Define the impulse response of the velocity potential as the integral of the Green's function over the radiating surface:

$$ h(y, z; t) = \int_0^\infty \int_0^{2\pi} \frac{\delta(t - \frac{R}{c})}{2\pi R} \, ds \, d\theta $$

For a circular piston: (cylindrical symmetry assumed)

- Piston radius \( R \)
- Observer coords. \((r, z)\)
- Source coords. \((\xi, \theta)\)

R - distance from source to observer
\( \xi \) - distance from origin to source point
\( \theta \) - angle between source point & the r-axis
\( d \) - intermediate variable

Zoom in on triangle on the piston face

\[ d^2 = r^2 + \xi^2 - 2r\xi \cos \theta \]
\[ d^2 + z^2 = R^2 \]
\[ R^2 = z^2 + r^2 + \xi^2 - 2r\xi \cos \theta \]
Near field analytical response is known for on-axis case, so start with that result.

\[ y = 0 \ (\text{on-axis}): \quad R = \sqrt{z^2 + b^2} \]

\[ h(r=0, z; t) = \int_0^{2\pi} \int_0^a \delta\left(t - \frac{\sqrt{z^2 + b^2}}{c}\right) \frac{1}{2\pi \sqrt{z^2 + b^2}} \ d\theta \ d\phi \]

\[ h(r=0, z; t) = \int_0^{\frac{2\pi}{2}} \frac{2\pi}{2\pi} \delta\left(t - \frac{\sqrt{z^2 + b^2}}{c}\right) \frac{1}{\sqrt{z^2 + b^2}} \ d\phi \]

Let \( \tau = t - \frac{\sqrt{z^2 + b^2}}{c} \)

\[ d\tau = -\frac{1}{c} \frac{1}{2} \left(\frac{z^2 + b^2}{z^2 + b^2}\right)^{-\frac{1}{2}} \int dz \]

\[ h(r=0, z; t) = \int_{t - \frac{2\pi}{2}}^{t} \delta(\tau) d\tau \]

\[ t = \frac{2\pi}{2} \]

Integral of \( \delta(\tau) d\tau \) is equal to 1 only if \( t_1 \leq \tau \leq t_2 \)

so \( h(r=0, z; t) = \begin{cases} -c & \text{if } \frac{2\pi}{2} \leq t \leq \frac{2\sqrt{z^2 + b^2}}{c} \\ 0 & \text{elsewhere} \end{cases} \)

Initial arrival time \( \frac{2\pi}{2} \) corresponds to shortest
To determine axial pressure distribution, evaluate

\[ p = -\rho \frac{\partial}{\partial t} \left[ u(t) \ast h(r, z; t) \right] \]

convolution

with \( u(t) = U_0 e^{j\omega t} \) as the excitation

for the time-harmonic case. Plot it-

\[ \frac{1}{c} \left[ h(r=0, z; t) \right] = \frac{1}{c} \begin{cases} 1 \quad & t = \frac{z}{c} \\ 0 \quad & t \neq \frac{z}{c} \end{cases} \]

\[ u(t) \ast h(r=0, z; t) = U_0 \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(r=0, z; \tau) d\tau \]

\[ u \ast h = U_0 \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} (-\tau) d\tau \]
\[ u \otimes h = \frac{V_0 e^{j\omega t}}{c} \sqrt{\frac{z^2 + a^2}{c}} e^{-j\omega z/c} \]

\[ u \otimes h = \frac{V_0 e^{j\omega t}}{c} \left[ e^{-j\omega \sqrt{z^2 + a^2}/c} - e^{-j\omega z/c} \right] \]

\[ p(r = 0, z; t) = -p_0 \frac{\partial}{\partial t} [u \otimes h] \]

\[ p(r = 0, z; t) = -p_0 j \omega e^{j\omega t} \left[ e^{-j\omega \sqrt{z^2 + a^2}/c} - e^{-j\omega z/c} \right] \]

\[ p(r = 0, z; t) = e^{j\omega t} p_0 c V_0 \left[ e^{-j\omega z/c} - e^{-j\omega \sqrt{z^2 + a^2}/c} \right] \]

same as 7.4.4 for circular piston on-axis

Have now verified that impulse response provides the same answer we obtained previously in the near field for a circular piston.

circular piston, off-axis: need impulse response because analytical expressions are not available in the near field & point source superposition is very slow.
Off-axis with \( r < a \):

Clever idea: treat piston as 2 separate sections

- one circular section specified s.t. the observation point is “on-axis” (shaded area below)
- remaining surface of circular piston outside shaded area

\( \begin{align*}
\text{shaded region shares a tangent with larger piston} \\
(\rho, z) \text{- coords. of observation pt.}
\end{align*} \)

\( a \): piston radius  
\( a - r \): radius of shaded region

Can use previous “on-axis” result for shaded region, must use \( (a - r) \) with this result as this is the radius of the shaded region

\[
h_{\text{shaded}}(t) = \int_{\frac{t - \sqrt{z^2 + (a - r)^2}}{c}}^{\frac{t - \sqrt{z^2 + (a - r)^2}}{c} + \frac{\tau}{c}} \delta(\tau) \, d\tau \quad \text{(only change is here)}
\]
remaining portion - sweep out arcs of radius $r$ outside of the shaded region ($r > a-r$)

