Chapter 3: Load and Stress Analysis

The careful text-books measure (Let all who build beware!)
The load, the shock, the pressure
Material can bear.
So when the buckled girder
Lets down the grinding span
The blame of loss, or murder
is laid upon the man.
Not on the stuff - The Man!

Rudyard Kipling, “Hymn of
Breaking Strain”

Introduction

- **Statics (and Dynamics)**
- **Mechanics of Materials**
  Understanding the Relationships between external loads and internal reactions on a body.
  Deals with the integrity, deformation and stability of a body.
- **Pressure, Force & Moment**
  - Stresses
- **Displacement, Deformation & Distortions**
  - Strains
- **Material Behavior**

Statics

- **External Force**
  - Surface Force & Traction
  - Body Force (weight)
- **Reactions**
- **Free Body Diagram**

### Statics

- **Equation of Equilibrium**
  - Balance of Force: \( \sum F_x = 0 \)
  - \( \sum F_y = 0 \)
  - \( \sum F_z = 0 \)
  - \( \sum M_x = 0 \)
  - \( \sum M_y = 0 \)
  - \( \sum M_z = 0 \)
Types of Loads

- Axial Load (tensile)
- Axial Load (compressive)
- Shear Load
- Bending (Moment) Load
- Torsion (Twisting Moment) Load

Boundary Conditions

- Two-force member
- In equilibrium

Shear Force and Bending Moment in Beam

\[ \Sigma F_y = -V + q(x)dx + (V + dV) = 0; \]
\[ \frac{dV}{dx} = q \]
\[ \Sigma M = -M + qdx(xdx) - Vdx + (M + dM) = 0 \]
\[ \frac{dM}{dx} = V \]
\[ \frac{dV}{dx} = \frac{d^2M}{dx^2} = q \]

Summary on Beam

\[ M = \int q \, dx \]
\[ V = \int q \, dx \]
\[ \frac{d^2y}{dx^2} = \frac{M}{EI} \]
\[ \frac{d^3y}{dx^3} = \frac{V}{EI} \]
Singularity Functions defined ME222

\((x - a)^n = (x - a)^0\) if \(x > a\)
\[= 0\] if \(x \leq a\)

Note \((x - a)^0 = 1\) if \(x > a\)
\[= 0\] if \(x \leq a\)

\[f(x - a)^n \, dx = \frac{1}{n + 1}(x - a)^{n+1}\]
\[\frac{d}{dx}(x - a)^n = n(x - a)^{n-1}\]

Stress on an Oblique Plane

\[\sum F_x = \sigma_x A A - (\sigma_x A A \sin \theta) \cos \theta - (\sigma_y A A \sin \theta) \sin \theta\]
\[\sum F_y = \tau_{xy} A A + (\tau_{xy} A A \sin \theta) \cos \theta - (\sigma_y A A \sin \theta) \cos \theta\]

Using trigonometry identities:
\(\sin 2\theta = 2 \sin \theta \cos \theta\),
\(\sin^2 \theta = \frac{(1 - \cos 2\theta)}{2}\),
\(\cos^2 \theta = \frac{(1 + \cos 2\theta)}{2}\)

\[\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta\]
\[\tau_{xy}' = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta\]
\[\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta\]

Stress Elements

Stress element showing general state of three-dimensional stress.

(a) Three-dimensional state of stress. (b) Plane view.

Tensor:
\[\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}\]

Stress Plot

Stresses vs. Angle

Series 1
Series 2
Series 3

\(3\text{ksi}\)
\(8\text{ksi}\)
\(5\text{ksi}\)
Mohr’s Circle-Plane Stress

\[
\sigma_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\tau_{xy} = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

\[
\left[ \sigma_x - \frac{\sigma_x + \sigma_y}{2} \right]^2 + \tau_{xy}^2 = \left( \tau_{xy} - \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2
\]

Equation for a circle: \((x-C)^2 + y^2 = R^2\)

Mohr’s Circle Convention

- Clockwise shear stress is positive.
- Counterclockwise shear stress is negative.

Applications

Axial Load

Cylindrical Vessel

Torsion Load

Spherical Vessel

Mohr’s Circle for Plane Stress

- Plot two points representing normal and shear stresses
- Draw a straight line
- Find the center, C
- Draw a circle with the radius of R

Mohr’s Circle for Plane Stress

- Clockwise shear stress is positive.
- Counterclockwise shear stress is negative.

