Quantifying Fatigue Failure: Stress-Life Method for Non-Zero Mean Stress
(Failure Surface = “Goodman Diagram”)

6-11 Characterizing Fluctuating Stresses

Variation in cyclic stress.

Mean stress: \( \sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2} \)
Stress Amplitude: \( \sigma_a = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \)
Stress ratio: \( R = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \)
Amplitude ratio: \( A = \frac{\sigma_a}{\sigma_m} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{\sigma_{\text{max}} + \sigma_{\text{min}}} = \frac{1 - R}{1 + R} \)

Terms for Stress Cycling

maximum stress, \( \sigma_{\text{max}} \)
minimum stress, \( \sigma_{\text{min}} \)
load reversals (2 reversals = 1 cycle)
stress amplitude, \( \sigma_a \)
mid-range stress, \( \sigma_m \)

Follow a material point A on the outer fiber

Schematic: Effect of Midrange Stress \( S_m = \sigma_m |_{N_f=\infty} \)

Alternating stress amplitude \( (\sigma_a) \) at failure, or fatigue strength \( S_a \)

Mid-range stress \( (\sigma_m) \) at failure, or mid-range strength \( S_m \)
Data: Effect of Midrange Stress $\sigma_m$

That is: the endurance limit drops as $\sigma_m|_{N_f} = S_m$ increases.

Simple Model to Account for Midrange Stress $\sigma_m$

That is: the endurance limit drops as $\sigma_m|_{N_f} = S_m$ increases.

Goodman Diagram for Fatigue

This line is known as the Goodman diagram.

Goodman diagram = "endurance limit as a function of mean stress".

Goodman diagram = drop in $S_f$ for rise in tensile $S_m$.

Goodman Diagram: Fatigue Failure with $\sigma_m \neq 0$

Alternating stress amplitude, $\sigma_a$, as fatigue strength $S_f$.

Equation of Goodman line:

$$\frac{S_f}{S_m} = \frac{\sigma_a}{\sigma_m}$$

"Goodman Criterion" for finite life:

$$\frac{S_f}{S_m} + \frac{S_m}{S_f} = 1$$

"Goodman Criterion" for $\infty$-life:

$$\frac{\sigma_a}{\sigma_m} + \frac{\sigma_m}{\sigma_a} = 1$$

1. Failure surface
2. Design Equation (i.e., shrink failure surface until you hit service loads).
Goodman Diagram: Fatigue Failure with $\sigma_m \neq 0$

Goodman Diagram for Non-Zero Mean Stress

E. g. 3. Goodman Diagram for Non-Zero Mean Stress

Shaft A: 1010 hot-rolled steel
$K_p = 1.6$ at the 3-mm fillet
$F = 0.5$ to 2 kN cycle
$S_a = 320$ MPa
Find safety factor against failure of shaft A.

Calculate the shear-stress concentration factor with $K_p = 1.6$ at the toe of the 3-mm fillet:

$$K_p = 1 + q_s (K_{ps} - 1)$$

$$K_{ps} = 1 + q_s (K_{ps} - 1) = 1 + 0.775(1.6 - 1) = 1.46$$

$$= 1.46(15.9) = 23.26$$

or 23.2 MPa $\leq$ $r_{fail}$ $\leq$ 92.9 MPa
**Safety factor based on Goodman criterion:**

Convert ideal to actual endurance limit:  
\[ S_e' = k_c k_0 S_e \]

(95% stress area is the same as R. R. Moore specimen)

\[ k_c = 1.24(20)^{-0.207} = 0.9 \]

\[ k_0 = 0.59 \]

\[ S_e = S_{el} \frac{k_0 + 1}{n} = 0.91(0.9)(0.59)(160) = 77.9 \text{ MPa} \]

Stress amplitude and mid-range stress:

\[ S_a = 0.67 S_{el} = 0.67(320) = 214.4 \text{ MPa} \]

\[ \frac{S_a}{S_{el}} = \frac{77.9}{214.4} \]

\[ \frac{1}{n} = 1.39 \]

\[ n = \frac{1}{1.39} = 0.72 \]

