1. Introduction

[2] It is now widely recognized that hydrogeological properties such as hydraulic conductivity vary significantly over a wide range of spatial scales [Gelhar, 1993]. This variability reflects the aggregate effect of different geological processes acting over extended time periods. It is tempting for analytical reasons to adopt a simplified description of spatial variability which imposes some sort of regular structure on the subsurface environment. Such a description can be deterministic (e.g., the subsurface consists of homogeneous layers) or stochastic (e.g., the subsurface properties are stationary random fields). Although these simplified descriptions do not capture the true nature of hydrogeologic variability, they may provide reasonable approximations for particular applications.

[3] In certain situations, variables such as hydraulic conductivity exhibit local trends that extend over scales comparable to the overall scale of interest in a given problem. Trends are a natural result of the sedimentation processes that create deltas, alluvial fans, and glacial outwash plains [Freeze and Cherry, 1979]. In such cases it seems reasonable to represent hydrogeologic variability as a stationary random field superimposed on a deterministic trend [Neuman and Jacobson, 1984; Rajaram and McLaughlin, 1990]. This note is concerned with predicting the velocity variances caused by unmodeled small-scale heterogeneity in the presence of systematic trends in mean hydraulic conductivity.

[4] Stochastic approaches for analyzing groundwater flow in heterogeneous trending media generally divide into (1) analytical techniques [e.g., Loaiciga et al., 1993; Rubin and Seong, 1994; Li and McLaughlin, 1995; Neuman and Orr, 1993; Indelman and Abramovich, 1994], which provide convenient closed form expressions for quantities such as the velocity variances and effective hydraulic conductivities but depend on restrictive assumptions (e.g., linear trends) and (2) numerical techniques [e.g., Smith and Freeze, 1979; McLaughlin and Wood, 1988; Li et al., 2003, 2004a], which make fewer assumptions but are more difficult to apply in practice. Here we present simple, closed form analytical solutions that relax the assumptions required in existing analytical theories of stochastic groundwater flow. This enables us to account for the effect of general trends while retaining the convenience of closed form results.

2. Problem Formulation

[5] To illustrate the basic concept we consider steady flow in a heterogeneous multidimensional porous medium with a systematic trend in the log hydraulic conductivity. There are many ways to partition a particular log conductivity function into a “trend” and a “random fluctuation.” Here we require that the fluctuation have a spatial average of zero and no obvious nonstationarities. The trend should vary more smoothly than the fluctuation. In practice, the trend can be estimated from a sample conductivity function by applying a low-pass or moving window filter [Rajaram and McLaughlin, 1990]. Given these general requirements, we assume that the log hydraulic conductivity is a locally isotropic, random field with a known spatially variable ensemble mean equal to the value of the trend at each point in space. We also assume that the fluctuation about the log conductivity mean is a wide-sense stationary random field...
with a known spectral density function. The random log conductivity fluctuation is approximately related to the piezometric head and velocity fluctuations by the following first-order, mean-removed flow equations:

\[
\frac{\partial^2 h}{\partial x \partial x} + \mu_l(x) \frac{\partial h}{\partial x} = J_i(x) \frac{\partial h}{\partial x} \quad x \in D, 
\]

\[
u_i' = K_g(x) \left[ J_i(x)f(x) - \frac{\partial h}{\partial x} \right] \quad x \in D, 
\]

\[
\nabla h(x) = 0 \quad x \in \Gamma_D, 
\]

where \( J_i(x) = -\partial h/\partial x \) is the mean head gradient and \( \mu_i(x) = \partial f/\partial x \) is the mean log conductivity gradient. The notation \( K_g(x) \) is the geometric mean conductivity. These equations are written in Cartesian coordinates, with the vector location symbolized by \( x \) and summation implied over repeated indices. The point values of the log hydraulic conductivity fluctuation \( f(x) \), head fluctuation \( h(x) \), and velocity fluctuation \( \nu_i(x) \), \( i = 1, 2, 3 \), are defined throughout the domain \( D \). A homogeneous condition is defined on the specified head boundary \( \Gamma_D \) and the specified flux boundary \( \Gamma_N \). Note that we consider \( f \) is solely the source of uncertainty applied in the aquifer system.

3. Spectral Solution

[6] Spectral methods offer a particularly convenient way to derive velocity statistics from linearized fluctuation equations. Taking advantage of scale disparity between the mean and fluctuation processes and invoking locally the spectral representation, one can solve (1) and (2) and obtain the following expression for predicting nonstationary velocity variances in heterogeneous trending media:

\[
\sigma_n^2(x) = C(\mu_l(x)) \sigma_f^2(x) K_g^2(x) J^2(x), 
\]

where

\[
C(\mu_l(x)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( 1 - \frac{s_{\mu_l} \omega_1}{\omega^2 - i\omega_1\dot{\omega}_f(x)} \right) \left( 1 - \frac{s_{\mu_l} \omega_2}{\omega^2 + i\omega_2\dot{\omega}_f(x)}\right) s_{\nu_i}(\omega)d\omega_1 d\omega_2, 
\]

Note \( s_{\nu_i} \) is the dimensionless spectral density function of \( f(x) \), \( s_{\nu_i} = s_{\nu_i} \sigma_f^2 \), \( \sigma_f^2 \) is the log conductivity variance, \( \omega_i \) is the wave number, \( \omega^2 = \omega_1^2 + \omega_2^2 \), \( s_{\mu_i} \sigma_1^2 \) and \( s_{\mu_i} \sigma_2^2 \) are respectively the longitudinal and transverse velocity variances. The detailed derivation process of (3) is very similar to that used by Gelhar [1993] for homogeneous media and is not repeated here.

