

Analysis of A Class of Two-Time-Scale Bond Graph Models

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Abstract

Eliminating derivative causality by inserting parasitic elements produces modified systems that are two-time-scale systems. The computational accuracy and simulation efficiency of the solution of the modified system are strongly influenced by the choice of the parasitic element parameters. This paper demonstrates how the singular perturbation method can be used to gain insight about the behavior of the two-time-scale modified system. A choice for the perturbation parameter ϵ that helps to reveal the time-scale properties of the modified system is given; a suggested singular perturbation standard form is shown, and an important characterization of the fast sub-system model is derived.

INTRODUCTION

Bond graph models with derivative causality produce mathematical models with algebraically-coupled derivative terms in the system equations. (See, for example, Rosenberg 1971; Karnopp et al., 2000, ch.5.) Normally it is preferable to eliminate the derivative causality at the model level, in order to enable an algorithmic derivation of explicit state-space equations. This approach provides the user with an easy-to-interpret formulation and enables him or her to apply the many numerical solution tools available for such models. One way to achieve this goal is to reduce the storage field associated with the derivative causality to an equivalent form having integral causality. If the field is linear but has more than two or three elements, the algebra may be unwieldy. If the field is nonlinear, then the algebraic manipulation may be very difficult; in fact, it may not be possible to develop an equivalent closed-form expression. An alternative approach that applies generally is that of inserting parasitic elements to eliminate the derivative causality effect. Karnopp and Margolis (1979) demonstrated a method to eliminate derivative causality in bond graph models and applied it to the

case of planar mechanisms. They broke the derivative causality (or algebraic) coupling between the coupled energy-storage elements by inserting parasitic compliance elements into the original model. Consequently, a modified model that approximates the original model was produced. This modified model is intended to be a two-time-scale (stiff) model, so that the dynamic behavior of the original state variables would not be greatly affected. Subsequently Zeid generalized their approach (1988). It is his results that are developed further in this work. Figure 1 shows a bond graph summary for this approach that was given by Zeid (1988). The inserted (so-called "parasitic") elements relieve the derivative causality constraint; two state variables are added to the original system model for each such intervention. The choice of values for the parasitic elements strongly influences both computational efficiency and approximation accuracy of the numerical solution. In fact, approximation accuracy can be traded off with the system stiffness. There are some important aspects of the modified two-time-scale system that have not been investigated in detail. Edstrom (1999) explored some of aspects for the simpler approach of inserting a parasitic resistance only. Our work investigates model properties for the more general case of inserting a pair of parasitic elements of the nature shown in Figure 1.

TWO-TIME-SCALE SYSTEMS AND THE SINGULAR PERTURBATION METHOD

It was stated above that the "Karnopp-Margolis-Zeid" algorithm for eliminating derivative causality produces a modified model that is intended to be a two-time-scale (stiff) model. The singular perturbation (SP) method has proved to be very useful in analysis and design of multi-time-scale dynamic systems (Kokotovic et al., 1986). The SP method decomposes a two-time-scale system into slow and fast sub-systems by assuming a small singular perturbation parameter (ϵ) that can meaningfully approach zero. Tools to analyze the slow and fast sub-system models and their interrelationship are available. There are two major issues related to the use of the SP method. The first issue is the

determination of an appropriate perturbation parameter (ϵ). The second issue is the transformation of a singularly perturbed model to a standard singular perturbation form. Solving the first issue is a prerequisite for solving the second one.

Singular perturbation theory has undergone significant development and has many useful applications. However, theory users have a major concern. The time-scale separation is not always easily detectable. In many situations a good understanding of the physical process is needed in order to detect the separation and to choose a suitable perturbation parameter (or parameters) (Kokotovic et al., 1986). This task becomes more difficult as system complexity increases. It should be noted that the perturbation parameter of a two-time-scale system is not necessarily unique; there might be more than one possible appropriate perturbation parameter.