Recall that for \( r_m = 0 \) (E.g. 2),

\[ \text{Mod. Goodman} \]

\[ \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_a} = 1 \]

**6-12 Fatigue Failure Criteria**

- **Mod. Goodman**
  \[ \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_a} = 1 \]
- **Soderberg**
  \[ \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_a} = 1 \]
- **Gerber**
  \[ \left( \frac{\sigma_a}{S_e} \right)^2 + \left( \frac{\sigma_m}{S_a} \right)^2 = 1 \]
- **ASME-elliptic**
  \[ \left( \frac{\sigma_a}{S_e} \right)^2 + \left( \frac{\sigma_m}{S_a} \right)^2 = 1 \]

**Table 6-6**

**Intersecting Equations**

<table>
<thead>
<tr>
<th>Intersecting Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_e + S_m = 1 )</td>
<td>( S_e + S_m = 1 )</td>
</tr>
<tr>
<td>( S_e + S_m = 1 )</td>
<td>( S_e + S_m = 1 )</td>
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<td>( S_e + S_m = 1 )</td>
<td>( S_e + S_m = 1 )</td>
</tr>
</tbody>
</table>

\[ \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_a} = 1 \]

**First-Cycle Yield**: Goodman Fatigue Line and Langer Yield Line

Table 6-6 lists the coordinates on the \( S_e, S_m, S_0 \) diagram where the load line intersects the Langer and the Goodman lines.

- **Goodman Fatigue Line**
  \[ S_e + S_m = 1 \]

- **Langer Static Yield**
  \[ \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_a} = 1 \]
Quantifying Fatigue Failure: Stress-Life Method for Non-Zero Mean Stress and Combined Loading
Combined Loading: Choice of Von Mises as “Effective” Stress

Using xyz components of three-dimensional stress, the von Mises stress can be written as

$$\sigma' = \sqrt{\frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2}}$$  \hspace{0.5cm} (5-14)

and for plane stress,

$$\sigma' = \sqrt{(\sigma_x - \sigma_y)^2 + (\tau_{xy})^2}^{1/2}$$  \hspace{0.5cm} (5-15)

Adapt for fatigue in tension/compression, bending and torsion of shafts:

1. Stresses are higher on average due to stress concentrations; mid-range stress is adjusted:

$$\sigma'_l = \left[ (K_f)_{\text{bending}}(\sigma_l)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_l)_{\text{axial}} \right]^2 + 3 \left[ (K_f)_{\text{torsion}}(\tau_l)_{\text{torsion}} \right]^2$$  \hspace{0.5cm} (6-56)

2. Axial misalignment affects the axial stress alternating about the mean, and hence applied to \(\sigma'_l\) only:

$$\sigma'_a = \left[ (K_f)_{\text{bending}}(\sigma_a)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_a)_{\text{axial}} \right]^2 + 3 \left[ (K_f)_{\text{torsion}}(\tau_a)_{\text{torsion}} \right]^2$$  \hspace{0.5cm} (6-55)

Combined Loading: Another Way to View Effective Stress Amplitude

$$\sigma'_e = \left[ \left( (K_f)_{\text{bending}}(\sigma_e)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_e)_{\text{axial}} \right)^2 + \left[ (K_f)_{\text{bending}}(\tau)_{\text{bending}} \right]^2 \right]^{1/2}$$

In other words,

$$\sigma'_e = \left[ \left( (K_f)_{\text{bending}}(\sigma_e)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_e)_{\text{axial}} \right)^2 + \left[ (K_f)_{\text{bending}}(\tau)_{\text{bending}} \right]^2 \right]^{1/2}$$

effective 1-D fatigue stress amplitude

where loading factor \(k_c\):

- rotating bending
  - pure axial \(k_c = 0.85\)
  - pure torsional \(k_c = 0.59\)

