results on the characteristic function for the FPT of an OU process. Section IV uses the characteristic function to find properties of moments of the FPT, and optimal parameters that lead to desired moments. A more general control problem that explores the parameter space to reach a desired FPT distribution is studied in Section V. Finally, conclusions and future work are discussed in Section VI.

II. PROBLEM DESCRIPTION

Consider an OU process defined by the following stochastic differential equation

\[ dx = -\theta x dt + \sigma \sqrt{\theta} dw_t. \] (1)
Here $x$ is the state, $\theta \in \mathbb{R}_{>0}$, and $\sigma \in \mathbb{R}_{>0}$ are parameters, and $dW_t$ are i.i.d. Wiener increments. We will refer to $\theta$ as the drift and $\sigma$ as the relative noise strength. Let $a$ and $b$ denote two thresholds such that $a < b$. The FPT, $\tau$, for $x(t)$ to cross either of the thresholds is mathematically defined as

$$\tau = \inf\{t : x(t) \notin (a, b) | x(0) = x_0 \in (a, b)\}. \quad (2)$$

Our aim is to investigate optimal drift $\theta$ and relative noise strength $\sigma$ that lead to desired FPT moments. Such problems could be of relevance in many contexts wherein a desired mean FPT and at least a tolerable CV is required. This problem can be generalized further by demanding a FPT distribution that is as close to a desired distribution as it could be. The thresholds $a$ and $b$ could be both absorbing, or one absorbing and the other reflecting.

To handle both these problems in an unified manner, we use the characteristic function of the OU process. Not only the moments can be easily computed from the characteristic function, but it also provides a useful way to characterize the distance between two probability distribution functions. Let $\tau$ denote the FPT and $\psi_\tau(\alpha)$ be its characteristic function. Then

$$\psi_\tau(\alpha) = \mathbb{E} \left[ e^{i\alpha \tau} \right], \quad \alpha \in \mathbb{R}. \quad (3)$$

A $m$-th order moment $\mathbb{E}[\tau^n]$ can be computed as

$$\mathbb{E}[\tau^n] = i^{-m} \left[ \frac{d^m}{d\alpha^m} \psi_\tau(\alpha) \right]_{\alpha = 0}. \quad (4)$$

Furthermore, the following result due to Parseval–Plancherel provides a metric to quantify the difference between two probability density functions in terms of their characteristic functions: Let $f_\tau(t)$ be the probability density function of the FPT, and $f_d(t)$ be a desired probability distribution function. The distance between these functions can be quantified in terms of their characteristic functions $\psi_\tau(\alpha)$ and $\psi_d(\alpha)$ as

$$\int_0^\infty |f_\tau(t) - f_d(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\psi_\tau(\alpha) - \psi_d(\alpha)|^2 d\alpha, \quad (5)$$

provided that the integrals exist [17].

III. BACKGROUND RESULTS ON FPT OF AN OU PROCESS

In this section, we provide background results on FPT of the OU process (1). For completeness, we provide detailed computation of the characteristic function using standard tools from the theory of stochastic processes (see, [5], [12], [16], [18], [19]). We consider two thresholds at $a$ and $b$, both of which could be absorbing or one of them could be reflecting.

A. When both thresholds are absorbing

To derive the characteristic function, $\psi_\tau(\alpha)$, for the OU process in (1), we define $g(y)$ as

$$g(y) = \mathbb{E} \left[ e^{i\alpha \tau(y)} \right], \quad (6)$$

where $y$ represents an initial condition, and $\tau(y)$ denotes the FPT starting from an initial condition $y$. Note that the characteristic function is related to $g(y)$ as $\psi_\tau = g(x_0)$. The computation of $g(y)$ using first principles is discussed below.