The general form of the standard singular perturbation model is

$$\dot{x} = f(x, \mathbf{h}, \mathbf{e}, t), \quad x(t_0) = x^0, \quad x \in R^n, \quad (1)$$

$$\mathbf{e}\dot{\mathbf{h}} = g(x, \mathbf{h}, \mathbf{e}, t), \quad \mathbf{h}(t_0) = \mathbf{h}^0, \quad \mathbf{h} \in R^m \quad (2)$$

where ϵ is a small positive scalar; a super-dot denotes a time derivative; and f and g are functions assumed to be sufficiently continuously differentiable. This model is the most commonly studied SP model in the literature and the first one to be used in control and systems theory (Kokotovic et al., 1986). Consequently, numerous analysis and design tools are available for two-time-scale systems that are cast in the standard form of Eqs. (1) and (2). In addition, there are procedures to transform a two-time-scale model from a non-standard form to the standard form. Expressing a model in the standard form does not change its time-scale properties, but it does reveal them more clearly.

Zeid (1988) suggested that the modified system that results from removing derivative causality by inserting parasitic elements could be transformed to a standard singular perturbation form. However, he did not develop a method to transform the system and hence he made no attempt to use singular perturbation analysis tools. The objectives of this work are

- (1) to show a choice for the perturbation parameter (ϵ) that reveals the two-time-scale properties of the modified system, and
- (2) to show how the general procedures outlined in Kokotovic et al. (1986) for transforming singularly perturbed models to the standard form can be used to obtain a preferred standard singular perturbation form (PSSPF) and to gain insight into the problem.

SUMMARY OF THE TRANSFORMATION PROCEDURE

The transformation procedure can be summarized as follows:

- (1) Put the singularly perturbed system in the form

$$\mathbf{e}\dot{\mathbf{n}} = F(\mathbf{e}) * \mathbf{n}, \quad \mathbf{n} \in R^{n+m} \quad (3)$$

where n is the order of the slow sub-system and m is the order of the fast sub-system. Let

$$F(\mathbf{e}) = F_0 + \mathbf{e} * F_1(\mathbf{e}). \quad (4)$$

For two or more time-scales to appear the following condition must be satisfied:

$$\det F(0) = 0.$$

- (2) To select a fast-variable vector \mathbf{h} choose m linearly independent vectors in R^{n+m} orthogonal to the null-space N of F_0 and arrange them as the rows of an $m \times (n+m)$ matrix Q . Note that N of F_0 is called sometimes right null space of F_0 and is defined as

$$N(F_0) \equiv \{y: F_0 y = 0\}$$

- (3) To select a slow-variable vector x find P (the left null space of F_0) such that

$$PF_0 = 0$$

The change of coordinates

$$x = P\mathbf{n}, \quad \mathbf{h} = Q\mathbf{n}; \quad T = \begin{bmatrix} P \\ Q \end{bmatrix} \quad (5)$$

will transform the system into the standard form

$$\dot{x} = A_{11}(\mathbf{e})x + A_{12}(\mathbf{e})\mathbf{h} \quad (6)$$

$$\mathbf{e}\dot{\mathbf{h}} = \mathbf{e}A_{21}(\mathbf{e})x + A_{22}(\mathbf{e})\mathbf{h} \quad (7)$$

provided the transformation matrix T is nonsingular. In addition, $A_{11}(0)$ contains the n slow eigenvalues and $A_{22}(0)$ contains the m fast eigenvalues.

APPLYING THE TRANSFORMATION PROCEDURE

To illustrate how to apply the transformation procedure, we use the electrical circuit example shown in Figure 2 (a). The corresponding bond graph model of part (b) has derivative causality. Fig. 3 shows a schematic diagram and a bond graph for the model after inserting the parasitic elements to eliminate the

derivative causality. Following the SCAP (Rosenberg et al. 1983), the homogeneous original system equations are

$$\frac{dp1}{dt} = -m \left(mR1 \frac{p1}{I1} + \frac{dp2}{dt} \right) \quad (8)$$

$$\frac{p2}{I2} = m \frac{p1}{I1} \quad (9)$$

and the homogeneous modified system equations are

$$\frac{dp1}{dt} = -\frac{m^2(R+R1)}{I1} p1 + \frac{mR}{I2} p2 - \frac{m}{C} q \quad (10)$$

$$\frac{dp2}{dt} = \frac{mR}{I1} p1 - \frac{R}{I2} p2 + \frac{1}{C} q \quad (11)$$

$$\frac{dq}{dt} = \frac{m}{I1} p1 - \frac{1}{I2} p2 \quad (12)$$

Next, we will follow the transformation procedure to convert the modified system to the standard form.