Note: When \(1/k_c\) is applied to amplify the service load in combined loading, we do not apply \(k_c\) to reduce the strength in the Miner equation (See Sections 6-9 and 6-14)

6-13,14 (Summary) Torsional Fatigue Strength under fluctuating Stresses & Combinations

- Torsional Fatigue Strength under fluctuating stresses
  - In Goodman diagram
    - \(S_{aw} = 0.67S_{aw}\)
    - \(S_{aw} = 0.577S_{aw}\)

- Combinations
  - In Goodman diagram
    - \(S_{aw} = 0.67\)
    - \(S_{aw} = 0.577\)

E.g. 4. Clutch

The steel shaft rotates at a constant speed \(ω\) while the axial load is applied linearly from zero to \(P\) and then released. The cycle is repeated. Given that \(T = |P(D+d)/4|\), find the maximum allowable load, \(P\), for infinite life of the shaft.

Given: friction coefficient \(f = 0.3\), and for the present \(r = 3\)-mm fillet, \(K_f = 2.84\) and \(K_{aw} = 1.8\)

$$K_f = 1 + q(K_f - 1) = 1 + 0.92(2.84 - 1) = 2.84$$
$$K_{aw} = 1 + q(K_{aw} - 1) = 1 + 0.93(1.8 - 1) = 1.74$$
Problem 5. Clutch

The steel shaft rotates at a constant speed or while the axial load is applied linearly from zero to \( P \) and then released. The cycle is repeated.

Given that \( T = P(D+d)/4 \), find the maximum allowable load, \( P \), for infinite life of the shaft.

For the present \( r = 3 \text{ mm fillet}, K_p=3 \text{ and } K_w=1.8 \), \( K_p = 1 + 0.92(3 - 1) = 2.84 \), \( K_w = 1 + 0.93(1.8 - 1) = 1.74 \)

Fatigue effective stress amplitude:

\[
\sigma_e = \sqrt{\left(\frac{\sigma_{\max}}{2}\right)^2 + \left(\frac{\sigma_{\min}}{2}\right)^2} \cdot \left(1000 \text{ P} \right) \cdot \text{S}
\]

\[
\sigma_e = \sqrt{\left(\frac{\sigma_{\max}}{2}\right)^2 + \left(\frac{\sigma_{\min}}{2}\right)^2} \cdot \left(1000 \text{ P} \right) \cdot \text{S} = 2216 \text{ P} \cdot \text{S} \]

\[
\sigma_{\max} = 800 \text{ MPa}, \quad \sigma_{\min} = 1000 \text{ MPa}
\]

E.g. 4. Clutch, continued

\[
\sigma_e = \sqrt{\left(\frac{\sigma_{\max}}{2}\right)^2 + \left(\frac{\sigma_{\min}}{2}\right)^2} = 4508 \cdot P \text{ [MPa]} \quad \sigma_e = \sqrt{\left(\frac{\sigma_{\max}}{2}\right)^2 + \left(\frac{\sigma_{\min}}{2}\right)^2} = 4332 \cdot P \text{ [MPa]}
\]

\[
S_e = 0.5S_{\sigma} = 0.5(1000 \text{ P} \cdot \text{S}) = 500 \text{ MPa}
\]

\[
S_e = k_e k_k k_j k_k k_j S_e
\]

\[
k_e = a S_e^{1/3} = 4.5 \cdot (1000 \text{ P} \cdot \text{S})^{1/3} = 0.723
\]

\[
k_j = 1.24 \cdot (30)^{0.163} = 0.862
\]

\[
k_k = 1 \quad \text{Note: } 1/k_3 \text{ has been applied to amplify the service load already}
\]

\[
S_e = k_e k_j k_k k_j S_e = (0.723)(0.862)500 = 312 \text{ MPa}
\]

\[
\frac{\sigma_{e1}}{S_e} + \frac{\sigma_{e2}}{S_e} = \frac{1}{n} \Rightarrow \frac{4508P}{312} + \frac{4332P}{1000} = \frac{1}{n}
\]

\[
P = \frac{0.0533 \text{e6}}{n} \text{ Newtons} = \frac{53.3 \text{e3}}{n} \text{ Newtons}
\]

The max. load for fatigue failure (\( n=1 \)) is 53.3 kN.