Consider the evolution of the OU process starting from $y$ in an infinitesimal time interval $h$. Denote $x_h = x(h) = y - \theta y h + \sigma \sqrt{h} W_h$. It follows that

$$g(y) = \mathbb{E}_{x_0} \mathbb{E}_{x_h} \left[ e^{i\alpha (h + \tau(x_h))} \right] = e^{i\alpha h} \mathbb{E}_{x_h} [g(x_h)]$$

$$= e^{i\alpha h} \left( g(y) - \theta y h \frac{dg}{dy} + \frac{1}{2} \sigma^2 \theta h \frac{d^2 g}{dy^2} + O(h^2) \right). \quad (7c)$$

Taking the limit $h \to 0$ results in

$$\frac{1}{2} \sigma^2 \theta \frac{d^2 g(y)}{dy^2} - \theta y \frac{dg(y)}{dy} + i\alpha g(y) = 0. \quad (8)$$

We are interested in the solution to the above differential equation which can be obtained using the series method. Let

$$g(y) = \sum_{n=0}^\infty c_n y^n. \quad (9)$$

Plugging this in (8) results in

$$\frac{1}{2} \sigma^2 \theta \sum_{n=0}^\infty (n+2)(n+1)c_{n+2}y^n - \theta \sum_{n=0}^\infty nc_n y^n + i\alpha \sum_{n=0}^\infty c_n y^n = 0. \quad (10)$$

It is straightforward to see that (10) results in the following recursive relation in the coefficients

$$c_{n+2} = \frac{2(-i\alpha + n\theta)}{\sigma^2 \theta (n+2)(n+1)} c_n. \quad (11)$$

The above recursion yields the following solution

$$c_n = \frac{2^n \Gamma \left[ \frac{1}{2} (n - \frac{\alpha}{\theta}) \right] c_0}{\Gamma \left[ -\frac{\alpha}{2\theta} \right] \sigma^n n!}, \quad n = 0, 2, 4, \ldots \quad (12a)$$

$$c_n = \frac{2^{n-1} \Gamma \left[ \frac{1}{2} (n - \frac{\alpha}{\theta}) \right] c_1}{\Gamma \left[ -\frac{\alpha + \theta}{2\theta} \right] \sigma^{n-1} n!}, \quad n = 1, 3, 5, \ldots \quad (12b)$$

where $\Gamma$ is the Gamma function. A general solution to (8) can be given by (9), with the coefficients given by (12). Simplifying the series in (9) via symbolic manipulation in Mathematica yields

$$g(y) = c_0 F_1 \left( -\frac{i\alpha}{2\theta} \frac{1}{2}, \frac{y^2}{\sigma^2} \right) + c_1 y F_1 \left( \frac{\theta - i\alpha}{2\theta} \frac{3}{2}, \frac{y^2}{\sigma^2} \right), \quad (13)$$

where $F_1$ represents the Kummer's confluent hypergeometric function.

The solution in (13) consists of two unknown coefficients $c_0$ and $c_1$ which can be computed using the boundary conditions. When both thresholds $a$ and $b$ are absorbing, the boundary conditions are given by $g(a) = 1$ and $g(b) = 1$. Using these boundary values, $c_0$ and $c_1$ can be determined by solving

$$c_{01} F_1 \left( -\frac{i\alpha}{2\theta} \frac{1}{2}, \frac{a^2}{\sigma^2} \right) + c_1 a_1 F_1 \left( \frac{\theta - i\alpha}{2\theta} \frac{3}{2}, \frac{a^2}{\sigma^2} \right) = 1, \quad (14a)$$

$$c_{01} F_1 \left( -\frac{i\alpha}{2\theta} \frac{1}{2}, \frac{b^2}{\sigma^2} \right) + c_1 b_1 F_1 \left( \frac{\theta - i\alpha}{2\theta} \frac{3}{2}, \frac{b^2}{\sigma^2} \right) = 1. \quad (14b)$$
Using these coefficients in (13) and evaluating \( g(x_0) \) results in the following for the characteristic function

\[
\psi_{\tau}(\alpha) = \frac{N_{\psi}}{D_{\psi}},
\]

where

\[
N_{\psi} = -x_0 F_1 \left( \frac{-i \alpha \ 1 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{\theta - i \alpha \ 3 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right)
+ \alpha F_1 \left( \frac{\theta - i \alpha \ 3 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right)
- b F_1 \left( \frac{-i \alpha \ 3 \ b^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right)
+ x_0 F_1 \left( \frac{-i \alpha \ 1 \ b^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{\theta - i \alpha \ 3 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right),
\]

\[
D_{\psi} = \alpha F_1 \left( \frac{-i \alpha \ 3 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ b^2}{2 \theta \ 2 \ \sigma^2} \right)
- b F_1 \left( \frac{-i \alpha \ 3 \ b^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right).
\]

The hypergeometric functions \( F_1 \) can be converted to other special functions, such as Hermite functions, parabolic cylinder functions, etc. [20]. Results on FPT of OU are presented in some of these forms in standard texts [16].

