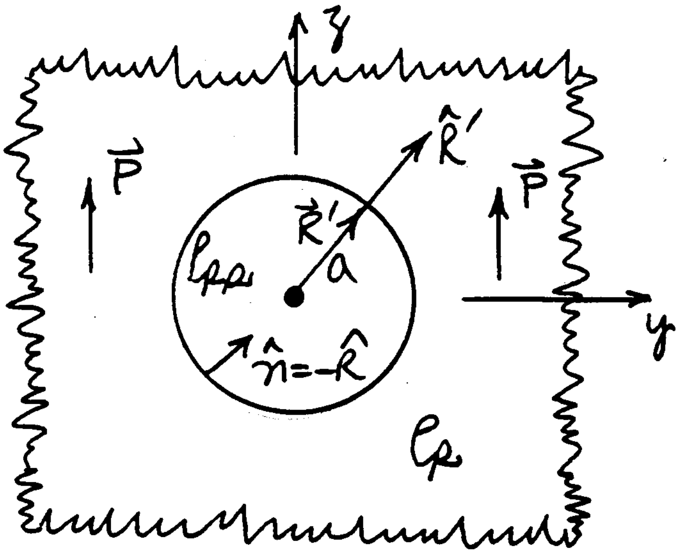
PROBLEM SET 5

P. 3-23 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\vec{P}$ exists.



assume $\vec{P} = \hat{z}P_0$ = uniform

$\rho_p = -\nabla\cdot\vec{P} = 0$ ... for $\vec{P}$ = const.

$\rho_{ps} = \hat{n}\cdot\vec{P} = -\hat{R}\cdot\hat{z}P_0 = -P_0\cos\theta$

$\vec{R} = 0, \quad \vec{R}' = \hat{R}'a$

$\vec{R}-\vec{R}' = -\hat{R}'a, \quad |\vec{R}-\vec{R}'| = a$

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\int_S \rho_{ps}(\vec{R}')\frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|^3}\,ds'$$

variable

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} -P_0\cos\theta'\ \frac{-\hat{R}'a}{a^3}\ a^2\sin\theta'\,d\theta'\,d\phi'$$

$$= \frac{P_0}{4\pi\epsilon_0}\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\cos\theta'\left[\hat{x}\sin\theta'\cos\phi' + \hat{y}\sin\theta'\sin\phi' + \hat{z}\cos\theta'\right]\times\sin\theta'$$

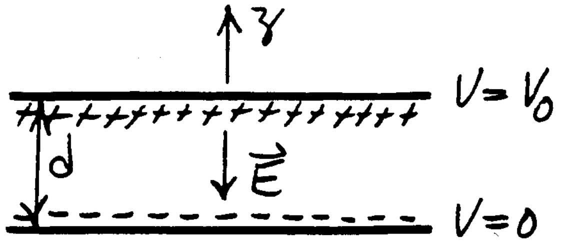
0 by integration
0

$$= \frac{\hat{z}P_0}{2\epsilon_0}\int_0^{\pi}\cos^2\theta'\sin\theta'\,d\theta' = \frac{\hat{z}P_0}{2\epsilon_0}\,\frac{-1}{3}\cos^3\theta'\Big|_0^{\pi}$$

$$\boxed{\vec{E}(\vec{0}) = \hat{z}\,\frac{P_0}{2\epsilon_0}}$$

P. 3-24 Solve the following problems:

a) Find the breakdown voltage of a parallel-plate capacitor, assuming that conducting plates are 50 (mm) apart and the medium between them is air.



$V = V(z)$ ... by planar symmetry

$\therefore \vec{E} = -\nabla V = -\hat{z}\dfrac{\partial V}{\partial z} = \hat{z} E_z(z)$

$\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0} = 0$ ... in space region

$$\frac{\partial E_z}{\partial z} = 0 \rightarrow E_z = C = \text{Const.}$$

$$V_0 = -\int_0^d E_z dz = -C\int_0^d dz = -Cd \rightarrow C = -\frac{V_0}{d}$$

$$E_z = -\frac{V_0}{d}, \quad \therefore \; E_b = \frac{(V_0)_{max}}{d}$$

$$(V_0)_{max} = E_b d = \frac{3\,kV}{mm} \times 50\,mm$$

$$\boxed{(V_0)_{max} = 150\,kV}$$

b) Find the breakdown voltage if the entire space between the conducting plates is filled with plexiglass, which has a dielectric constant 3 and a dielectric strength 20 (kV/mm).