Three Dimensional Mohr’s Circle

Mohr’s circle for triaxial stress state. (a) Mohr’s circle representation; (b) principal stresses on two planes.
Principal Stresses and Directions: Eigen-values and -vectors

- Mohr’s circle diagrams. (a) Triaxial stress state when \( \sigma_1 = 23.43 \text{ ksi}, \sigma_2 = 4.57 \text{ ksi}, \text{ and } \sigma_3 = 0 \); (b) biaxial stress state when \( \sigma_1 = 30.76 \text{ ksi}, \sigma_2 = -2.76 \text{ ksi} \); (c) triaxial stress state when \( \sigma_1 = 30.76 \text{ ksi}, \sigma_2 = 0 \), and \( \sigma_3 = -2.76 \text{ ksi} \).

4-7 3-D Stress

Eigen Value Problem

\[
\begin{vmatrix}
\sigma_x - \alpha & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - \alpha & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - \alpha
\end{vmatrix} = (\sigma_x - \alpha)(\sigma_y - \alpha)(\sigma_z - \alpha) + r_x r_y r_z + r_x r_z + r_y
\]

\[
(\sigma_x - \alpha)^2 - (\sigma_y - \alpha)^2 - (\sigma_z - \alpha)^2 = 0
\]

Example

\[
\begin{vmatrix}
\sigma_x - \alpha & 0 & 0 \\
0 & \sigma_y - \alpha & 0 \\
0 & 0 & \sigma_z - \alpha
\end{vmatrix} = 0
\]

\[
\alpha = \frac{2}{3}(\sigma_x + \sigma_y + \sigma_z)
\]

\[
\alpha_1, \alpha_2, \alpha_3 = \frac{2}{3}(\sigma_x + \sigma_y + \sigma_z)
\]

\[
\nu = \frac{2}{3}(2\sigma_x + \sigma_y + \sigma_z)
\]

\[
\nu_1 = \frac{1}{2}\sqrt{5}(-e_z + 2e_x)
\]

General Three-Dimensional Stress

Principal Stresses and Directions: Eigen-values and -vectors

\[
\begin{vmatrix}
\sigma_x - \alpha & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - \alpha & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - \alpha
\end{vmatrix} = \alpha^3 - (\sigma_x + \sigma_y + \sigma_z)\alpha^2
\]

\[
(\sigma_x - \alpha)^2 - (\sigma_y - \alpha)^2 - (\sigma_z - \alpha)^2 = 0
\]

Example

\[
\begin{vmatrix}
2 - \sigma & 0 & 0 \\
0 & 3 - \sigma & 4 \\
0 & 4 & -3 - \sigma
\end{vmatrix} = 0
\]

\[
\alpha = \frac{2}{3}(\sigma_x + \sigma_y + \sigma_z)
\]

\[
\nu = \frac{2}{3}(2\sigma_x + \sigma_y + \sigma_z)
\]

\[
\nu_1 = \frac{1}{2}\sqrt{5}(-e_z + 2e_x)
\]

Relationship between Stresses and Strains
Plane Stress

Plane Strain

\[ E = 2G(1 + \nu) \]

3-8 Elastic Strain

\[ \sigma = E\varepsilon \]

Important Formula

- No resultant force on the surface: Neutral Plane (N.P.)

\[ \sum F_x = 0; \quad \int_A \sigma_x dA = -\sigma_{\text{max}} \int y dA = 0 \]

- The integral of the elemental moment over the surface must equal to the bending moment

\[ \sum M = 0; \quad M = -\int_A \sigma_x y dA = -\frac{E}{\rho} \int y^2 dA \]

- Moment of Inertia:

\[ I = \int_A y^2 dA \]

- Flexure Formula:

\[ \sigma_x = -\frac{My}{I} = -\frac{Ey}{\rho} = -\frac{\sigma_{\text{max}} y}{c} \]
Moments of Areas

First Moment of An Area: Centroid of An Area

First Moment of Area A w.r.t. x-axis:
\[ Q_x = \int_A y \, dA = A \bar{y} \]

First Moment of Area A w.r.t. y-axis:
\[ Q_y = \int_A x \, dA = A \bar{x} \]

Centroid of Area A: \( C \) (\( x, y \))

Example: Rectangle
\[ Q_x = A \bar{y} = \frac{b h}{2} \]
\[ Q_y = A \bar{x} = \frac{h b}{2} \]

Second Moment (Moment of Inertia)

Second Moment w.r.t. x-axis:
\[ I_x = \int_A y^2 \, dA \]

Second Moment w.r.t. y-axis:
\[ I_y = \int_A x^2 \, dA \]

Polar Moment of Inertia:
\[ J_o = \int_A r^2 \, dA = I_x + I_y \]

Parallel-Axis Theorem
\[ I_x = \int_A y'^2 \, dA = \int_A (y' + d)^2 \, dA = \int_A y^2 \, dA + 2 \int_A y' \, dA + \int_A d^2 \, dA = I_x + 2dQ_x + Ad^2 = I_x + Ad^2 \]
\[ \therefore Q_x = A \bar{y} = A(0) = 0 \]