An Evolutionary Based Bayesian Design Optimization Approach Under Incomplete Information

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Abstract

Design optimization in the absence of complete information about uncertain quantities has been recently gaining consideration, as expensive repetitive computation tasks are becoming tractable due to invent of faster and parallel computers. This work uses Bayesian inference to quantify design reliability when only sample measurements of the uncertain quantities are available. A generalized Bayesian Reliability Based Design Optimization (BRBDO) algorithm has been proposed and implemented for numerical as well as engineering design problems. The approach uses an evolutionary algorithm (EA) to obtain a trade-off front between design objective and reliability. The Bayesian approach provides a well-defined link between the amount of available information and the reliability through a *confidence* measure, and the EA acts as an efficient optimizer for a discrete and a multi-dimensional objective space. Additionally, a GPU-based parallelization study shows computational speed-up of close to 100 times in a simulated scenario wherein the constraint qualification checks may be time consuming and render a sequential implementation impractical for large sample sets. These results show promise for the use of a parallel implementation of EAs in handling design optimization problems under uncertainties.

1 Introduction

Deterministic design assumes that there is no uncertainty in the design variables and/or parameters and the optimal designs obtained from an optimizer can be successfully realized. These results typically lie on a constraint surface, or at the intersection of more than one constraint surfaces. They do not account for uncertainty in design variables due to manufacturing imperfections or tolerance limits. Thus, deterministic optimal results are usually unreliable because there exists inherent input and parameter variation that results in output variation.

Uncertainties can be classified into two major types [18]. *Aleatory* uncertainties are those due to unpredictable variability in the value of a quantity, and a characterization of this variability is usually available. *Epistemic* uncertainties are due to lack of information about the variability, so that such a variation can not be characterized explicitly. Input variation due to aleatory uncertainty is fully accounted for in Reliability-Based Design Optimization (RBDO). Probability distributions describe the stochastic nature of the design variables and model parameters. Variations are represented by standard deviations (typically assumed constant) and a mean performance measure is optimized subject to probabilistic constraints. RBDO has been extensively studied because it provides optimum designs in the presence of uncertainty [9, 11, 28, 16, 19].

RBDO does not, however, consider the fact that in actual design, much of the information regarding the uncertain quantities is available in the form of a set of finite samples. These samples are usually not enough to infer probability distributions, and in many cases there is no reason to assume that they would follow any standard distribution. Also, collecting more samples is often not possible due to cost or time constraints. For such situations, methods like Bayesian methods [18, 29], possibility-based methods [24, 30] and evidence-based methods [3, 2] have been suggested. Using such methods, it is possible to use samples or interval-based information to evaluate reliability. Extending our earlier preliminary study [26], in this work, we have implemented the Bayesian approach to reliability estimation using an evolutionary algorithm. We have then combined the Bayesian approach with an RBDO technique to yield a general Bayesian RBDO (BRBDO) algorithm, which is then applied to numerical and engineering design problems. Using an evolutionary algorithm, our results show how a decision making process can be facilitated for such a problem. For a scenario when such an analysis becomes too time-consuming due to large number of available samples or the complexity of the constraints, GPU-based parallelization has been shown to be a viable alternative and speed-ups of up to 100 times have been obtained for simulated cases.

In the remainder of the paper, in Section 2, the reliability based design optimization methodology is reviewed. Section 3 makes an overview of different design optimization techniques for handling uncertain decision variables and parameters. Thereafter, in Section 4, the Bayesian inference method is described. Section 5 presents a basic Bayesian approach implemented within an evolutionary algorithm. The algorithm is applied to a two-variable numerical optimization problem to show its working principle. Thereafter, in Section 6, the proposed Bayesian RBDO methodology is described in detail. The methodology is applied to two engineering design problems and results are described in detail. To reduce the computational complexity of the proposed procedure, the BRBDO procedure is parallelized using the emerging CUDA technology in Section 7. Results on engineering design problems show substantial savings in computational time. Finally, conclusions derived from this study are made in Section 8.

2 Reliability Based Design Optimization for Complete Information

In traditional reliability-based optimization, uncertainties are embodied as random design variables $\mathbf{X}$ and random design parameters $\mathbf{P}$, leading to the problem being formulated as [11, 20, 7]:

$$\begin{aligned} \underset{\mu_{\mathbf{X}}}{\text{minimize}} \quad & f\left(\mu_{\mathbf{X}}, \mu_{\mathbf{P}}\right), \\ \text{subject to:} \quad & Pr\left[g_j\left(\mathbf{X}, \mathbf{P}\right) \leq 0\right] \geq R_j,\ j = 1, \ldots, J. \end{aligned} \tag{1}$$

While the objective of the problem remains the same as the deterministic case – minimization of a function f – it must now be minimized with respect to the means (μ's) of the random variables given the standard deviation of the random parameters. This also changes the behavior of the constraints ($g_j(\mathbf{X}, \mathbf{P}) \leq 0$ for $j = 1, \ldots, J$), since the probability of design feasibility must now be constrained to be greater than or equal to R_j for the j-th constraint. The parameter R_j is the pre-specified target reliability for the j^{th} probabilistic constraint. A solution to a reliability-based optimization problem is called an optimal-reliable design. Evidently, the complication in this new formulation which takes into account the uncertainties is the calculation of so-called *chance* constraint function: $Pr\left[g_j\left(\mathbf{X}, \mathbf{P}\right) \leq 0\right]$. The probability that the constraint is satisfied under uncertainties now must be set at least equal to the specified reliability. The value of this probability can be expressed as follows:

$$Pr\left[g_j\left(\mathbf{X},\mathbf{P}\right) \leq 0\,\right] = 1 - \int_{g_j(\mathbf{X},\mathbf{P})>0} f_{\mathbf{Z}}\left(\mathbf{Z}\right) d\mathbf{Z}, \tag{2}$$

where $f_{\mathbf{Z}}$ is the joint probability density function of $\mathbf{Z} = (\mathbf{X}, \mathbf{P})$. Since it is usually impossible to find an analytical expression for above integral, several approximate and alternative techniques are proposed to incorporate reliability consideration in optimization procedures [9, 11, 13]. These techniques can be broadly classified as follows:

- **Simulation methods:** Using techniques such as Monte Carlo Simulations (MCS) for reliability was discussed as early as [5]. In such a method, the probabilistic constraint would be converted to a deterministic one, expressed as a ratio of successful simulations to total number of simulations being greater than the desired reliability value. However, the use of alternative techniques was sought since a large number of simulations would be required for values of reliability close to one.

