Nonlinear Systems and Control
Lecture # 10
The Invariance Principle

Example: Pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\,a\sin x_1 - bx_2\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = -\,bx_2^2$$

The origin is stable. $\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

However, near the origin, the solution cannot stay identically in the set $\{x_2 = 0\}$

Definitions: Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall \, t \in R$$

Examples:

- Equilibrium points
- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall \, t \geq 0$$

Example: The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\mathrm{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\mathrm{dist}(x(t), M) < \varepsilon, \ \ \forall\, t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \longrightarrow \infty$

Notice, however, that $x(t)$ does converge to any specific point on the limit cycle

Lemma: If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$

LaSalle's theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$

Proof:

$$\dot{V}(x) \leq \text{ in } \Omega \quad \Rightarrow \quad V(x(t)) \text{ is a decreasing}$$

$$V(x) \text{ is continuous in } \Omega \quad \Rightarrow \quad V(x) \geq b = \min_{x \in \Omega} V(x)$$

$$\Rightarrow \quad \lim_{t \to \infty} V(x(t)) = a$$

$$x(t) \in \Omega \quad \Rightarrow \quad x(t) \text{ is bounded} \quad \Rightarrow \quad L^+ \text{ exists}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \to \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n \to \infty} t_n = \infty$ such that $x(t_n) \to p$ as $n \to \infty$

$$V(x) \text{ is continuous} \quad \Rightarrow \quad V(p) = \lim_{n \to \infty} V(x(t_n)) = a$$

$V(x) = a$ on L^+ and L^+ invariant $\Rightarrow$ $\dot{V}(x) = 0, \ \forall\, x \in L^+$

$$L^+ \subset M \subset E \subset \Omega$$

$x(t)$ approaches L^+ $\Rightarrow$ $x(t)$ approaches M (as $t \to \infty$)

Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable

- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction

- Furthermore, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2)\end{aligned}$$

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \text{for } 0 < |y| < a$$

$$V(x) = \int_0^{x_1} h_1(y)\, dy \; + \; \tfrac{1}{2}x_2^2$$

$$D = \{-a < x_1 < a, \;\; -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h_2(x_2) = 0 \;\Rightarrow\; x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The only solution that can stay identically in S is $x(t) \equiv 0$

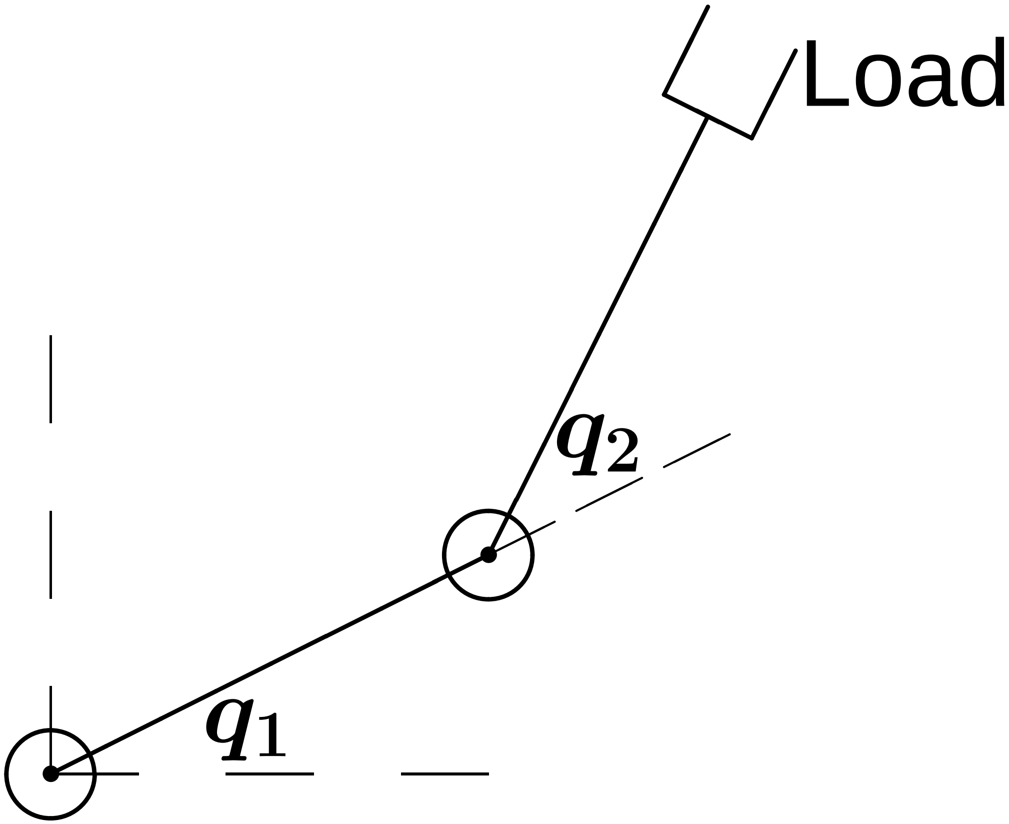
Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z)\, dz \ \rightarrow \ \infty \ \text{as} \ |y| \rightarrow \ \infty$