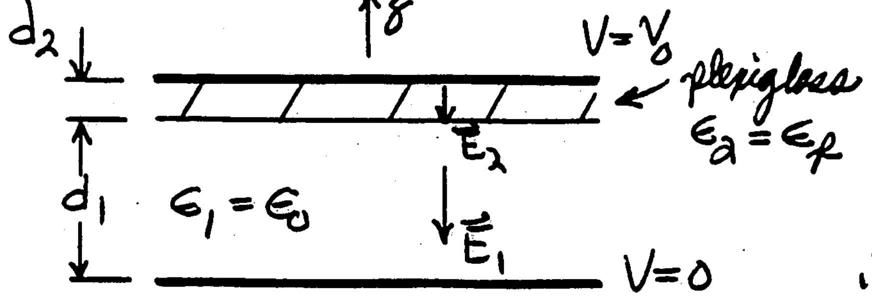
$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} = 0$ ... in capacitor, so analysis doesn't change and

$E_z = -\frac{V_0}{d}$, $\therefore$ $E_b = \frac{(V_0)_{max}}{d}$

$(V_0)_{max} = E_b d = 20 \frac{kV}{mm} \times 50\ mm$

$$\boxed{(V_0)_{max} = 1000\ kV}$$

c) If a 10 (mm) thick plexiglass is inserted between the plates, what is the maximum voltage that can be applied to the plates without a breakdown?



$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} = 0$
... both regions

$\therefore$ $\frac{\partial E_{z1}}{\partial z} = \frac{\partial E_{z2}}{\partial z} = 0$

$\therefore$ $E_{z1} = C_1$, $E_{z2} = C_2$ ... constants in both regions

$$V_0 = -\int_0^d E_z\,dz = -C_1\int_0^{d_1} dz - C_2\int_{d_1}^{d_1+d_2} dz$$

$$= -C_1 d_1 - C_2 d_2$$

$$\hat{n}\cdot(\vec{D}_2-\vec{D}_1) = \cancel{\rho_{s}}^{\,0} \longrightarrow \hat{z}\cdot(\epsilon_p \vec{E}_2 - \epsilon_0 E_1) = 0$$ ... at interface between plexiglass and air

↑ B.C. for normal $\vec{D}$

$$\epsilon_p E_{z2} = \epsilon_0 E_{z1} \longrightarrow \epsilon_p C_2 = \epsilon_0 C_1$$

$$C_2 = \frac{\epsilon_0}{\epsilon_p} C_1 = \frac{C_1}{\epsilon_{rp}}$$

$$V_0 = -\left(d_1 + \frac{d_2}{\epsilon_{rp}}\right) C_1 \rightarrow E_{z1} = -\frac{V_0}{d_1 + \frac{d_2}{\epsilon_{rp}}}$$

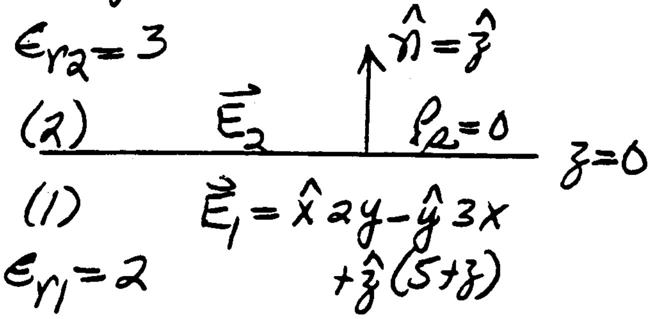
note: $E_{z2} = \dfrac{E_{z1}}{\epsilon_{rp}} < E_{z1}$ and $E_{bp} > E_{ba}$

$\therefore$ air region breaks down first and

$$(V_0)_{max} = (E_b)_{air}\left(d_1 + \frac{d_2}{\epsilon_{rp}}\right) = 3\,\frac{kV}{mm}\left(40 + \frac{10}{3}\right) mm$$

$$\boxed{(V_0)_{max} = 130\ kV}$$

P. 3-25 Assume that the $z=0$ plane separates two lossless dielectric regions with $\epsilon_{r1}=2$ and $\epsilon_{r2}=3$. If we know that $\vec{E}_1$ in region (1) is $\hat{x}2y-\hat{y}3x+\hat{z}(5+z)$, what do we also know about $\vec{E}_2$ and $\vec{D}_2$ in region 2? Can we determine $\vec{E}_2$ and $\vec{D}_2$ at any point in region 2? Explain.