  Recently simulation techniques using meta-models have been proposed. However, using models tends to introduce variability in the reliability estimates. To counter this problem, techniques such as smoothing of indicator functions [27] have been proposed. Studies such as [8] and [4] have explored the issue of appropriate selection of points for the training of the meta-model, which is quite crucial for the accuracy of the model.

- **Double loop methods:** As is evident from their name, double loop methods use an inner loop to find the reliability at a point by formulating and solving an optimization problem, and an outer loop to optimize the design considering the reliability as a constraint. Thus, it is a nested optimization task. Studies such as [23] used a reliability index in the inner loop lending the name RIA (Reliability Index Approach) to such methods. Other studies such as [20] used the Performance Measure Approach (PMA) which imposes an equality constraint on the performance of the design in the inner level optimization.

- **Single loop methods:** In [7], an approach to avoid the nested optimization task or double loop was suggested in which the probabilistic constraint would be replaced with a deterministic one defined using the constraint function and depending on the variables only. This removed the need for the inner level optimization. Similarly, it was proposed in [21] to replace the inner loop with an approximately optimum value of the variables and parameters evaluated at the mean design point using the desired reliability index. These methods are faster, but less accurate than the double loop methods.

- **Decoupled methods:** These approaches utilize the information extracted from the reliability analysis task in the outer optimization task thereby improving the numerical efficiency. Some of these methods utilize the sensitivity information of the reliability index to reach an approximate probability value such as [1]. Another approach that has become popular is the so-called Sequential Optimization and Reliability Assessment (SORA) method [13]. This replaces the double loop with a *dual* optimization task wherein the two optimizations are performed one after the other sequentially, each one benefiting from the other.

If the uncertainty in the variables and parameters can be confidently expressed as probability distributions (aleatory uncertainty), the RBDO formulation as described so far is sufficient

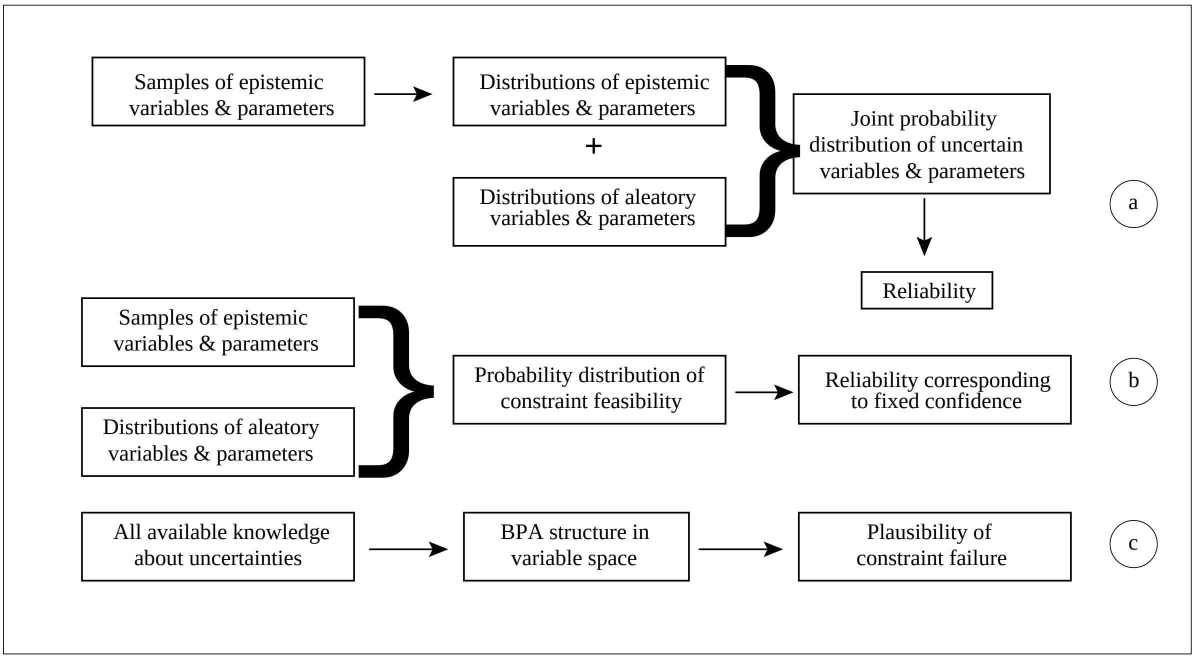

Figure 1: Various methods of handling incomplete information in RBDO.

for reliability analysis. However, it is often found that the uncertainty associated with the variables and parameters of a design optimization problem cannot be expressed as a probability distribution. The only information available might be a certain number of physical realizations of the variables, or expert opinions about the uncertainty. The above RBDO formulation cannot utilize this type of information and therefore, a different approach to uncertainty analysis is thus called for.

3 Handling Incomplete Information

As stated earlier, several RBDO techniques have been proposed and extensively investigated assuming the uncertainties to be aleatory – having a well-defined, exact probability distribution. In practice too, the general approach is either to ignore the lack of knowledge about epistemic uncertainties, or deal with them in an expensive but still inaccurate manner. The approach labeled 'a' in Figure 1 is generally adopted, wherein the samples of variables and parameters considered epistemic are fit to probability distributions so as to combine them with the aleatory ones, and facilitate a conventional RBDO approach (typically Monte Carlo simulations are performed).

The above approach is not preferable since it does not capture the aspect of incomplete information about the uncertainties effectively. A large number of samples are required to fit any probability distribution with confidence over the data, which is expensive. The uncertainty itself may not of the form of the general distributions assumed, and this approach will lead to misrepresentation of the uncertainty in such a case. Also, there is no clear relationship between the information available and the reliability of results obtained. For the above reasons, the methods labeled 'b' (Bayesian) and 'c' (evidence based) are better for handling epistemic uncertainties.