4. Approximate Closed Form Solution

[7] Equation (3) applies to flows in general, mildly heterogeneous trending media. To obtain explicit results, one must in general evaluate the associated double integrals numerically. In most cases, these evaluations can be quite difficult. Most prior research focused on limited special cases for which (4) can be reduced to a form that allows exact, closed form integration [Loaiciga et al., 1993; Li and McLaughlin, 1995; Gelhar, 1993].

[8] In this paper, we evaluate (4) approximately under general trending conditions. Our approximate solution is based on the following observations:

[9] 1. Trends in conductivity influence the variance dynamics through \( C(\mu_l(x)) \) and \( K_g^2(x) J(x) \) (see (3)).

[10] 2. It is the evaluation of \( C(\mu_l(x)) \) for a general, multidimensional trend distribution \( \mu_l(x) \) that is difficult. The \( \mu_l(x) K_g J(x) \) term in (4) makes the integration analytically intractable.

[11] 3. It is, however, predominantly \( K_g J(x) \) that controls the nonstationary spatial variance dynamics.

[12] 4. For most trending situations, change in the mean log conductivity over the characteristic length of small-scale heterogeneity (a correlation scale \( \lambda \)) is small (or \( \mu_l^2 \lambda^2 \ll 1 \)) since the mean conductivity is expected to be much smoother than the fluctuation [Gelhar, 1993].

[13] To enable variance modeling under general, complex conditions, we propose approximating \( C(\mu_l(x)) \) via the following Taylor’s expansion-based expression:

\[
C(\mu_l(x)) = C(0) + \frac{\partial C(0)}{\partial \mu_l} \mu_l + \ldots \approx C(0) 
\]

Essentially we suggest that the small, but hard-to-evaluate contribution to variance nonstationarity from \( C(\mu_l(x)) \) be ignored relative to the more important contribution from \( K_g J(x) \). This assumption may seem quite crude at first sight but proves to be highly effective and makes general, approximate variance modeling in nonstationary trending media possible. Previous studies investigating the effects of trending are all based on full rigorous integration of (4) or full solution of (1) that is intractable unless the trends are assumed to be of special forms [Li and McLaughlin, 1991, 1995; Loaiciga et al., 1993; Gelhar, 1993]. These highly restrictive assumptions severely limit the practical utilities of the results.

[14] Substituting (5) into (3), we obtain

\[
\sigma_n^2(x) = \sigma_f^2(x) K_g^2(x) J^2(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( 1 - \frac{s_{\mu_l} \omega_1}{\omega^2} \right) s_{\nu_i}(\omega)d\omega_1 d\omega_2 
\]

Equation (6) can be easily integrated in the polar coordinate system. The result is the following explicit expressions:

\[
\sigma_n^2(x) = 0.375 \sigma_f^2(x) K_g^2(x) J^2(x), 
\]

\[
\sigma_n^2(x) = 0.125 \sigma_f^2(x) K_g^2(x) J^2(x), 
\]

These expressions are independent of the specific form of the log conductivity spectrum or covariance function for the isotropic case.

[15] Note that (7) and (8) are of the same form as the equations derived by previous researchers [e.g., Mizell et al., 1982; Gelhar, 1993] for statistically uniform flow, except that \( K_g \) and \( J \) are now allowed to vary over space as a function of the mean log conductivity gradient. These
general equations are simpler than the nonstationary expressions developed by Li and McLaughlin [1995], Loaiciga et al. [1993], and Gelhar [1993] for simple linear trending media.

In the following section, we illustrate how these simple closed form expressions can be used to quantify robustly groundwater velocity uncertainty with surprising accuracy, even in the presence of complex and strongly nonstationary trending hydraulic conductivity.

5. Illustrative Examples

Our examples consider two-dimensional steady state flows in a confined aquifer. The hydraulic conductivity is assumed to exhibit both a randomly varying small-scale fluctuation and a systematic large-scale nonstationarity. The small-scale fluctuation is represented as a random field characterized by a simple exponential and isotropic covariance function with a log conductivity variance of 1.0 and a correlation scale \( \lambda \) of 1.0. The large-scale nonstationarity is represented as a deterministic trend.