Another interpretation of this "clever idea": evaluate the diffraction integral in the observer's coordinate system (Note that arcs are centered at a distance $r$ from the center of the piston, where $r$ is the radial coordinate of the observer)

$$h_{\text{remaining}}(t) = \int_{a-r}^{a+r} \int_{-1}^{1} \frac{\delta(t - \frac{B}{c})}{2\pi \tau} \delta \, d\delta d\Theta$$

$\uparrow$ to be determined

$(a-r)$: radius of arc at outer edge of shaded region (start of remaining portion)

$(a+r)$: radius of arc at most distant portion of the piston

want to sweep out arcs at all distances $r$ in between $(a-r)$ and $(a+r)$
Need law of cosines to determine starting and stopping angles

\[ a^2 = b^2 + r^2 - 2br \cos \theta \]

\[ \theta = \cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right) \]

For each value of \( \theta \), we will sweep out an arc from \(-\theta\) to \(\theta\), where \(\theta\) is defined as above.

As before, \( R = \sqrt{x^2 + b^2} \), so

\[ h_{\text{remaining}}(t) = \int_{a-r}^{a+r} \int_{-\cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right)}^{\cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right)} \frac{8(t - \sqrt{b^2 + z^2})}{2\pi \sqrt{b^2 + z^2}} \, dz \, dh \]

\( \theta \) integration is trivial (note no \( \theta \)-terms in integrand)

\[ h_{\text{remaining}}(t) = \int_{a-r}^{a+r} 2 \cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right) \frac{8(t - \sqrt{b^2 + z^2})}{2\pi \sqrt{b^2 + z^2}} \, dz \, d\theta \]
Again, \( \tau = t - \frac{\sqrt{\delta^2 + z^2}}{c} \); \( c^2(t - \tau)^2 = \delta^2 + z^2 \)

\[
d^\tau = -\frac{1}{c} \left( \delta^2 + z^2 \right)^{-\frac{1}{2}} \frac{\tau}{2} \, 2\delta \, d\delta
\]

\[
h_{\text{remaining}}(t) = -\frac{c}{\pi} \int \delta(\tau) \cos^{-1} \left( \frac{c^2(t - \tau)^2 - z^2 + r^2 - a^2}{2r \sqrt{c^2(t - \tau)^2 - z^2}} \right) \frac{t - \sqrt{z^2 + (a-r)^2}}{c} \, d\tau
\]

Combining these results gives for \( r < a \):

\[
h(r,a,t) = \begin{cases} 
- \frac{c}{\pi} \cos^{-1} \left( \frac{c^2t^2 - z^2 + r^2 - a^2}{2r \sqrt{c^2t^2 - z^2}} \right) \\
\text{for} \quad \frac{c}{\pi} \leq t \leq \frac{\sqrt{z^2 + (a-r)^2}}{c}
\end{cases}
\]

\[
h(r,a,t) = \begin{cases} 
- \frac{c}{\pi} \cos^{-1} \left( \frac{c^2t^2 - z^2 + r^2 - a^2}{2r \sqrt{c^2t^2 - z^2}} \right) \\
\text{for} \quad \frac{\sqrt{z^2 + (a-r)^2}}{c} \leq t \leq \frac{\sqrt{z^2 + (a+r)^2}}{c}
\end{cases}
\]

\[
h(r,a,t) = 0 \quad \text{all other times (} t < \frac{c}{\pi}, t > \frac{\sqrt{z^2 + (a+r)^2}}{c} \text{)}
\]
Outside of the piston radius, it is not possible to draw a circle and find an "on-axis" contribution. However, the "remaining area" is described by the same expression as before.

Shortest distance from point projected onto plane containing circular piston

\( r + a \): longest distance from projected observation point to another point on the piston face

\[
h(r > a, z; t) = \int_{r-a}^{r+a} \int_{-\cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right)}^{\cos^{-1}\left(\frac{b^2 + r^2 - a^2}{2br}\right)} \delta\left(t - \frac{\sqrt{b^2 + z^2}}{c}\right) \, d\theta \, dp
\]

this is the same as the remaining for \( r < a \) except that the sign is switched on the
lower limit of the integral for 6.
Since this quantity is squared when we use the variable transformation
\( \tau = \frac{t - \frac{\sqrt{z^2 + z^2}}{c}}{c} \), there really isn’t any difference.

\[
h(r > a, z; t) = -\frac{c}{\pi} \int_{t - \frac{\sqrt{z^2 + (r+a)^2}}{c}}^{t} \frac{c}{2r} \sqrt{c^2(t-\tau)^2 - z^2 + r^2 - a^2} \cos^{-1}\left\{ \frac{c^2 t^2 - z^2 + r^2 - a^2}{2r \sqrt{c^2(t-\tau)^2 - z^2}} \right\} d\tau
\]

\[
h(r > a, z; t) = \left\{ \begin{array}{ll}
-\frac{c}{\pi} \cos^{-1}\left\{ \frac{c^2 t^2 - z^2 + r^2 - a^2}{2r \sqrt{c^2(t-\tau)^2 - z^2}} \right\} & \\
& \text{for } c \frac{1}{\sqrt{z^2 + (r-a)^2}} \leq t \leq \frac{c}{\sqrt{z^2 + (r+a)^2}} \\
0 & \text{all other times}
\end{array} \right.
\]

(i.e. \( t > \frac{\sqrt{z^2 + (r+a)^2}}{c} \))

\( t < \frac{\sqrt{z^2 + (r-a)^2}}{c} \)