Bayesian, Possibility and Evidence based approaches have recently come up to address the issues discussed before. The Bayesian approach has been discussed and applied in [18] to obtain design having a fixed reliability and various levels of confidence in design. In [29], an approach for Bayesian reliability analysis with evolving and subjective data sets has been explored. Possibility approach was compared with the probabilistic approach in [24] and integrated with RBDO in [30]. Since the possibility approach is applicable to lower level of information than the Bayesian approach, it has been applied widely by several studies in the past decade. The use of evidence theory was suggested by [3] for uncertainty quantification for large scale structures. In [2], surrogate models were proposed to represent the measures of uncertainty in evidence theory as continuous functions, so that conventional sequential optimization techniques could be used. Various other studies ([22] being a good example) have dealt with epistemic uncertainty using evidence theory to incorporate the effects of epistemic uncertainties in design optimization.

The above techniques, however, have not been explored in an evolutionary context to the best of our knowledge. In [11], it was shown that evolutionary algorithms can provide very useful solutions in the form of well-distributed trade-off fronts between design objective and reliability for aleatory uncertainties. For a decision maker, such form of solutions are extremely useful, since he/she can now pick a solution by observing the over-all variation of the objectives with respect to each other. Due to this capability of performing multi-objective optimization and the non-requirement of gradients, evolutionary algorithms are well suited for handling epistemic uncertainties as well. Our preliminary study [26] showed an indication that EAs can be used efficiently in Bayesian design optimization studies. Our detailed study here and as we shall discuss in a later section, the ability of EAs to be used with a parallel computing platform makes EAs computationally efficient algorithms for optimization under uncertainties.

4 Bayesian Inference Method

In this section, we outline the Bayesian approach as suggested in [18]. For a Bernoulli sequence, the probability of having r successful trials out of N trials follows a Binomial distribution, where p is the probability of success in each trial. Thus,

$$Pr\left(r\,|\,N,p\right) = \binom{N}{r} p^r \left(1-p\right)^{N-r}. \tag{3}$$

When the probability of success p is unknown, its distribution can be calculated using Bayes' rule by a process known as Bayesian inference. Given r successes out of N trials and a prior distribution $Pr\left(r\,|\,p\right)$, the posterior distribution of p given by $Pr\left(p\,|\,r\right)$ can be calculated as

$$Pr\left(p\,|\,N,r\right) = \frac{Pr\left(p\right) \times Pr\left(r\,|\,p\right)}{\int_0^1 Pr\left(p\right) \times Pr\left(r\,|\,p\right) dp}. \tag{4}$$

When the trials are done for the first time, no previous information is available and $Pr\left(r\,|\,p\right) = U\left(0,1\right)$, a uniform distribution can be used. If a prior distribution is known from a previous set of trials, the same can be used to calculate the posterior distribution. Thus, the distribution of p can be *updated* as more and more trials are done and more information is available. For the uniform prior distribution, the posterior distribution is the following Beta distribution where $\alpha = r+1$ and $\beta = N-r+1$,

$$Pr\left(p\,|\,N,r\right) = Beta(p;\alpha,\beta) = \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} p^{\alpha-1}\left(1-p\right)^{\beta-1}. \tag{5}$$

For a design optimization problem, we can partition the random variables and parameters in our formulation into two vectors: $\mathbf{X} = [\mathbf{X_t}, \mathbf{X_s}]$ and $\mathbf{P} = [\mathbf{P_t}, \mathbf{P_s}]$. The vectors $\mathbf{X_t}$ and $\mathbf{P_t}$ are those random (aleatory) variables and parameters whose probability density functions (pdf's) are known. The vectors $\mathbf{X_s}$ and $\mathbf{P_s}$, on the other hand, are those random (epistemic) variables and parameters whose pdf's are not known, and instead, only some samples are available.

For a probabilistic constraint $Pr\left[\, g_j\left(\mathbf{X},\mathbf{P}\right) \leq 0 \,\right] = P_{g_j}\left(0\right)$, we can obtain a distribution estimate of $P_{g_j}\left(0\right)$ using $E_j(r)$, the expected number of feasible realizations of the design given by

$$E_j(r) = \sum_{k=1}^{N} Pr\left[\, g_j\left(\mathbf{X_t},\mathbf{P_t}\right) \leq 0 \,|\, (\mathbf{X_s},\mathbf{P_s})_k \,\right]. \tag{6}$$

The posterior distribution estimate of $P_{g_j}(0) \in [0,1]$ is thus given by the Beta distribution as follows:

$$Pr[P_{g_j}(0)] = Beta\left(P_{g_j}(0); E_j\left(r\right)+1, N-E_j\left(r\right)+1\right). \tag{7}$$

For any design, the confidence of the design with respect to the j^{th} inequality constraint is defined to be the probability that it will meet or exceed the reliability target.

$$\zeta_j = Pr\left[P_{g_j}\left(0\right)|_{\mu_{\mathbf{X}}} \geq R_j\right]; \quad j = 1,\, \ldots,\, J. \tag{8}$$

In Figure 2, the area of the shaded region represents the confidence for a hypothetical case when the desired R_j is 0.90 and $E_j(r) = 21$ for a case of 25 samples. A $\zeta_j = 0$ means that the design is completely unreliable, while a $\zeta_j = 1$ means that the design meets or exceeds the target. The confidence given in Equation 8 can also be written as:

$$\zeta_j\left(\mu_{\mathbf{X}}\right) = 1 - \phi_{B_j}\left(R_j\right), \tag{9}$$

where $\phi_{B_j}\left(.\right)$ is the cumulative density function (cdf) of the j^{th} Beta distribution, for all $j = 1,\, \ldots,\, J$.