To systematically test the effectiveness of the closed form solutions, we consider a range of trending conductivity situations, from relatively simple and mild trends to trends that are much stronger, more complex, strongly multidimensional, and even discontinuous. For each problem, we first solve the mean deterministic groundwater flow equation without accounting for the small-scale heterogeneity, use the resulting head to evaluate the mean hydraulic gradient, and then substitute it into the explicit expressions in (7) and (8) to obtain the local variance values. We verify the accuracy of the closed form variance solutions by comparing them with the corresponding numerical solutions obtained from the first-order nonstationary spectral method [Li and McLaughlin, 1991; Li et al., 2004a, 2004b] and Monte Carlo simulation (based on 10,000 realizations). In all test examples, the simple closed form solutions proves to be robust, can capture complex spatial nonstationarities, and match well with the corresponding numerical perturbation solutions and Monte Carlo simulation.

We present in this section results from two selected examples involving relatively difficult trending situations. We select these examples because we believe that if a methodology is able to handle such distinctly different mean log conductivity trends, it should be able to handle perhaps most trends that can be practically represented using field data in real-world groundwater modeling. Table 1 presents detailed information defining the large-scale mean trends, statistics describing the small-scale heterogeneity, and other inputs used in the examples.

Table 1. Parameter Definitions for the Examples

<table>
<thead>
<tr>
<th></th>
<th>Continuous Trend</th>
<th>Discontinuous Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>lnK variance</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>lnK correlation scale ( \lambda )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Geometric mean conductivity</td>
<td>a replicate from a random field with a correlation scale of 40( \lambda ), variance of 0.5 (see Figure 1)</td>
<td>(see Figure 2)</td>
</tr>
<tr>
<td>Domain length</td>
<td>80x</td>
<td>80x</td>
</tr>
<tr>
<td>West boundary condition</td>
<td>constant head 100 m</td>
<td>constant head 92 m</td>
</tr>
<tr>
<td>East boundary condition</td>
<td>constant head 100 m</td>
<td>constant head 92 m</td>
</tr>
<tr>
<td>North boundary condition</td>
<td>no flow</td>
<td>no flow</td>
</tr>
<tr>
<td>South boundary condition</td>
<td>no flow</td>
<td>no flow</td>
</tr>
</tbody>
</table>

[20] The first example involves complex multidimensional trends artificially generated by a random field generator with a large correlation scale. We have purposely made the trends more complicated than that may typically be delineated with scattered field data in order to test the robustness of the closed form formulas. The second example involves discontinuous trends in the mean conductivity. Although we do not expect the closed form solutions to apply right at the discontinuities where the trending slope \( \mu_i(x) \) is infinite, we are interested in determining if and to what degree the closed form solutions apply overall and away from the discontinuities. The ability for a stochastic method to model discontinuous trends is important since many practical applications require working with zones of distinct materials, sharp geological boundaries, and multiple aquifers with different mean conductivities.

6. Results

Figures 1 and 2 illustrate the mean conductivity distributions for the trending examples and the distributions of corresponding steady state, mean head distributions. Although the driving head difference in both examples is uniform, the strong trends in the mean log conductivity...
yield nonuniform flow patterns which in turn cause strong nonstationarity in the velocity variances. Figures 3 and 4 show the complex, nonuniform distributions of the predicted velocity standard deviations for, respectively, the continuous and discontinuous trending examples. The results are presented in a profile along the domain centerline and obtained using the approximate closed form formulas, the numerically oriented nonstationary spectral method, and the Monte Carlo simulation. These plots clearly show that, despite the simplifications and the strong multidimensional trending nonstationarities, the simple closed form solutions reproduce well the corresponding first-order, nonstationary spectral solutions and allow capturing both the spatial structure and the magnitude of the highly nonstationary, complex uncertainty distributions. The closed form predictions of the velocity uncertainty also match reasonably well with those obtained from the Monte Carlo simulation for both examples. For the discontinuous trending case, the results becomes inaccurate at the discontinuities but the inaccuracies appear to be very localized.

7. Conclusions

[23] In this note, we have developed and demonstrated approximate closed form formulas to predict velocity variances for flow in heterogeneous porous media with systematic trends in log conductivity. Examples involving nonstationary flows in complex trending media are used to illustrate the approximate methodology. The results reveal that, despite the gross simplifications, the closed form expressions are highly effective and can reproduce the corresponding numerical, nonstationary spectral and Monte Carlo solutions. The results also show how trends in hydraulic conductivity produce complex structural

![Figure 2](image1.png)

**Figure 2.** Spatial distribution of the geometric mean conductivity and corresponding mean head distribution for the discontinuous trending case.

![Figure 3](image2.png)

**Figure 3.** Centerline profile of the predicted velocity standard deviations using the closed form formulae, nonstationary numerical spectral method, and Monte Carlo simulation for continuous trending case.
changes in the spatial distributions in the velocity variances. The closed form formulas make it possible to model the velocity uncertainty in nonstationary trending media. The analysis represents a step closer to our ultimate goal to include a systematic uncertainty analysis as a part of routine groundwater modeling. We are currently in the process of extending the approximate methodology to general unsteady flow in both confined and unconfined aquifers in the presence of complex sources and sinks.

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References

Figure 4. Centerline profile of the predicted velocity standard deviations using the closed form formulae, nonstationary numerical spectral method, and Monte Carlo simulation for discontinuous trending case.