1) R3-1

Write the differential form of the fundamental postulates in free space:

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \]

\[ \nabla \times \vec{E} = 0 \]

2) R3-2

Under what conditions will the electric field be solenoidal & irrotational?

For static fields, \( \nabla \times \vec{E} = 0 \)
For a source-free region, \( \nabla \cdot \vec{E} = 0 \)
So if the "region under test" is source-free and contains only static fields,

\[ \nabla \times \vec{E} = 0 \quad \text{(irrotational)} \]

\[ \nabla \cdot \vec{E} = 0 \quad \text{(solenoidal)} \]

An example would be a region of space where a static-field passes through that region, but the region itself contained no sources.

\[ \text{COPY A} \]
Write the integral form of the fundamental postulates of E-statics in free space, and state their meaning in words.

\[ \Phi E \cdot ds = \frac{q}{\varepsilon_0} \quad (1) \]

"The surface-integral of E over any closed surface is equal to the total charge enclosed divided by the permittivity of the medium."

This is also Gauss's law.

\[ \Phi E \cdot dl = 0 \quad (2) \]

"The line integral of E over any closed path is zero." This is the basis of Kirchoff's circuit law (KVL), which says that the voltage drop over a closed path is zero. E is a conservative field!
The Mathematical Form of Coulomb's Law is:
\[ F_{ij} = \frac{q_i q_j}{4\pi\varepsilon_0 R^2} \]

which, in words, means that charges \( q_1 \) and \( q_2 \) experience force \( F_{ij} \) when placed in the vicinity of one another, and the magnitude of \( F_{ij} \) is inversely proportional to the distance \( R \) between the charges, is directly proportional to the value of the charges, and is directed from \( q_1 \) to \( q_2 \) or vice-versa.

State Gauss's Law:
\[ \Phi_E = q \text{ (already explained in Prob 3)} \]

It is useful when a symmetric surface with a constant, known normal vector \( \mathbf{\hat{n}} \) is specified around a charge distribution.

Ex: Spheres (The only applicable cases of Gauss's law):
- Cylinder of charge, normal = \( \mathbf{\hat{z}} \)
- Sphere of charge, normal = \( \mathbf{\hat{R}} \)
- Infinite sheet of charge, normal = \( \mathbf{\hat{k}} \)
Find the force between a charged circular loop of radius \( b \) and charge density \( \rho_c \) at a point charge \( q \) on the loop axis at a distance \( h \) from the plane of the loop. What is \( \vec{F} \) when \( h \gg b \) and \( h = 0 \)?

\[
\vec{F}_{\infty} = \frac{q}{
\begin{array}
\frac{\hat{R}}{h^2 + b^2} & \hat{h} - \hat{b} \\
\end{array}
\right] \\
\hat{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_c}{R^3} d\ell'
\]

\( \hat{R} = \hat{r} - \hat{r}' \) \( \hat{r} = \text{field point} = z \hat{h} \) \( \hat{r}' = \text{source point} = \hat{b} \)

\[
\hat{R} = \hat{r} - \hat{r}' = \hat{b}h - \hat{b}'b \\
|\hat{R}| = \sqrt{h^2 + b^2}
\]

We integrate this around a loop:

\[
d\ell' = dr' + r'd\phi' + dz' = dr' + b d\phi' + dz'
\]

\( dr' = 0 \to \text{constant loop} \)

\( dz' = 0 \to \text{loop is fixed at } z = 0 \)

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{b}h}{(h^2 + b^2)^{3/2}} d\phi'
\]

Remember: \( \hat{r}' \) is a variable unit base vector.

And can be parameterized as:

\[
\hat{r}' = \hat{x}' \cos \phi' + \hat{y}' \sin \phi'
\]

as \( \hat{r}' \) is integrated from \( \phi' = 0, 2\pi \), \( \hat{r}' \) goes to zero!