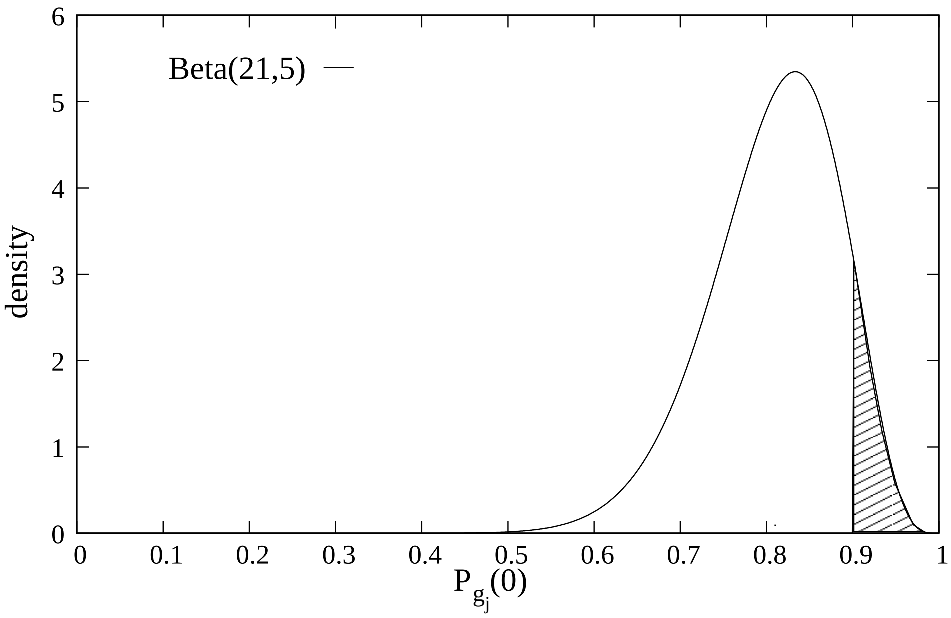

Figure 2: Obtaining confidence (shaded region) from the distribution of $P_{g_j}(0)$.

The ζ_j's can be lumped into $\zeta_s\left(\mu_{\mathbf{X}}\right)$, a quantity called the overall confidence of a design. Here, for simplicity, we use the minimum of all ζ_j as ζ_s. Using this measure the multi-objective problem becomes:

$$\begin{aligned} \underset{\mu_{\mathbf{X}}}{\text{minimize}} \quad & f\left(\mu_{\mathbf{X}}, \mu_{\mathbf{P}}\right), \\ \underset{\mu_{\mathbf{X}}}{\text{maximize}} \quad & \zeta_s\left(\mu_{\mathbf{X}}\right), \\ \text{subject to:} \quad & 0 \leq \zeta_s\left(\mu_{\mathbf{X}}\right) \leq 1. \end{aligned} \tag{10}$$

It has been shown elsewhere [17] that the maximum confidence is related to the number of samples and the reliability desired by the equation:

$$\zeta_s^{max} = 1 - R^{N+1}. \tag{11}$$

This is because the maximum value of $E_j(r)$ for any set of samples is N, which corresponds to the rightmost distribution of $P_{g_j}(0)$ given by $Beta(P_{g_j}(0); N+1, 1)$. The maximum confidence for a given reliability will be equal to the confidence obtained from this rightmost distribution.

5 A Basic Evolutionary Algorithm Based on Bayesian Approach

In this section, the above basic Bayesian approach is implemented with a multi-objective EA, namely, the elitist non-dominated sorting GA or NSGA-II [12]. Following the study in [18], the two-objective optimization problems described in equation 10 is solved using NSGA-II and the entire trade-off frontier is targeted.

As a proof-of-principle study, the following problem borrowed from an earlier study [17] is chosen:

$$\begin{aligned} \text{minimize} \quad & f(X_1, X_2) = X_1 + X_2, \\ \text{subject to:} \quad & g_1(X_1, X_2) \equiv 1 - \frac{X_1^2 X_2}{20} \leq 0, \\ & g_2(X_1, X_2) \equiv 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120} \leq 0, \\ & g_3(X_1, X_2) \equiv 1 - \frac{80}{X_1^2 + 8X_2 - 5} \leq 0. \end{aligned} \tag{12}$$

In this problem, the complete information about the uncertainty in variable X_1 is not known and a few samples is considered to be available at every point. Thus, this variable is an epistemic variable. To supply the samples, we use the following artificial probability distribution: $Pr_{X_1} = \dfrac{e^{-5X_1^4+1.5X_1^2+0.5X_1}}{1.614}$. The variable X_2 is also considered uncertain (aleatory), but is expected to follow a Beta distribution as follows: $Beta(X_2; 1.5, 5)$. These pdfs at other mean values are obtained by shifting the distributions accordingly.

The equivalent stochastic problem with chance constraints becomes as follows:

$$\begin{aligned} \underset{\mu_{\mathbf{X}}}{\text{minimize}} \quad & f(\mu_{\mathbf{X}}) = \mu_{X_1} + \mu_{X_2}, \\ \text{subject to:} \quad & Pr\left[g_1 : 1 - \frac{X_1^2 X_2}{20} \leq 0 \right] \geq R_1, \\ & Pr\left[g_2 : 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120} \leq 0 \right] \geq R_2, \\ & Pr\left[g_3 : 1 - \frac{80}{X_1^2 + 8X_2 - 5} \leq 0 \right] \geq R_3. \end{aligned} \tag{13}$$

The constraint-wise reliabilities of $R_1 = R_2 = R_3 = 0.95$ are specified.

To evaluate the second objective (minimum confidence level) using Equation 8, first, we need to compute the $E_j(r)$ value using Equation 6. For a given solution ($\bar{X}_1$,$\bar{X}_2$), the probability that $g_j \leq 0$ can be computed for a fixed sample value of X_1^k by using the known Beta distribution of X_2. For example, for the first constraint, the feasibility condition yields: $X_2 \geq 20/{X_1^k}^2$. The mean value of $Beta(x; 1.5, 5)$ distribution occurs at $x = 0.230769$. Thus, at a given $\bar{X}_2$, the upper limit of X_2 variation is $\bar{X}_2 + 1 - 0.230769$ or $\bar{X}_2 + 0.769231$. The probability that above condition is true can be computed as follows:

$$Pr\left[g_1(X_2) \leq 0 \,|\, {X_1}^k \right] = \int_{20/{X_1^k}^2}^{\bar{X}_2+0.769231} Beta(X_2 - (\bar{X}_2 - 0.230769); 1.5, 5) dX_2.$$

The above probability is illustrated in Figure 3 by the shaded region. The variable X_1 is uncertain and known with a set of samples around $\bar{X}_1$ and the variable X_2 is uncertain with a known Beta probability distribution.

