Nonlinear Systems and Control
Lecture # 2
Examples of Nonlinear Systems
Pendulum Equation

\[
ml\ddot{\theta} = -mg \sin \theta - kl\dot{\theta}
\]

\[
x_1 = \theta, \quad x_2 = \dot{\theta}
\]
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2
\end{align*}
\]

Equilibrium Points:

\[
\begin{align*}
0 &= x_2 \\
0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2
\end{align*}
\]

\((n\pi, 0)\) for \(n = 0, \pm 1, \pm 2, \ldots\)

Nontrivial equilibrium points at \((0, 0)\) and \((\pi, 0)\)
Pendulum without friction:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1
\end{align*}
\]

Pendulum with torque input:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T
\end{align*}
\]
Tunnel-Diode Circuit

\[ i_C = C \frac{dv_C}{dt}, \quad v_L = L \frac{di_L}{dt} \]

\[ x_1 = v_C, \quad x_2 = i_L, \quad u = E \]
\[ i_C + i_R - i_L = 0 \Rightarrow i_C = -h(x_1) + x_2 \]

\[ v_C - E + Ri_L + v_L = 0 \Rightarrow v_L = -x_1 - Rx_2 + u \]

\[ \dot{x}_1 = \frac{1}{C} [-h(x_1) + x_2] \]

\[ \dot{x}_2 = \frac{1}{L} [-x_1 - Rx_2 + u] \]

**Equilibrium Points:**

\[ 0 = -h(x_1) + x_2 \]
\[ 0 = -x_1 - Rx_2 + u \]
\[ h(x_1) = \frac{E}{R} - \frac{1}{R} x_1 \]
Mass–Spring System

\[ m \ddot{y} + F_f + F_{sp} = F \]

Sources of nonlinearity:
- Nonlinear spring restoring force \( F_{sp} = g(y) \)
- Static or Coulomb friction
\[ F_{sp} = g(y) \]

\[ g(y) = k(1 - a^2y^2)y, \quad |ay| < 1 \quad \text{(softening spring)} \]

\[ g(y) = k(1 + a^2y^2)y \quad \text{(hardening spring)} \]

\( F_f \) may have components due to static, Coulomb, and viscous friction.

When the mass is at rest, there is a static friction force \( F_s \) that acts parallel to the surface and is limited to \( \pm \mu_s mg \) \((0 < \mu_s < 1)\). \( F_s \) takes whatever value, between its limits, to keep the mass at rest.

Once motion has started, the resistive force \( F_f \) is modeled as a function of the sliding velocity \( v = \dot{y} \).
(a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect
Negative-Resistance Oscillator

\[ h(0) = 0, \quad h'(0) < 0 \]

\[ h(v) \to \infty \text{ as } v \to \infty, \quad \text{and} \quad h(v) \to -\infty \text{ as } v \to -\infty \]
\[ i_C + i_L + i = 0 \]
\[ C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^{t} v(s) \, ds + h(v) = 0 \]

Differentiating with respect to \( t \) and multiplying by \( L \):
\[ CL \frac{d^2v}{dt^2} + v + Lh'(v) \frac{dv}{dt} = 0 \]

\( \tau = \frac{t}{\sqrt{CL}} \)

\[ \frac{dv}{d\tau} = \sqrt{CL} \frac{dv}{dt}, \quad \frac{d^2v}{d\tau^2} = CL \frac{d^2v}{dt^2} \]
Denote the derivative of $v$ with respect to $\tau$ by $\dot{v}$

$$\ddot{v} + \varepsilon h'(v) \dot{v} + v = 0, \quad \varepsilon = \sqrt{L/C}$$

Special case: Van der Pol equation

$$h(v) = -v + \frac{1}{3}v^3$$

$$\ddot{v} - \varepsilon (1 - v^2) \dot{v} + v = 0$$

State model: $x_1 = v$, $x_2 = \dot{v}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2$$
Another State Model: \[ z_1 = i_L, \quad z_2 = v_C \]

\[
\begin{align*}
\dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\
\dot{z}_2 &= -\varepsilon [z_1 + h(z_2)]
\end{align*}
\]

Change of variables: \[ z = T(x) \]

\[
\begin{align*}
x_1 &= v = z_2 \\
x_2 &= \frac{dv}{d\tau} = \sqrt{CL} \frac{dv}{dt} = \sqrt{\frac{L}{C}}[-i_L - h(v_C)] \\
&= \varepsilon [-z_1 - h(z_2)]
\end{align*}
\]

\[
T(x) = \begin{bmatrix}
-h(x_1) - \frac{1}{\varepsilon} x_2 \\
x_1
\end{bmatrix}, \quad T^{-1}(z) = \begin{bmatrix}
z_2 \\
-\varepsilon z_1 - \varepsilon h(z_2)
\end{bmatrix}
\]
Adaptive Control

*Plant*: \[ \dot{y}_p = a_p y_p + k_p u \]

*Reference Model*: \[ \dot{y}_m = a_m y_m + k_m r \]

\[ u(t) = \theta_1^* r(t) + \theta_2^* y_p(t) \]

\[ \theta_1^* = \frac{k_m}{k_p} \text{ and } \theta_2^* = \frac{a_m - a_p}{k_p} \]

When \( a_p \) and \( k_p \) are unknown, we may use

\[ u(t) = \theta_1(t) r(t) + \theta_2(t) y_p(t) \]

where \( \theta_1(t) \) and \( \theta_2(t) \) are adjusted on-line
Adaptive Law (gradient algorithm):

\[
\dot{\theta}_1 = -\gamma(y_p - y_m)r \\
\dot{\theta}_2 = -\gamma(y_p - y_m)y_p, \quad \gamma > 0
\]

State Variables: 
\[e_o = y_p - y_m, \quad \phi_1 = \theta_1 - \theta_1^*, \quad \phi_2 = \theta_2 - \theta_2^*\]

\[
\dot{y}_m = a_p y_m + k_p(\theta_1^*r + \theta_2^*y_m) \\
\dot{y}_p = a_p y_p + k_p(\theta_1 r + \theta_2 y_p)
\]

\[
\dot{e}_o = a_p e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 y_p - \theta_2^* y_m) \\
= \cdots \cdots + k_p[\theta_2^* y_p - \theta_2^* y_p] \\
= (a_p + k_p \theta_2^*) e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 - \theta_2^*) y_p
\]
Closed-Loop System:

\[
\begin{align*}
\dot{e}_o &= a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)] \\
\dot{\phi}_1 &= -\gamma e_o r(t) \\
\dot{\phi}_2 &= -\gamma e_o [e_o + y_m(t)]
\end{align*}
\]