So

\[
\vec{E} = \frac{\rho_c}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{\hat{b}h}{(h^2 + b^2)^{3/2}} d\phi' = \frac{h b}{4\pi\varepsilon_0 (h^2 + b^2)^{3/2}} \int_0^{2\pi} d\phi'
\]

\( \rho_c = 1, 4 \)
\[ E = \frac{Q}{2} \frac{bh^3}{2\varepsilon_0 (h^2 + b^2)^{3/2}} \]

\[ \vec{F}_{12} = Q \vec{E} = \frac{Qab h Pe}{2} \frac{2abhPe}{\varepsilon_0 (h^2 + b^2)^{3/2}} \]

It is important to remember that \( d\vec{r}' \) in this case is a scalar quantity, since we are applying \( d\vec{r}' \) to a scalar quantity \( Pe \). Also, \( \vec{E} \) integrates to zero around a loop, because for every \( \vec{E} \) in one direction, there is another \( \vec{E} \) in the opposite direction that cancels it out.

When \( h \gg b \),

\[ \lim_{h \to \infty} \vec{F} = \frac{Qab h Pe}{\varepsilon_0 (h^2 + b^2)^{3/2}} = \frac{Qab h Pe}{\varepsilon_0 h^3} = \frac{Qab Pe}{\varepsilon_0 h^2} \]

When \( h = 0 \), \( \vec{F} = 0 \Rightarrow \) There are forces on all sides of equal magnitude.
Find \( \vec{E} \) at the center of a line charge of radius \( b \) semi-circular.

\[
\vec{E} = \frac{1}{4\pi \varepsilon_0} \int \frac{\vec{P}_x}{|\vec{r} - \vec{r}'|^3} \, d\vec{r}'
\]

\( \vec{r}' = \text{field point} = (0, 0, 0) \Rightarrow \text{center of loop} \)

\[
\vec{r} = b\hat{n}' = b [\hat{x}\cos\varphi' + \hat{y}\sin\varphi'] = \text{source point}
\]

\[
\vec{r} - \vec{r}' = -b [\hat{x}\cos\varphi' + \hat{y}\sin\varphi'] \quad |\vec{r} - \vec{r}'| = b
\]

\[
dl' = \rho d\varphi' + b \rho d\varphi' = b d\varphi'
\]

We now have a semi-circle, so we integrate.

From \( \varphi' = \pi \), \( \pi + \pi' \) we can choose any start angle, as long as the stop angle is \( \pi \) rads, beyond it.

\[
\vec{E} = \frac{\rho_e}{4\pi \varepsilon_0 b} \int_{\pi/2}^{3\pi/2} \frac{-b [\hat{x}\cos\varphi' + \hat{y}\sin\varphi']}{{b}^3} \, d\varphi'
\]

\[
= \frac{-\rho_e}{4\pi \varepsilon_0 b} \left[ \hat{x}\sin\varphi' - \hat{y}\cos\varphi' \right] \frac{3\pi/2}{\pi/2}
\]

\[
= \frac{-\rho_e}{4\pi \varepsilon_0 b} \left[ -2\hat{x} - \hat{y} \cdot 0 \right] = \frac{\rho_e \hat{x}}{2\pi \varepsilon_0 b}
\]

\( \text{P} \cdot \text{C} \)
For this problem, we could have chosen any coordinate system. Let's reflect on this result: the field is directed in the positive x-direction (remember how we defined our coordinate system).

This makes sense! Think about every point on the ring as an infinitesimally small point charge, and think about which directions don't cancel!
The uniform charge densities form an equilateral triangle, as below:

\[ P_1 = 2P_2 = 2P_3 \]
\[ P_2 = P_3 = \frac{1}{2} P_1 \]

For Fig. 1, the field at \( P \) is 0! (because everything is symmetric)

For Fig. 2,

\[ h = \frac{L \cdot \sin 60^\circ}{2} = \frac{\sqrt{3} L}{4} \]

\[ \vec{E} = \hat{\lambda} \frac{P_1}{2 \cdot 4 \pi \varepsilon_0} \int_{-L/2}^{L/2} \frac{h}{(x^2 + h^2)^{3/2}} \, dx \]

\[ = \hat{\lambda} \int_{-L/2}^{L/2} \left[ \frac{x}{(x^2 + h^2)^{3/2}} \right] \, dx \]

\[ = \frac{P_1}{2 \cdot 4 \pi \varepsilon_0 \sqrt{3} L} \]

\[ \vec{E} = \frac{P_1}{2 \cdot 4 \pi \varepsilon_0 \sqrt{3} L} \]

\[ \vec{r} = (0, h) \]
\[ \vec{r}' = (x', 0) \]
\[ |\vec{r}| = (x^2 + h^2)^{1/2} \]