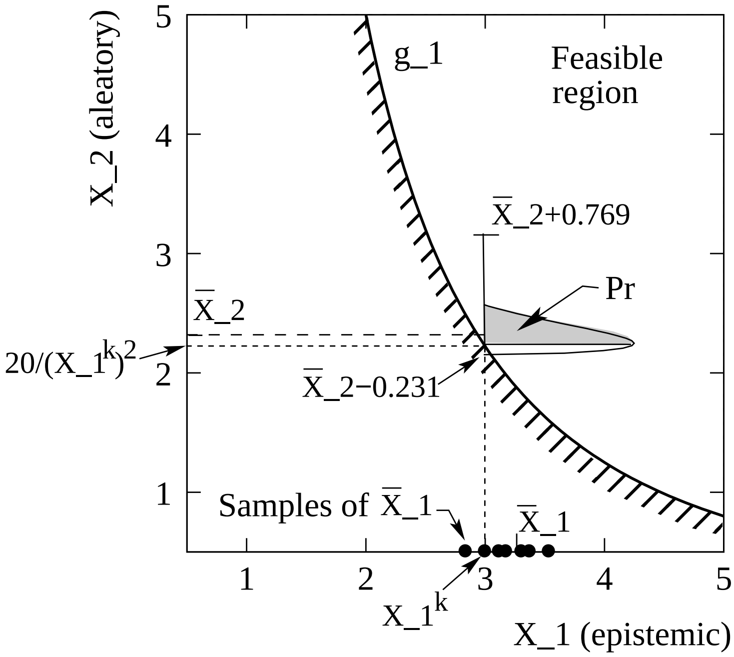

Figure 3: Computation of $Pr\left[g_1(X_2) \le 0 \,|\, {X_1}^k\right]$ for g_1 constraint on a hypothetical $(\bar{X}_1, \bar{X}_2)$ solution for a particular sample of $\bar{X}_1$ $(= \bar{X}_1^k)$ is illustrated.

The above computation can be repeated for all samples for X_1 and the the above probability terms can be added to computer $E_1(r)$. Thereafter, the probability distribution of $P_{g_1}(0)$ can computed by using Equation 7 and finally ζ_1 can be computed using Equation 8. It is worth mentioning here that the above procedure for computing $E_j(r)$ cannot generalized in a computationally tractable manner for a larger number of epistemic and aleatory variables. We shall suggest a generic procedure for this purpose in Section 6. But for the above two-variable problem, the above simplistic approach provides a easier understanding of the Bayesian inference procedure.

Thereafter, the above procedure can be repeated for second and third constraints and respective ζ_2 and ζ_3 values can be computed. The minimum of these confidence values will then be used as the second objective function value.

Figure 4 shows the results obtained when 25 and 135 samples of X_1 are used in two different runs. The evolutionary algorithm results in a trade-off front for each case. As evident from the figure, the front represents the trade-off between objective function value and the confidence in the design at that design point. For a small confidence requirement, a small objective value is obtained, whereas to achieve a solution with a high confidence value, a compromise on the objective value is needed. Also for a fixed confidence value, a higher number of samples allows a smaller objective value to be found, as a large number of samples provide a more accurate idea of the underlying uncertainty in X_1 and the NSGA-II can exploit this aspect to provide a better optimal solution. As Equation 11 predicts, it is possible to have almost 100% confidence at some design points using a larger number (135) of samples for the given reliability target.

The figure also shows the bi-objective trade-off fronts obtained by another study [17] on the same problem, but obtained with a different set of samples. It is interesting to note that

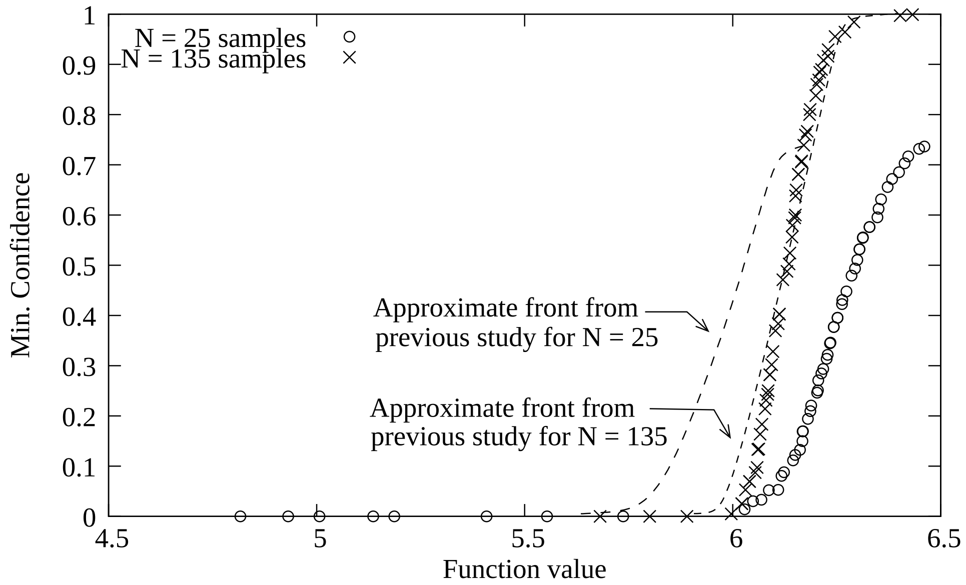

Figure 4: Trade-off of confidence level vs. objective function value for fixed reliability (95%) for all constraints (two variable numerical problem). Results are compared with that of a previous study [17].

when a small sample set (25) is used, the obtained front largely depends on the exact samples used to compute the confidence value. However, when a large enough sample set is used (135, for example), the fronts by our proposed approach and the previous approach are more or less the same and the exact location of the samples does not matter much.

5.1 Need for a Practical Approach

In the above section, the Bayesian approach is used to obtain designs with varying levels of confidence for a particular desired reliability. In practice, however, designers will be interested in knowing how optimal solutions would change with different reliability values, while fixing the confidence level to a high value. Thus, to make the approach more pragmatic, we propose an alternative approach wherein, we ask the designer to fix the confidence level he or she desires in the design, and perform a multi-objective optimization to yield a trade-off front between the objective function value and the maximum reliability that can be achieved with the given confidence level. This will allow the designer to make a more practical decision when choosing a design. Most designers will prefer a high confidence value (depending on the number of samples, but limited by the maximum reliability they seek, according to (11)) and would prefer to see how much they will need to sacrifice on the objective function value in order to have a more reliable design.