Then, $D = R^2$ and $V(x) = \int_0^{x_1} h_1(y)\, dy \ + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in R^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Example: m-link Robot Manipulator



Two-link Robot Manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q is an m-dimensional vector of joint positions

u is an m-dimensional control (torque) inputs

$M = M^T > 0$ is the inertia matrix

$C(q,\dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces

$$(\dot{M} - 2C)^T = -(\dot{M} - 2C)$$

$D\dot{q}$ accounts for viscous damping; $D = D^T \geq 0$

$g(q)$ accounts for gravity forces; $g(q) = [\partial P(q)/\partial q]^T$

$P(q)$ is the total potential energy of the links due to gravity

Investigate the use of the (PD plus gravity compensation) control law

$$\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_p(\boldsymbol{q} - \boldsymbol{q}^*) - \boldsymbol{K}_d\, \dot{\boldsymbol{q}}$$

to stabilize the robot at a desired position $\boldsymbol{q}^*$, where $\boldsymbol{K}_p$ and $\boldsymbol{K}_d$ are symmetric positive definite matrices

$$\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}^*, \quad \dot{\boldsymbol{e}} = \dot{\boldsymbol{q}}$$

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{e}} &= \boldsymbol{M}\ddot{\boldsymbol{q}} \\ &= -\boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{D}\,\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{u} \\ &= -\boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{D}\,\dot{\boldsymbol{q}} - \boldsymbol{K}_p(\boldsymbol{q} - \boldsymbol{q}^*) - \boldsymbol{K}_d\,\dot{\boldsymbol{q}} \\ &= -\boldsymbol{C}\,\dot{\boldsymbol{e}} - \boldsymbol{D}\,\dot{\boldsymbol{e}} - \boldsymbol{K}_p\,\boldsymbol{e} - \boldsymbol{K}_d\,\dot{\boldsymbol{e}} \end{aligned}$$

$$\boldsymbol{M\ddot{e} = -C\,\dot{e} - D\,\dot{e} - K_p\,e - K_d\,\dot{e}}$$

$$\boldsymbol{V = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e}$$

$$\begin{aligned}
\boldsymbol{\dot{V}} &= \boldsymbol{\dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p\dot{e}} \\
&= \boldsymbol{-\dot{e}^T C\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e - \dot{e}^T K_d\dot{e}} \\
&\quad \boldsymbol{+ \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p\dot{e}} \\
&= \boldsymbol{\frac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T(K_d + D)\dot{e}} \\
&= \boldsymbol{-\dot{e}^T(K_d + D)\dot{e} \;\leq\; 0}
\end{aligned}$$

$(\boldsymbol{K_d} + \boldsymbol{D})$ is positive definite

$$\dot{\boldsymbol{V}} = -\dot{\boldsymbol{e}}^T (\boldsymbol{K_d} + \boldsymbol{D}) \dot{\boldsymbol{e}} = \boldsymbol{0} \;\Rightarrow\; \dot{\boldsymbol{e}} = \boldsymbol{0}$$

$$\boldsymbol{M} \ddot{\boldsymbol{e}} = -\boldsymbol{C}\,\dot{\boldsymbol{e}} - \boldsymbol{D}\,\dot{\boldsymbol{e}} - \boldsymbol{K_p}\,\boldsymbol{e} - \boldsymbol{K_d}\,\dot{\boldsymbol{e}}$$

$$\dot{\boldsymbol{e}}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow \ddot{\boldsymbol{e}}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow\; \boldsymbol{K_p e}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow\; \boldsymbol{e}(\boldsymbol{t}) \equiv \boldsymbol{0}$$

By LaSalle’s theorem the origin $(\boldsymbol{e} = \boldsymbol{0}, \dot{\boldsymbol{e}} = \boldsymbol{0})$ is globally asymptotically stable