Assume $\rho_s=0$

Use: $E_{t1}=E_{t2}$

$\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=0$ at $z=0$

$D=\epsilon E$

$E_{t1}=E_{t2} \longrightarrow E_{x2}=E_{x1}=2y,\; E_{y2}=E_{y1}=-3x$

$\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=0 \longrightarrow \hat{z}\cdot(\epsilon_2\vec{E}_2-\epsilon_1\vec{E}_1)=0,\; \epsilon_{r2}E_{z2}=\epsilon_{r1}E_{z1}$

or $E_{z2}=\dfrac{\epsilon_{r1}}{\epsilon_{r2}}E_{z1}=\dfrac{2}{3}(5)$

$$\boxed{\begin{aligned}\vec{E}_2&=\hat{x}2y-\hat{y}3x+\hat{z}\tfrac{10}{3}\\ \vec{D}_2&=\epsilon_2\vec{E}_2=3\epsilon_0\left(\hat{x}2y-\hat{y}3x+\hat{z}\tfrac{10}{3}\right)\end{aligned}}$$ ... at $z=0$

$(\vec{E}_2,\vec{D}_2)$ unknown for $z>0$, because boundary conditions only relate surface values.

$$V_0 = -\int_0^d E_z\,dz = -c_1\int_0^{d_1} dz - c_2\int_{d_1}^{d_1+d_2} dz$$

$$= -c_1 d_1 - c_2 d_2$$

$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \rho_s \overset{=0}{} \rightarrow \hat{z}\cdot(\epsilon_p \vec{E}_2 - \epsilon_0 E_1) = 0 \ldots$$ *at interface between plexiglass and air*

↑ B.C. for normal $\vec{D}$

$$\epsilon_p E_{z2} = \epsilon_0 E_{z1} \longrightarrow \epsilon_p c_2 = \epsilon_0 c_1$$

$$c_2 = \frac{\epsilon_0}{\epsilon_p} c_1 = \frac{c_1}{\epsilon_{rp}}$$

$$V_0 = -\left(d_1 + \frac{d_2}{\epsilon_{rp}}\right)c_1 \rightarrow E_{z1} = -\frac{V_0}{d_1 + \frac{d_2}{\epsilon_{rp}}}$$

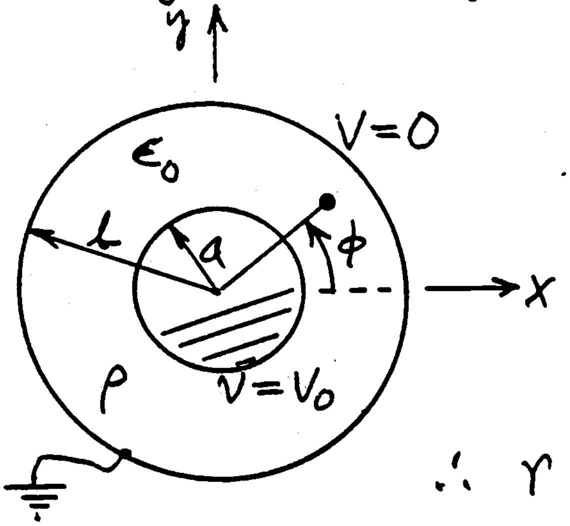
note: $E_{z2} = \frac{E_{z1}}{\epsilon_{rp}} < E_{z1}$ and $E_{bp} > E_{ba}$

$\therefore$ air region breaks down first and

$$(V_0)_{max} = (E_b)_{air}\left(d_1 + \frac{d_2}{\epsilon_{rp}}\right) = 3\,\frac{kV}{mm}\left(40 + \frac{10}{3}\right) mm$$

$$\boxed{(V_0)_{max} = 130\ kV}$$

P.4-6 Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $a<r<b$, where a and b are the radii of the inner and outer conductors, respectively. The inner conductor is maintained at a potential V_0, and the outer conductor is grounded. Determine the potential distribution in the region $a<r<b$ by solving Poisson's equation.



$V(r,\phi,z) = V(r)$ ... by cylindrical symmetry

$\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ ... Poisson's eq.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{A}{\epsilon_0 r}$$

$$\therefore\ r\frac{\partial V}{\partial r} = -\frac{A}{\epsilon_0}r + c_1$$

$$\frac{\partial V}{\partial r} = -\frac{A}{\epsilon_0} + \frac{c_1}{r} \longrightarrow V(r) = -\frac{A}{\epsilon_0}r + c_1\ln r + c_2$$

$$V(r=b) = 0 \longrightarrow c_2 = \frac{Ab}{\epsilon_0} - c_1\ln b$$

$$V(r) = -\frac{A}{\epsilon_0}(r-b) + c_1\ln\left(\frac{r}{b}\right)$$

$$V(r=a) = V_0 \longrightarrow \frac{A(b-a)}{\epsilon_0} + c_1\ln\left(\frac{a}{b}\right) = V_0$$

$$c_1 = \frac{V_0 - A(b-a)/\epsilon_0}{\ln(a/b)} = \frac{A(b-a)/\epsilon_0 - V_0}{\ln(b/a)}$$