Such a multi-objective optimization task can be readily performed using an evolutionary algorithm like NSGA-II. For this purpose, the desired ζ_j is used to obtain R_j for the obtained $P_{g_j}(0)$ distribution, which is the value of R_j for which the cumulative density of $P_{g_j}(0)$ becomes equal to $(1-\zeta_j)$. We then define $R_s = \min.\ R_j$ for $j = 1, \ldots, J$, using which the optimization problem now becomes:

$$
\begin{aligned}
\underset{\mu_{\mathbf{X}}}{\text{minimize}} \quad & f\left(\mu_{\mathbf{X}}, \mu_{\mathbf{P}}\right), \\
\underset{\mu_{\mathbf{X}}}{\text{maximize}} \quad & R_s\left(\mu_{\mathbf{X}}\right), \\
\text{subject to:} \quad & g_j\left(\mathbf{X}, \mathbf{P}\right) \le 0, \quad j = 1, \ldots, J.
\end{aligned} \tag{14}
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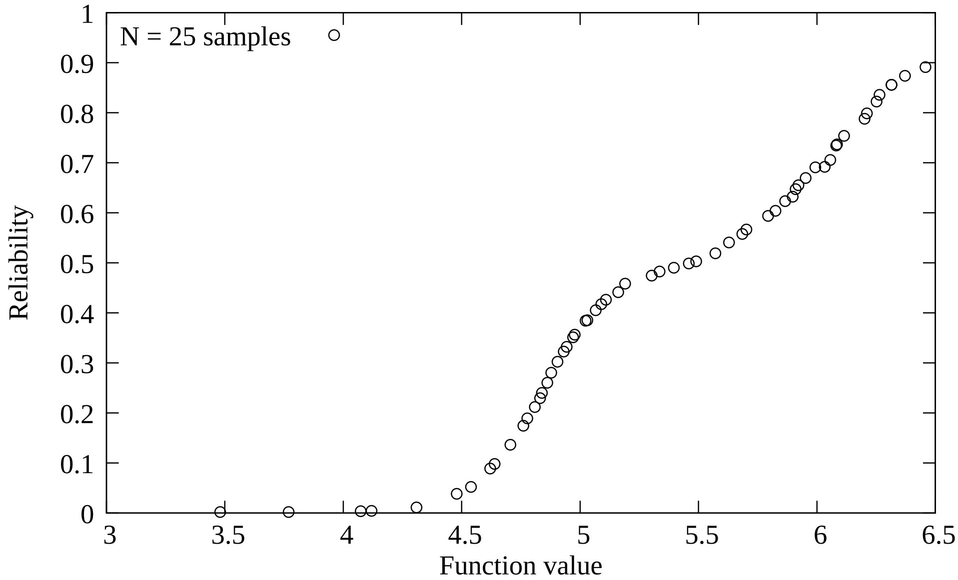

Figure 5: Bayesian reliability front for 95% confidence in each constraint and 25 samples of X_1 (two variable numerical problem).

Figure 5 shows a sample result of the same two-variable problem used in the previous section obtained using NSGA-II applied on a population size of 60 for 50 generations. The confidence level in this case is fixed at 95% and 25 samples of X_1 are used. The variable X_2 is still considered to be aleatory. The trade-off between objective function and reliability is clear from the figure. To achieve a larger reliability in a design, a sacrifice in the objective function value is called for. The use of NSGA-II not only enables us to find an entire gamut of trade-off solutions, but also provides us with the shape of the front that reveals interesting relationships of their variations.

To perform a more detailed analysis, it may be useful to compare the trade-offs between reliability and objective function value for various confidence levels. Such a comparison for the same two-variable problem is shown in Figure 6. It can be clearly seen that if one needs a higher confidence level to achieve the same reliability, one must sacrifice on the function value.

6 Proposed Bayesian RBDO algorithm

A general design optimization problem will involve several variables/parameters, some with aleatory uncertainty while others with epistemic uncertainty. Also, some constraints might involve variables with epistemic uncertainty while others may not. In such a situation, the computation of Equation 6 becomes a difficult task. To alleviate this difficulty, we propose the use of the Fast Reliability Index Approach (FastRIA [11]) to find the probability of feasibility of a constraint. FastRIA method is used to find a point on the constraint boundary which is closest to the solution, called the Most Probable Point (MPP [14]) of failure. Assuming linearity for the constraint in the vicinity of the MPP, a value for the reliability can be obtained which is a good approximation for non-linear constraints as well. This method has earlier been used for RBDO in [11] when the variables have aleatory uncertainty characterized by a normal distribution. For a sample $(X_s, P_s)_k$ for epistemic variables and parameters, the MPP point is found for each constraint by considering aleatory variables and parameters (X_t, P_t) in the FastRIA approach. The MPP point then provides the value of $Pr[g_j(\mathbf{X}) \leq 0|(X_s, P_s)_k]$ [11]. The remaining procedure of computing the probability distribution of $P_{g_j}(0)$ and then the

