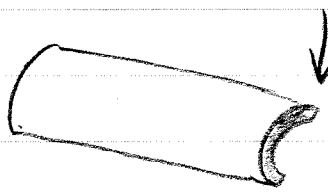
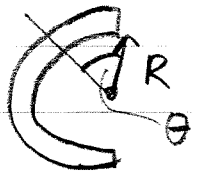
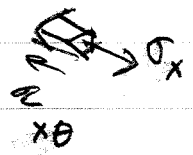
H.W 13 Solutions

① 

$$-\tau_{x\theta}\, t\, dx + \int_{A^*} \sigma_x \Big|_{\text{front}-\text{back}} dA = 0$$



$$\sigma_x \Big|_{\text{front back}} = \frac{dM\, y}{I}$$



$$\Rightarrow \quad \tau_{x\theta} = -\frac{1}{t}\int \frac{dM}{dx}\,\frac{y}{I}\,dA \overset{\frac{dM}{dx}=-V}{=} \frac{V}{It}\int_{A^*} y\, dA$$

$$I = \int_C y^2 dA = \int_0^{\pi} (R\cos\theta)^2\, tR\, d\theta = tR^3 \int_0^{\pi} \cos^2\theta\, d\theta = \frac{\pi t R^3}{2}$$

alternatively $I = \frac{1}{2} I_{\bigcirc} = \frac{1}{2}\left[\frac{\pi}{4}\left((R+t)^4 - R^4\right)\right] = \frac{\pi}{8}\left[R^4 + 4tR^3 + \dots - R^4\right] = \frac{\pi t R^3}{2}$

$$\int_{A^*} y\, dA = \int_0^{\theta^*} (R\cos\theta)\, tR\, d\theta = tR^2 \int_0^{\theta} \cos\theta\, d\theta = tR^2 \sin\theta^*$$

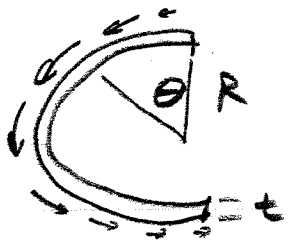


$dA = tR\, d\theta$

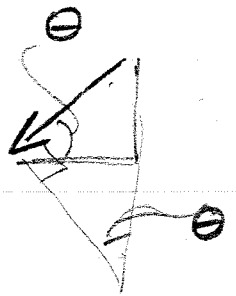
Comment: Centroid of A^* is not on A^* so $\bar{y}A^*$ is not so convenient



thus $\mathcal{Z}_{x\theta} = \dfrac{V \; t R^2 \sin\theta}{\left(\frac{\pi}{2} t R^3\right) t} = \dfrac{2V \sin\theta}{\pi t R}$



component in y direction is $\mathcal{Z}_{x\theta} \sin\theta$



$$\oint_C \mathcal{Z}_{x\theta} \sin\theta \, dA = \int_0^{\pi} \left(\frac{2V\sin\theta}{\pi t R}\right) \sin\theta \, t R \, d\theta$$

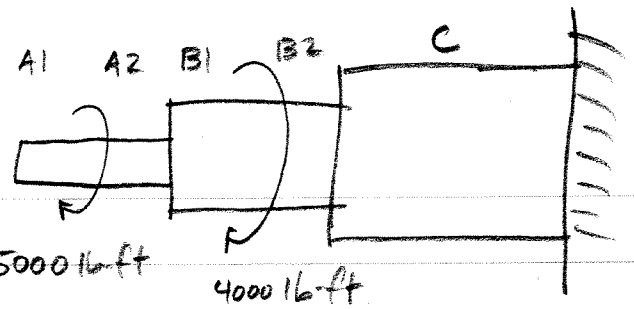
$$= \frac{2V}{\pi} \int_0^{\pi} \sin^2\theta \, d\theta = \frac{2V}{\pi} \left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi} = V \quad \checkmark$$

(2) $$V_e = \oint_C \mathcal{Z}_{x\theta} \, R \, dA = \int_0^{\pi} \frac{2V\sin\theta}{\pi t R} \, R \, t R \, d\theta$$