**Remark 1:** If the FPT characteristic function is desired for a single threshold, it could be computed as a special case of the two threshold case analyzed here. There are two possibilities: either the initial condition is above the threshold, or below it. If the initial condition is above the threshold, then we may analyze this case as two thresholds case by letting the threshold at \( b \rightarrow +\infty \), and considering \( a \) as our threshold of interest. In the other case when the initial condition \( x_0 \) is below the threshold, then we let \( a \rightarrow -\infty \) and assume the threshold of interest at \( b \).

**Remark 2:** Recall our definition of the FPT for two threshold case given in (2). The initial condition \( x_0 \) there is assumed to lie between the thresholds \( a \) and \( b \). If that were not the case, then the thresholds problem also becomes a single threshold problem. More specifically, if \( x_0 < a < b \), then the process will always reach \( a \) before \( b \). Therefore, the FPT is same as that for a single threshold at \( a \). Analogously, if \( x_0 > b > a \), then the process will hit the threshold \( b \) before the threshold \( a \), and the FPT is same as that for reaching a single threshold at \( b \).

**B. When one of the thresholds is reflecting**

Another possible situation of interest arises when one of the thresholds is reflecting. For example, we could assume that the threshold at \( a \) is not absorbing and the process is reflected back as soon as it hits \( a \). We are interested in computing the characteristic function of the first time at which the process reaches the threshold \( b \).

The computation follows the same principles as those for the two threshold case, and therefore reduces to solving the differential equation (8) for \( g(y) \). The general form of the solution in (13) can be used in this case, with appropriate boundary conditions given by \( g(b) = 1 \) and \( g'(a) = 0 \) (see [12], [19] for more details). Note that if \( a \) is the absorbing threshold and \( b \) is the reflecting threshold, then we will have the boundary conditions \( g(a) = 1 \) and \( g'(b) = 0 \). We do not analyze this case here.

Let us denote the FPT for this case by \( \tau_r \) and corresponding characteristic function by \( \psi_{\tau_r} \). Using the initial conditions to compute \( c_0 \) and \( c_1 \) in (13) and then evaluating \( g(x_0) \) results in the following for the characteristic function

\[
\psi_{\tau_r}(\alpha) = \frac{N_{\psi}}{D_{\psi_r}},
\]

where

\[
N_{\psi} = \frac{6 \alpha}{\theta} x_0 F_1 \left( \frac{1 - i \alpha \ 3 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{\theta - i \alpha \ 3 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right)
+ 2 \alpha^2 \left( 1 - i \alpha \theta \right) F_1 \left( \frac{3 \ 2 \ \sigma_0^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right)
+ 3 \alpha^2 \theta \left( \frac{1 - i \alpha \ 3 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right),
\]

\[
D_{\psi_r} = 2 \alpha^2 \left( 1 - i \alpha \theta \right) F_1 \left( \frac{3 \ 2 \ \sigma_0^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ b^2}{2 \theta \ 2 \ \sigma^2} \right)
+ 6 \alpha \theta b^2 F_1 \left( \frac{1 - i \alpha \ 3 \ a^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{\theta - i \alpha \ 3 \ b^2}{2 \theta \ 2 \ \sigma^2} \right)
+ 3 \alpha^2 \theta^2 F_1 \left( \frac{-i \alpha \ 1 \ b^2}{2 \theta \ 2 \ \sigma^2} \right) \Gamma \left( \frac{-i \alpha \ 1 \ x_0^2}{2 \theta \ 2 \ \sigma^2} \right).
\]

So far we have computed the characteristic functions for FPT of OU process in various scenarios. The characteristic function can now be used to explore how various parameters affect the FPT statistics, and how they could be tuned to achieve desired FPT behavior.