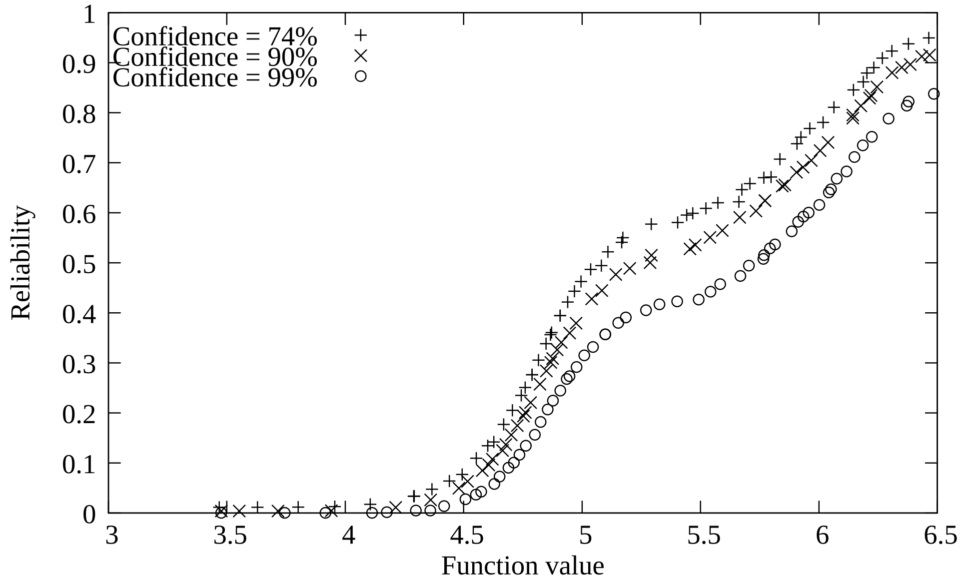

Figure 6: Comparison of Bayesian reliability fronts for different confidence values and 25 samples of X_1 (two variable numerical problem).

confidence level ζ_j or the reliability value R_j can be achieved as before. We describe the overall algorithm next.

6.1 Overall Algorithm

We now outline the overall algorithm for performing Bayesian based design optimization for a problem with several constraints and variables/parameters having different types of uncertainties. Algorithm 1 shows the steps to be followed. The outlined algorithm takes into account different types of constraints when some variables and parameters have aleatory uncertainty while others have epistemic uncertainty.

The outlined algorithm takes into account different types of constraints when some variables have aleatory uncertainty while others have epistemic uncertainty. When all the variables and parameters occurring in a constraint have aleatory uncertainty expressed as a probability distribution, the Bayesian approach is not required and traditional RBDO is used to find the reliability at the point. If some variables/parameters having epistemic uncertainty are involved, for which information is only available as sets of samples, we use the Bayesian approach to find the reliability with the specified confidence level. The minimum reliability is considered as an additional objective to be maximized in a bi-objective formulation, along with the original objective function. The result is obtained as a trade-off between objective function value and reliability when there is a single objective function. It is notable that due to lack of complete information, the reliability at each design point is with a confidence value greater than or equal to the minimum confidence level specified, depending on the type of variables involved in the closest constraint at that point.

We now illustrate the working of the above algorithm to a couple of engineering design problems.

6.2 Results on a Car Side-Impact Problem

The European enhanced vehicle-safety committee (EEVC) side-impact test procedure is a guideline used to test European vehicles for side impact protection. Effects of side impact on the

Algorithm 1: Proposed Bayesian RBDO algorithm.

foreach *constraint (say, j-th constraint)* **do**
  if *constraint involves variables with epistemic uncertainty* **then**
    foreach *sample set of* $(\mathbf{X_s}, \mathbf{P_s})_k$ **do**
      Use FastRIA to find $Pr\left[\, g_j\left(\mathbf{X_t}, \mathbf{P_t}\right) \leq 0 \,|\, (\mathbf{X_s}, \mathbf{P_s})_k \,\right]$;
    end
    Obtain $E_j(r)$ using Equation 6;
    Use $E_j(r)$ (sum of the reliabilities obtained) to Find the posterior distribution of $P_{g_j}(0)$ using Equation 7;
    Using the desired confidence value, calculate the reliability at the point using Equation 9;
  end
  else
    Use FastRIA to directly find the reliability at the point;
  end
end
Find minimum reliability $R_s\left(\mu_{\mathbf{X}}\right)$ over all constraints;
Perform a multi-objective optimization procedure with an additional goal to maximize $R_s\left(\mu_{\mathbf{X}}\right)$

dummy and the car are considered. The constraints related to the dummy's response include head injury criterion (HIC), abdomen load, pubic symphysis force, viscous criteria (VC), and rib deflections (upper, middle and lower). Car-related constraints pertain to the velocity of the B-pillar at the middle point (V_{MBP}) and the velocity of the front door at the B-pillar (V_{FD}). An increase in the dimensions of components may improve the dummy's performance on safety, but will increase the weight of the car and lower the fuel economy. The optimization problem [15] involves the minimization of the weight of the car subject to EEVC restrictions on safety performance. There are 11 design variables. For our study, we partitioned the 11-variable vector into three sets: variables with aleatory uncertainty $\mathbf{X_t} = \{X_1, X_2, X_3, X_4, X_6, X_7\}$, variable with epistemic uncertainty $\mathbf{X_s} = \{X_5\}$, and parameters with aleatory uncertainty $\mathbf{P_t} = \{X_8, X_9, X_{10}, X_{11}\}$. The variables/parameters with aleatory uncertainty are normally distributed about their means with standard deviations as given in Table 1, along with the description of variables and parameters and their upper and lower bounds. All quantities are in mm.

The parameters X_8, X_9, X_{10} and X_{11} are assumed to have a normal distribution with the given standard deviations and a fixed mean of 0.345, 0.192, 0 and 0 mm, respectively. Thus there are seven decision variables involved in the optimization task. The NLP formulation of the problem is as follows:

$$\begin{aligned}
\underset{\mathbf{X}}{\text{minimize}} \quad & f\left(\mathbf{X}\right) = \text{Weight} \\
\text{subject to:} \quad & g_1(\mathbf{X}) \equiv \text{Abdomen Load} \leq 1\text{kN}, \\
& g_2(\mathbf{X}) \equiv VC_u \leq 0.32\text{m/s}, \\
& g_3(\mathbf{X}) \equiv VC_m \leq 0.32\text{m/s}, \\
& g_4(\mathbf{X}) \equiv VC_l \leq 0.32\text{m/s}, \\
& g_5(\mathbf{X}) \equiv D_{ur} \text{ (upper rib deflection)} \leq 32\text{mm}, \\
& g_6(\mathbf{X}) \equiv D_{mr} \text{ (middle rib deflection)} \leq 32\text{mm}, \\
& g_7(\mathbf{X}) \equiv D_{lr} \text{ (lower rib deflection)} \leq 32\text{mm}, \\
& g_8(\mathbf{X}) \equiv F \text{ (Pubic force)} \leq 4\text{kN}, \\
& g_9(\mathbf{X}) \equiv V_{MBP} \leq 9.9\text{mm/ms}, \\
& g_{10}(\mathbf{X}) \equiv V_{FD} \leq 15.7\text{mm/ms}.
\end{aligned} \tag{15}$$

Table 1: Description of variables and parameters for the car side-impact problem.