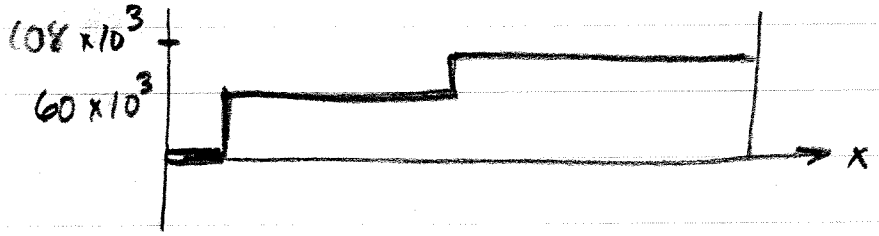
$$= \frac{2RV}{\pi} \int_0^{\pi} \sin\theta \, d\theta = \frac{2RV}{\pi} \left(-\left(\cos\pi - \cos 0\right)\right) = \frac{4RV}{\pi}$$

$$\Rightarrow \quad e = \frac{4R}{\pi} = 1.27\,R$$

(3)



T in lb-in



$$J = \frac{\pi}{2} R^4$$

$$J_A = \frac{\pi}{2} 1^4 = \frac{\pi}{2} \text{ in}^4$$

$$J_B = \frac{\pi}{2} (2)^4 = 8\pi \text{ in}^4$$

$$J_C = \frac{\pi}{2} (3)^4 = \frac{81}{2}\pi \text{ in}^4$$

$$\phi = \phi_{A1} + \phi_{A2} + \phi_{B1} + \phi_{B2} + \phi_C$$

$$\phi_{A1} = 0$$

$$\phi_{A2} = \frac{TL}{GJ} = \frac{(60 \times 10^3 \text{ lb-in})\, 6 \text{ in}}{10 \times 10^6 \frac{\text{lb}}{\text{in}^2} \cdot \frac{\pi}{2} \text{ in}^4} = \frac{60 \times 6}{10 \times 10^3 \times \frac{\pi}{2}} = \frac{9}{125\pi} \text{ rads}$$

$$\phi_{B1} = \frac{TL}{GJ} = \frac{(60 \times 10^3)\, 8}{(20 \times 10^6)\, 8\pi} = \frac{3}{1000\pi}$$

$$\phi_{B2} = \frac{TL}{GJ} = \frac{(108 \times 10^3)\, 8}{(20 \times 10^6)\, 8\pi} = \frac{27}{5000\pi}$$

$$\phi_C = \frac{TL}{GJ} = \frac{(108 \times 10^3)\, 15}{(15 \times 10^6) \frac{81}{2}\pi} = \frac{1}{375\pi}$$

$$\Rightarrow \phi = \frac{623}{7500\pi} \text{ rads} \approx .0265 \text{ rads}$$

$$\phi = \frac{623}{7500\pi} \cdot \frac{180^\circ}{\pi} = \frac{1869}{125\pi^2}^\circ \approx 1.51^\circ$$

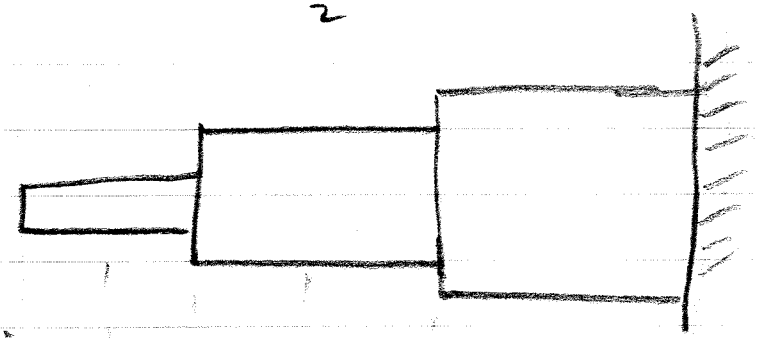
$$\tau_{max} = \frac{Tr}{J} \Rightarrow \tau_{max} = 0 \text{ in } A1$$