First, the modified system equations (10) - (12) can be expressed in the matrix form

$$\frac{d}{dt} \begin{bmatrix} p1 \\ p2 \\ Rq \end{bmatrix} = \begin{bmatrix} \frac{-m^2(R+R1)}{I1} & \frac{mR}{I2} & \frac{-m}{C} \\ \frac{mR}{I1} & \frac{-R}{I2} & \frac{1}{C} \\ \frac{m}{I1} & \frac{-1}{I2} & 0 \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ Rq \end{bmatrix} \quad (13)$$

At this point we take $e = \frac{1}{R}$ and take $a = R^2 C$ as a

constant. That is, we convert the parasitic parameters R and C to the SP parameters e and a . It is also useful to scale the variable q as Rq , so as to put the system of Eq. (13) in the form of Eq. (3). The substitutions and scaling produce

$$e \frac{d}{dt} \begin{bmatrix} p1 \\ p2 \\ Rq \end{bmatrix} = \begin{bmatrix} \frac{-m^2(1+eR1)}{I1} & \frac{m}{I2} & \frac{-m}{C} \\ \frac{m}{I1} & \frac{-1}{I2} & \frac{1}{C} \\ \frac{m}{I1} & \frac{-1}{I2} & 0 \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ Rq \end{bmatrix} \quad (14)$$

Comparing Eqs. (3) and (4) with Eq. (14), we see that

$$F_0 = \begin{bmatrix} \frac{-m^2}{I1} & \frac{m}{I2} & \frac{-m}{C} \\ \frac{m}{I1} & \frac{-1}{I2} & \frac{1}{C} \\ \frac{m}{I1} & \frac{-1}{I2} & 0 \end{bmatrix} \quad (15)$$

Next, to select the fast variables, find the right null space (y) of F_0 . Calculating y from $F_0 y = 0$ leads

$$\text{to } y = \begin{bmatrix} y1 \\ \frac{mI2}{I1} y1 \\ 0 \end{bmatrix}, \text{ where } y1 \text{ is an arbitrary constant.}$$

The next step is to choose two linearly independent vectors that are orthogonal to y . Clearly, these vectors are not unique. We suggest that the two orthogonal vectors be

$$\begin{bmatrix} \frac{m}{I1} \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (16)$$

This choice is recommended because the two fast variables are physically meaningful, Omara (2000). The two fast variables are

$$\begin{bmatrix} \frac{m}{I1} p1 - \frac{1}{I2} p2 \\ Rq \end{bmatrix} \quad (17)$$

The first fast variable can be understood as the error introduced into Eq. (9) of the original system model due to inserting the parasitic elements. The second fast variable (Rq) is a scaling of the charge of the parasitic capacitance; basically, it is the time integral of the first fast variable.

Third, select the slow variable. Find P (the left null space of F_0) such that $PF_0 = 0$. This will produce

$$P = [P1 \quad mP1 \quad 0] \quad (18)$$

where $P1$ is an arbitrary constant. We chose it equal to one; so that

$$P = \begin{bmatrix} 1 & m & 0 \end{bmatrix} \quad (19)$$

With this choice, the slow variable becomes

$$p1 + mp2 \quad (20)$$

This slow variable has physical meaning. If we had algebraically manipulated the original system equations (8) and (9) to formulate them in explicit state-space form, this variable would have emerged. With P and Q available, the transformation matrix T can be written as

$$T = \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 1 & m & 0 \\ m & -1 & 0 \\ I1 & I2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (21)$$

Consequently, the system matrix of the transformed system is

$$\tilde{A} = T \begin{bmatrix} -m^2(R+R1) & mR & -mR \\ I1 & I2 & a \\ mR & -R & R \\ I1 & I2 & a \\ mR & -R & a \\ I1 & I2 & 0 \end{bmatrix} T^{-1}, \text{ or}$$

$$\tilde{A} = \begin{bmatrix} \frac{-m^2 R1}{(I1+m^2 I2)} & \frac{-m^3 I2 R1}{(I1+m^2 I2)} & 0 \\ \frac{-m^3 R1}{I1(I1+m^2 I2)} & \frac{-m^4 I2^2 R1}{I1 I2 (I1+m^2 I2)} - R \left[\frac{m^2}{I1} + \frac{1}{I2} \right] & -R \left[\frac{m^2}{I1} + \frac{1}{I2} \right] \\ 0 & R & 0 \end{bmatrix}$$