**IV. OPTIMAL PARAMETERS FOR DESIRED FPT MOMENTS**

In this section, we investigate the effect of various parameters of the OU process on the FPT moments. Then, we examine how the parameters could be tuned so as to get a desired FPT moments.

**A. Scale invariance of the FPT distribution**

In the previous section, we derived characteristic functions of FPT distribution of the OU process under different scenarios (both thresholds absorbing or one of them reflecting). More generally, the characteristic function for other scenarios can also be derived from the generalized form in (13), with appropriate boundary conditions. An important point to note is that in both (15) and (16), the drift parameter \( \theta \) always appears as \( \alpha / \theta \). Therefore, if we consider the rescaled variable \( \tau = \tau \theta \), and find a general form similar to (13), it would be given by

\[
\bar{g}(y) = \mathbb{E} e^{i \alpha \tau (y)} = \mathbb{E} e^{i \alpha \theta \tau(y)}
\]

\[
= c_0 F_1 \left( \frac{-i \alpha \ 1 \ y^2}{2 \theta \ 2 \ \sigma^2} \right) + c_1 y F_1 \left( \frac{1 - i \alpha \ 3 \ y^2}{2 \theta \ 2 \ \sigma^2} \right).
\]

Thus, the general solution \( \bar{g}(y) \) for the rescaled variable \( \tau = \theta \tau \) would not depend on \( \theta \). As the coefficients \( c_0 \) and
An alternate way to infer this feature is to look at (1). As $dw$ is of the order of $\sqrt{\theta t}$, we can rescale time by $\theta$ (as in $\tilde{t} = \theta t$) and rewrite (1) as

$$dx = -\xi dx + \sigma dw \tau.$$  

(18)

Because $\theta$ does not appear in the new time scale, the characteristic function of the rescaled variable $\tau$

$$\psi(\alpha) = E\left[e^{i\alpha \tau}\right] = 1 + i\alpha E[\tau] + \frac{i^2 \alpha^2}{2!} E[\tau^2] + \ldots \tag{19}$$

Since $\theta$ does not appear in the above characteristic function, all moments of $\tau$ are independent of $\theta$. Furthermore, because $\tau = \theta \tau$, we have that

$$E[\tau^m] = \theta^m E[\tau^m], \quad m \geq 1.$$  

(20)

Because $E[\tau]$ does not depend upon upon $\theta$, this implies that

$$E[\tau^m] \propto \theta^m,$$  

(21)

and appropriately scaled moments of the FPT, $E[\tau^m]/(E[\tau])^m$, are independent of $\theta$. It follows that if we operate with normalized higher statistical moments such as the coefficient of variation (CV), skewness, kurtosis, etc., then changing the drift parameter $\theta$ only changes the mean FPT $E[\tau]$. The scale invariance has been observed in distributions of other quantities [21], and also of FPTs in other contexts [5], [22].

In terms of the characteristic function, we illustrate the scale invariance property in Fig. 1 for the case when both thresholds are absorbing. The real and imaginary parts of the characteristic function are plotted for the FPT. By varying the drift parameter $\theta$, the characteristic function $\psi(\alpha)$ does not change in shape and just scales with respect to the $\alpha$ axis. However, changing the relative noise strength $\sigma$ affects the shape of the characteristic function. Similar behavior is also seen in the case when one of the thresholds is reflecting, though the results are not shown in order to avoid repetition.

**B. Tuning FPT moments**

Recall the form of (1). Suppose that we are interested in tuning the two parameters ($\theta$ and $\sigma$) of the process so as to get desired moments of the FPT. Since $\theta$ only changes the mean, and the other quantities of interest (such as coefficient of variation (CV), skewness etc.) are independent of it, one could independently tune the mean FPT and one other quantity. Typically, the CV is the other quantity of interest because it represents the noise in the FPT.