Quantifying Fatigue Failure: Stress-Life Method for Stress Cycles of Unequal Amplitudes

- Palmgren-Miner Cyclic ratio summation rule

Miner’s rule (Linear damage Rule)

\[
\sum \frac{n_i}{N_j} = 1
\]

where \( n_i \) is the number of cycle at stress level \( \sigma_i \) and \( N_j \) is the number of cycle to failure at stress level \( \sigma_j \).

Assumption:
The stress sequence does not matter and the rate of damage accumulation at a particular stress level is independent of the stress history.

6-15 Varying, Fluctuating Stresses; Cumulative Fatigue Damage
Problem 5: Miner’s Rule

A rotating beam specimen (4130 steel) with an endurance limit of 50 kpsi and an ultimate strength of 125 kpsi is cycled 20% of the time at 70 kpsi, 50% at 55 kpsi and 30% at 40 kpsi. Estimate the number of cycle to failure.

Solution:

$$a = \frac{(N_L)^1}{(0.8631031)^1} = 0.86 (\text{Fig. 6-18})$$

$$b = \frac{\sigma_f}{(0.057562)} = 1.15$$

$$a = 70 \text{kpsi}, N_L = \left( \frac{70}{1.15} \right)^{0.86} \text{cycles} = 33883 \text{cycles}$$

$$a = 55 \text{kpsi}, N_L = \left( \frac{55}{1.15} \right)^{0.86} \text{cycles} = 283560 \text{cycles}$$

$$a = 40 \text{kpsi}, N_L = \left( \frac{40}{1.15} \right)^{0.86} \text{cycles} = 22012 \text{cycles}$$

Fig. 6-18: $f=0.867$

Problem 7: Shaft

1040 Steels ($S_u=85\text{ksi}$) rpm=1600rpm, $F_1=2500\text{lbf}$ and $F_2=1000\text{lbf}$

n of infinite life or

$$N = \frac{14750(1.625/2)}{1.625/64} = 85\text{ksi}$$

This stress is far below the yield strength of 71 kpsi, so yielding is not predicted.

Fig. A-15-9: $\frac{F}{d} = 0.0625/1.625 = 0.04, \frac{D}{d} = 1.875/1.625 = 1.15, K_y = 1.95$

Get the notch sensitivity either from Fig. 6-20, or from the curve-fit Eqs. (6-34) and (6-35a). We will use Fig. 6-20.

$$q = 0.76$$

Eq. (6-32): $K_y = 1+q(K_y-1) = 1+0.76(1.95-1) = 1.72$

Eq. (6-8): $S_u=0.58u=0.5(85)=42.5 \text{ksi}$

Problem 7: cont

Eq. (6-19) $k_2 = aS_u = 2.70(85)^{0.096} = 0.832$

Eq. (6-20) $k_2 = 0.879d^{-0.107} = 0.879(1.625)^{-0.107} = 0.835$

Eq. (6-26) $k_2 = 1$

Eq. (6-18) $S_k = k_1k_2k_3S_s = (0.832)(0.835)(1)(42.5) = 29.5\text{ksi}$

$$d_1 = \frac{S_k}{K \alpha'} \frac{29.5}{1.72(35)} = 0.49$$

No infinite life

Fig. 6-18: $f=0.867$

Eq. (6-14) $a = \frac{N}{(S_u)^{0.8631031}} = \frac{(0.867/85)^{0.86}}{29.5} = 184.1$

Eq. (6-15) $b = \frac{1}{3} \log \frac{S_k}{S_u} + \frac{1}{3} \log \left( \frac{(0.867/85)^{0.86}}{29.5} \right) = -0.1325$

Eq. (6-16) $N = \frac{(17.2(35))^{1.184.1}}{184.1} = 4611\text{cycles}$