Therefore, this system can be written in the standard singular perturbation form of Eqs. (6) and (7) with $x = p1 + mp2$,

$$\mathbf{h} = \begin{bmatrix} \frac{m}{I1} p1 - \frac{1}{I2} p2 \\ Rq \end{bmatrix}, \text{ and the submatrices}$$

$$A_{11} = \begin{bmatrix} -m^2 R1 \\ (I1+m^2 I2) \end{bmatrix}, \quad A_{12} = \begin{bmatrix} m^3 I2 R1 \\ (I1+m^2 I2) \end{bmatrix} \quad 0,$$

$$A_{21} = \begin{bmatrix} -m^3 R1 \\ I1(I1+m^2 I2) \\ 0 \end{bmatrix}, \text{ and}$$

$$A_{22} = \begin{bmatrix} \frac{-m^4 I2^2 R1 e}{I1 I2 (I1+m^2 I2)} - \left[\frac{m^2}{I1} + \frac{1}{I2} \right] & \frac{-1}{a} \left[\frac{m^2}{I1} + \frac{1}{I2} \right] \\ 1 & 0 \end{bmatrix}$$

The eigenvalue of the original system is equal to that implied by A_{11} , as expected. Furthermore, $A_{22}(0)$ implies the eigenvalues of the fast sub-system.

$$A_{22}(0) = \begin{bmatrix} -\left[\frac{m^2}{I1} + \frac{1}{I2} \right] & -\frac{1}{a} \left[\frac{m^2}{I1} + \frac{1}{I2} \right] \\ 1 & 0 \end{bmatrix} \quad (22)$$

Using equation (22), the eigenvalues (λ) of the fast sub-system can be calculated from

$$I^2 + \left(\frac{m^2}{I1} + \frac{1}{I2} \right) R I + \frac{1}{C} \left(\frac{m^2}{I1} + \frac{1}{I2} \right) = 0 \quad (23)$$

Equation (23) is the characteristic equation of a series

R-L-C circuit, with the term $\left(\frac{m^2}{I1} + \frac{1}{I2} \right)$ as the

equivalent inductance of L. This result shows that the as e tends to zero, the fast sub-system can be accurately approximated by the parasitic capacitance, the parasitic resistance, and the equivalent of the λ field at the port where the parasitic elements were inserted. Figure 4 shows a model for the fast sub-system as e tends to zero, where leq refers to the equivalent of the λ field. The result applies to nonlinear systems also, Omara (2000). Our choices placed the eigenvalues of the fast sub-system at a user specified location; more details will be forthcoming in a future publication.

THE TRANSFORMATION MATRIX

Assume that a linear system includes a derivative-causality energy-storage field together with other energy-storage elements that have integral causality. Further, assume that the coupled field includes only one dependent energy-storing element and that the algebraic equation obtained from the SCAP for its elements is in the general form of Eq. (24) below.

$$S_d * X_d = J_{di} * S_i * X_i \quad (24)$$

Note that X_i is the independent energy variables vector, X_d is the dependent energy variables vector, and S_i and S_d are inverse inertia matrices. Assume that the state vector of the energy storing elements

that are not among the coupled field elements is X_r , and its dimension is r . One way to recognize the elements of the coupled field is by the appearance of their parameters in J_{di} . Assume that the state vector of the corresponding modified explicit system is

$$X = \begin{bmatrix} X_r \\ X_i \\ X_d \\ q \end{bmatrix}, \text{ where } q \text{ is the state of the inserted}$$

parasitic capacitance. The transformation matrix that converts the modified system to the preferred standard singular perturbation form (PSSPF), as illustrated in the circuit example is

$$T = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_i & -J_{id} & 0 \\ 0 & J_{di}S_i & -S_d & 0 \\ 0 & 0 & 0 & R \end{bmatrix}, \quad (25)$$

where I_r is a unit matrix whose dimension is that of X_r , and I_i is a unit matrix whose dimension is that of X_i , Omara (2000).