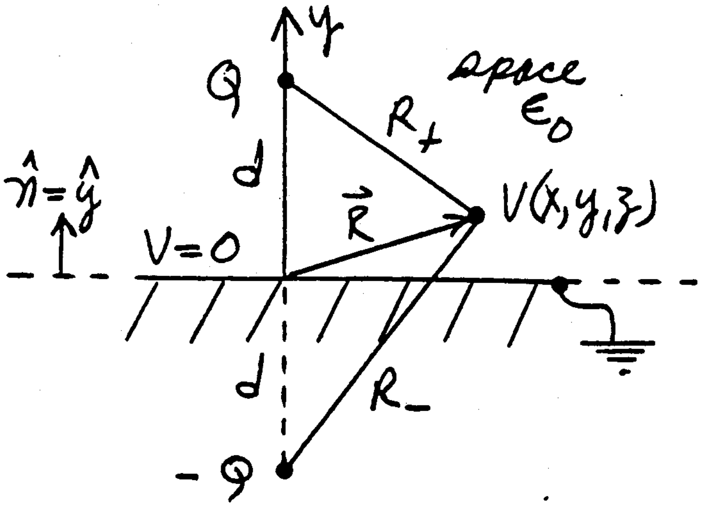
$$c_2 = \frac{Ab}{\epsilon_0} - c_1 \ln b = \frac{A/\epsilon_0\left[b\ln(b/a) - (b-a)\ln b\right] + V_0 \ln b}{\ln(b/a)}$$

$$= \frac{V_0 \ln b + A/\epsilon_0\,(a\ln b - b\ln a)}{\ln(b/a)}$$

$$\boxed{\begin{aligned} V(r) &= -\frac{A}{\epsilon_0} r + c_1 \ln r + c_2 \\ c_1 &= \frac{A(b-a) - V_0}{\ln(b/a)} \\ c_2 &= \frac{V_0 \ln b + A/\epsilon_0\,(a\ln b - b\ln a)}{\ln(b/a)} \end{aligned}}$$

P.4-7 A point charge exists at a distance d above a large grounded conducting plane. Determine

a) the surface charge density ρ_s,



$$V(R)=\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)$$

$$R_+=\sqrt{x^2+(y-d)^2+z^2}$$

$$R_-=\sqrt{x^2+(y+d)^2+z^2}$$

$$\rho_s=\epsilon_0(\hat{n}\cdot\vec{E})=-\epsilon_0(\hat{y}\cdot\nabla V)=-\epsilon_0\left.\frac{\partial V}{\partial y}\right|_{y=0}$$ ... at conductor surface

$$\rho_s=-\epsilon_0\frac{Q}{4\pi\epsilon_0}\frac{\partial}{\partial y}\left(R_+^{-1}-R_-^{-1}\right)\Big|_{y=0}$$

$$=-\frac{Q}{4\pi}\left[-R_+^{-2}\frac{\partial R_+}{\partial y}+R_-^{-2}\frac{\partial R_-}{\partial y}\right]\Big|_{y=0}$$

$$=-\frac{Q}{4\pi}\left[-\frac{1}{R_+^2}\frac{(y-d)}{R_+}+\frac{1}{R_-^2}\frac{(y+d)}{R_-}\right]\Big|_{y=0}$$

$$=-\frac{Q}{4\pi}\left[\frac{(y+d)}{R_-^3}-\frac{(y-d)}{R_+^3}\right]\Big|_{y=0}$$

$$\boxed{\rho_s=-\frac{2Qd}{4\pi(x^2+d^2+z^2)^{3/2}}}$$

b) the total charge induced on the conducting plane.

$$Q_{ind} = \int_S \rho_s \, ds = \iint_{-\infty}^{\infty} \rho_s(x,z)\, dx\, dz$$

Change to polar coordinates in the x-z plane, i.e., let $r^2 = x^2 + z^2$ and $ds = dx\,dy = r\,dr\,d\theta$, then

$$Q_{ind} = \int_{-\pi}^{\pi} \int_0^{\infty} \rho_s(r)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^{\infty} \frac{-Qd}{2\pi (r^2+d^2)^{3/2}}\, r\, dr$$

$$= -Qd \left. \frac{-1}{\sqrt{r^2+d^2}} \right|_{r=0}^{r=\infty} = -Q$$

$$\therefore \quad \boxed{Q_{ind} = -Q}$$