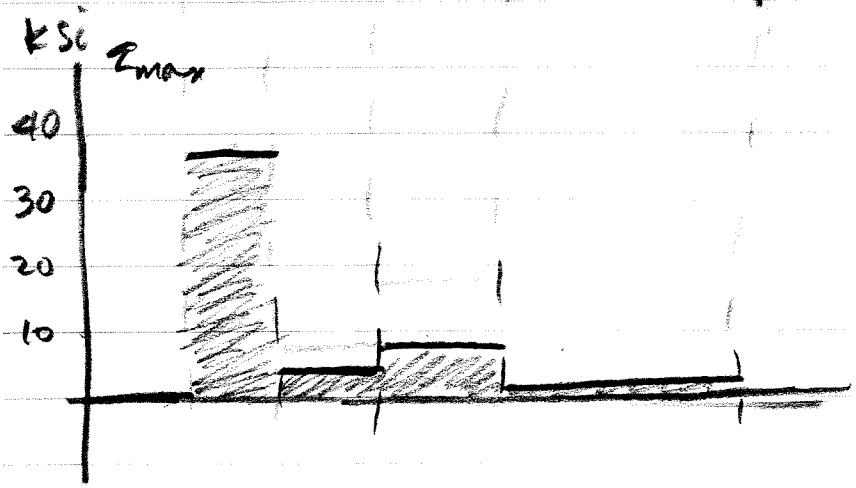
$$\tau_{max\ A2} = \frac{(60\times10^3 \text{ lb-in})\,1 \text{ in}}{\frac{\pi}{2} \text{ in}^4} = \frac{120\times10^3}{\pi} \text{ psi} = 38.2\times10^3 \text{ psi}$$

$$\tau_{max\ B1} = \frac{(60\times10^3)\,2}{8\pi} = \frac{15\times10^3}{\pi} \text{ psi} = 4.77\times10^3 \text{ psi}$$

$$\tau_{max\ B_2} = \frac{(108\times10^3)\,2}{8\pi} = \frac{27\times10^3}{\pi} \text{ psi} = 8.59\times10^3 \text{ psi}$$

$$\tau_{max\ C} = \frac{(108\times10^3)\,3}{\frac{81\pi}{2}} = \frac{8\times10^3}{\pi} = 2.55\times10^3 \text{ psi}$$





6.3 $T_A = \dfrac{T}{1 + \frac{aJ_b}{bJ_A}}$, $T_B = \dfrac{T}{1 + \left(\frac{aJ_b}{bJ_A}\right)^{-1}}$

$$\frac{aJ_b}{bJ_a} = \frac{a\,\frac{\pi}{2} r_b^4}{b\,\frac{\pi}{2} r_a^4} = \frac{a r_b^4}{b r_a^4} = 2\left(\frac{15}{20}\right)^4 = \frac{81}{128}$$

$$\tau_{max_A} = \frac{T_A r_a}{J_a} = \frac{2}{\pi r_a^3} T_A = \frac{2}{\pi (.01\,m)^3}\left(\frac{T}{1+\frac{81}{128}}\right) = \frac{256\times10^6}{209}\frac{T}{\pi}\,m^3$$

$$\tau_{max_B} = \frac{T_B r_b}{J_b} = \frac{2}{\pi r_b^3} T_B = \frac{2}{\pi (.0075)^3}\,\frac{T}{1+\frac{128}{81}} = \frac{384\times10^6}{209}\frac{T}{\pi}$$

require $150\ MPa > \max\left(\tau_{max_A}, \tau_{max_B}\right) = \dfrac{384\times10^6}{209}\dfrac{T}{\pi}\dfrac{1}{m^3}$

$$\Rightarrow T < \frac{209\pi}{384\times10^6}\,150\ MPa\cdot m^3 = \frac{(209)(150)}{384}\pi\frac{N}{m^2}\,m^3$$

$$\Rightarrow T < \frac{5225}{64}\pi\ N\cdot m \approx 256.5\ N\cdot m$$