It should be noted that one does not have to formulate the original system equations to be able to formulate the modified system and transform it to the PSSPF. To obtain the transformation matrix given in Eq. (25) one needs to know J_{id} , S_i , and S_d . The matrix J_{di} that relates the flow variables of the dependent and independent energy-storage elements can be obtained readily from the original system bond graph, and S_i and S_d are inertia matrices that are part of the original system model. It can be shown that the transformation matrix defined above is always non-singular, Omara (2000). It can be used to transform the modified system to the standard singular perturbation form readily without the need for experience with a transformation procedure. Consequently, the transformed modified system can benefit from several tools that are available for systems in the standard singular perturbation form.

CONCLUSIONS

This paper showed that a bond graph model with derivative causality can be converted to a standard singularly perturbed form by the use of an appropriate transformation matrix. The transformation procedure was illustrated, although the choice of values for key parameter was not motivated in any systematic way. Those details will be the subject of additional reporting in subsequent work.

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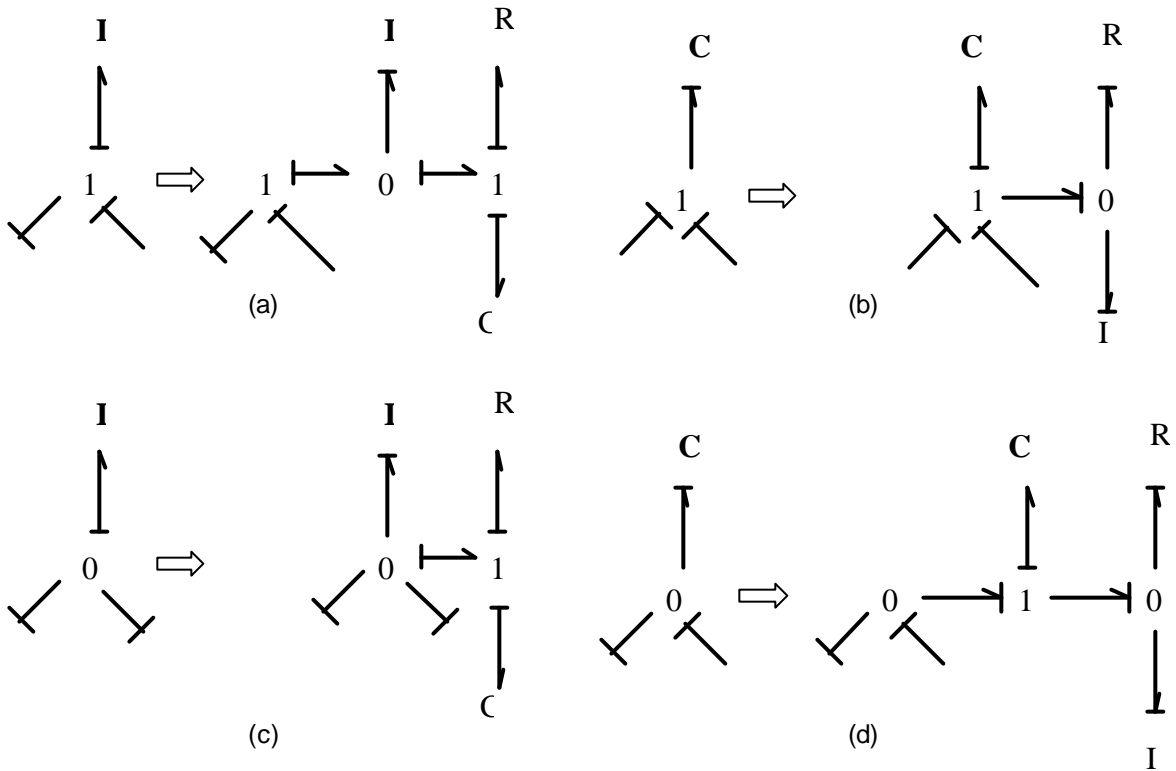


Figure 1. Summary of the modification algorithm under investigation.