What remains to be seen is how the relative noise strength $\sigma$ affects the mean and CV of the FPT. One could then choose appropriate $\sigma$ such that the CV is at a desired level, and then tune $\theta$ to get the desired mean. It turns out that both the mean and CV are decreasing functions of $\sigma$ for the two absorbing thresholds case. The CV eventually approaches to a limiting value

$$\sqrt{\frac{(x_0-a)^2 + (b-x_0)^2}{3(x_0-a)(b-x_0)}},$$  

(22)

which corresponds to the CV of the FPT for a diffusion with zero drift. In case when the threshold at $a$ is reflecting, the mean still decreases with increase in $\sigma$. The CV, on the other
hand, shows a slight dip before increasing to a limiting value
\[
\sqrt{\frac{2((x_0-a)^2+(b-a)^2)}{3(b-x_0)((x_0-a)+(b-a))}}
\]
(23)
that corresponds to the CV of the FPT for a diffusion with zero drift.
Collectively, these results show that if one were to tune \( \theta \), and \( \sigma \) then any desired mean FPT could be achieved, but there is a limit to the achievable CV. Achieving a low CV in the double thresholds case requires a high value of \( \sigma \) whereas for the case when one the barriers is reflecting, there is an optimal \( \sigma \) that minimizes the CV.

C. Effect of thresholds and initial condition

Our analysis thus far has assumed fixed thresholds and a given initial condition. In Fig. 3, we examine how the results change when one of these parameters are changed. As a first case, we consider a symmetric thresholds, i.e., \( a = -b \), and the initial condition to be at \( x_0 = 0 \). In this case, increasing \( b \) leads to increase in CV of FPT if both thresholds are absorbing. In contrast, if \( a \) is considered to be reflecting, then there is an optimal threshold \( b \) at which the CV hits a minimum. Increasing the threshold beyond a certain point does not affect the CV anymore. This corresponds to the situation when the absorbing threshold(s) is far from the initial condition and crossing it is dominated purely by noise (see Fig. 3, top).

Next, we consider the case when the initial condition \( x_0 \) is not symmetric. Assuming the threshold to be \( a = -b \), we take two cases: \( x_0 = -\frac{a}{2} \) and \( x_0 = \frac{b}{2} \). When both \( a \) and \( b \) are absorbing, increasing the threshold decreases CV of FPT and the CV seems to approach the limit of symmetric initial condition (Fig. 3, middle and bottom). However, when the threshold \( a \) is taken as reflecting, then the CV properties change depending upon \( x_0 \). More specifically, when \( x_0 \) is near the reflecting threshold, then increasing the threshold increases CV. In contrast, when \( x_0 \) is near the absorbing threshold, then increasing the threshold leads to reduction in CV of FPT.

To sum up, the FPT distribution for an OU process is scale invariant with respect to the drift parameter, and thereby allows independent tuning of the mean FPT and another statistical quantity that consists of appropriately scaled moments of the FPT. If one is interested in obtaining a FPT distribution that matches more than two statistical quantities of interest, it is not possible. A question of interest at this point is how close can the FPT distribution get to a given distribution?

V. OPTIMAL PARAMETERS FOR DESIRED FPT DISTRIBUTION

Suppose that instead of tuning the moments, we are interested in tuning the distribution of the FPT itself. More specifically, we are interested in choosing the parameters.
such that the FPT distribution is as close to a desired distribution as possible. In this section, we discuss the tuning of OU process to achieve such behavior.

To this end, we consider the relation between probability density function and the characteristic function stated in (5). Although the desired distribution could be specified as any distribution of interest, we consider the Gamma distribution here. The rationale behind this is that the Gamma distribution is that it is the distribution of a summation of exponential random variables and in a limiting case, it can even represent a degenerate (deterministic) distribution.

Future work will focus on analyzing the optimal parameters by not restricting $\theta$ to be positive, and thereby allowing the OU process to not just be mean reverting. Given the numerical tractability of characteristic functions, it would also be interesting to explore other stochastic processes the approach presented here and explore optimal/sub-optimal control strategies that result in desired FPT statistics.

**REFERENCES**