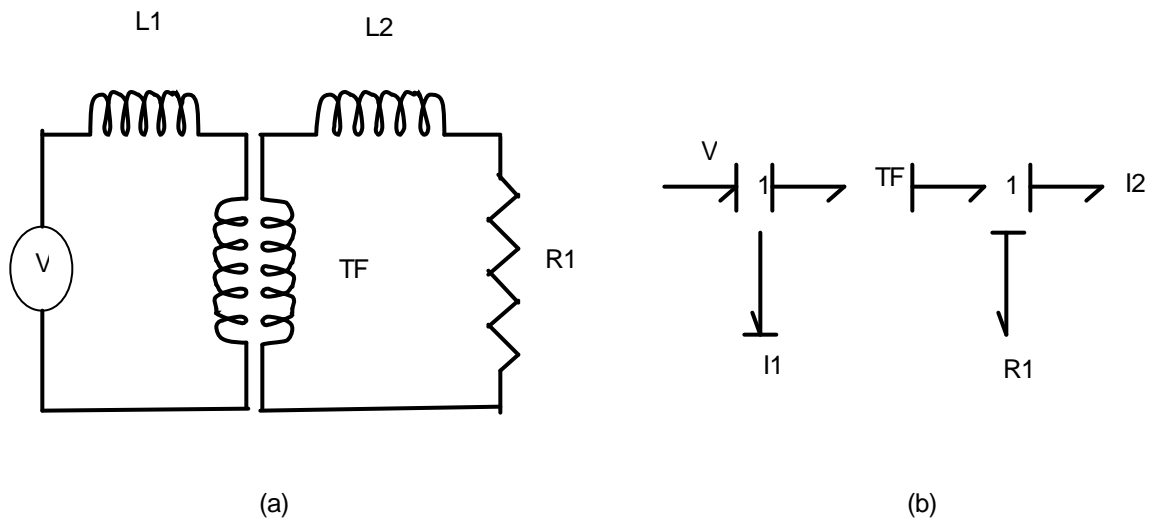


Figure 2. Electrical circuit diagram and bond graph models.

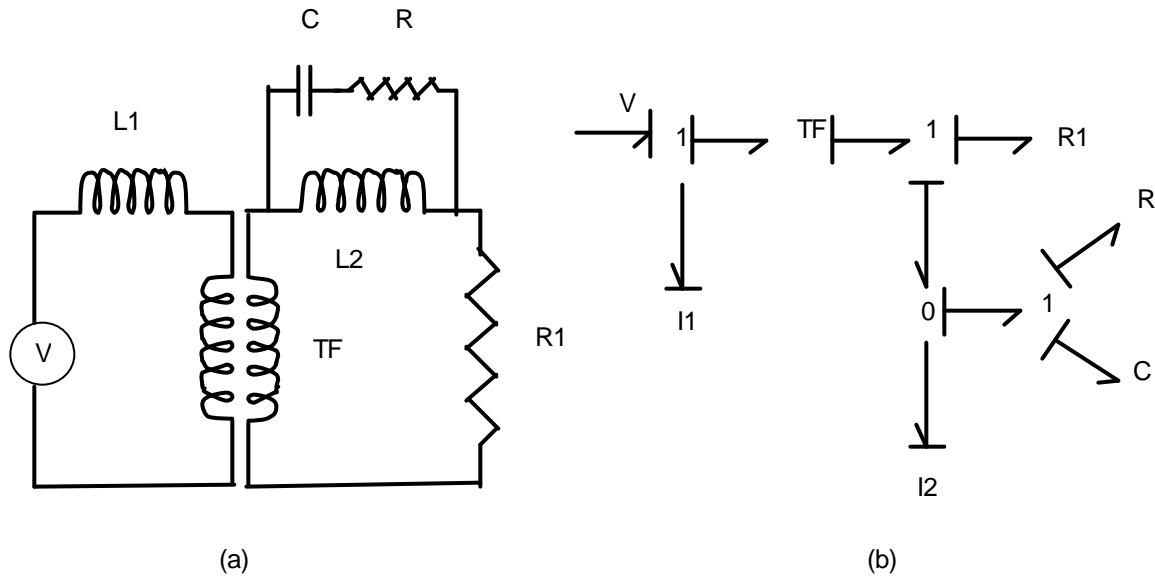


Figure 3. Circuit diagram and bond graph for the modified model of Figure 2.

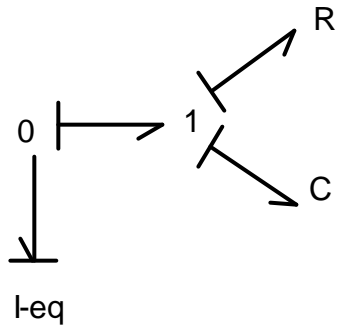


Figure 4. A bond graph model for the fast sub-system as ϵ tends to zero.