Variable	Description	Uncertainty	Standard Deviation	Lower Bound	Upper bound
X_1	B-pillar inner	Aleatory	0.03	0.5	1.5
X_2	B-pillar reinforcement	Aleatory	0.03	0.45	1.35
X_3	Floor side inner	Aleatory	0.03	0.5	1.5
X_4	Cross member	Aleatory	0.03	0.5	1.5
X_5	Door beam	Epistemic	–	0.875	2.625
X_6	Door belt line	Aleatory	0.03	0.4	1.2
X_7	Roof rail	Aleatory	0.03	0.4	1.2
X_8	Material of B-pillar inner	Aleatory	0.006	–	–
X_9	Material of floor side inner	Aleatory	0.006	–	–
X_{10}	Barrier height	Aleatory	10	–	–
X_{11}	Barrier hitting position	Aleatory	10	–	–

The functional form of the objective function and the constraints are as follows:

$$
\begin{aligned}
f(\mathbf{X}) =& 1.98 + 4.9X_1 + 6.67X_2 + 6.98X_3 + 4.01X_4 + 1.78X_5 \\
&+ 0.00001X_6 + 2.73X_7, \\
g_1(\mathbf{X}) =& 1.16 - 0.3717X_2X_4 - 0.00931X_2X_{10} - 0.484X_3X_9 \\
&+ 0.01343X_6X_{10}, \\
g_2(\mathbf{X}) =& 0.261 - 0.0159X_1X_2 - 0.188X_1X_8 - 0.019X_2X_7 \\
&+ 0.0144X_3X_5 + 0.87570.001X_5X_{10} + 0.08045X_6X_9 \\
&+ 0.00139X_8X_{11} + 0.00001575X_{10}X_{11}, \\
g_3(\mathbf{X}) =& 0.214 + 0.00817X_5 - 0.131X_1X_8 - 0.0704X_1X_9 \\
&+ 0.03099X_2X_6 - 0.018X_2X_7 + 0.0208X_3X_8 + 0.121X_3X_9 \\
&- 0.00364X_5X_6 + 0.000771X_5X_{10} - 0.0005354X_6X_{10} \\
&+ 0.00121X_8X_{11} + 0.00184X_9X_{10} - 0.018X_2^2,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
g_4(\mathbf{X}) =& 0.74 - 0.61X_2 - 0.163X_3X_8 + 0.001232X_3X_{10} \\
&- 0.166X_7X_9 + 0.227X_2^2, \\
g_5(\mathbf{X}) =& 28.98 + 3.818X_3 - 4.2X_1X_2 + 0.0207X_5X_{10} \\
&+ 6.63X_6X_9 - 7.77X_7X_8 + 0.32X_9X_{10}, \\
g_6(\mathbf{X}) =& 33.86 + 2.95X_3 + 0.1792X_{10} - 5.057X_1X_2 \\
&- 11X_2X_8 - 0.0215X_5X_{10} - 9.98X_7X_8 + 22X_8X_9, \\
g_7(\mathbf{X}) =& 46.36 - 9.9X_2 - 12.9X_1X_8 + 0.1107X_3X_{10}, \\
g_8(\mathbf{X}) =& 4.72 - 0.5X_4 - 0.19X_2X_3 - 0.0122X_4X_{10} \\
&+ 0.009325X_6X_{10} + 0.000191X_{11}X_{11}, \\
g_9(\mathbf{X}) =& 10.58 - 0.674X_1X_2 - 1.95X_2X_8 + 0.02054X_3X_{10} \\
&- 0.0198X_4X_{10} + 0.028X_6X_{10}, \\
g_{10}(\mathbf{X}) =& 16.45 - 0.489X_3X_7 - 0.843X_5X_6 + 0.0432X_9X_{10} \\
&- 0.0556X_9X_{11} - 0.000786X_{11}^2.
\end{aligned}
\tag{17}
$$

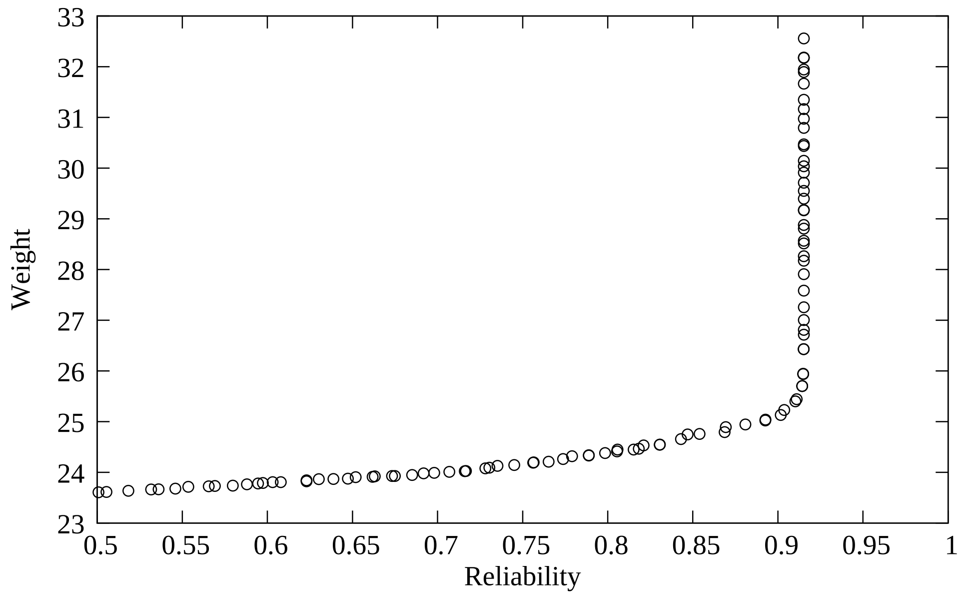

Figure 7: Weight versus reliability trade-off front using 25 samples of X_5 with 90% confidence (car side impact problem).

The Bayesian RBDO procedure is applied to the problem using NSGA-II for the optimization task. 150 generations are run for a population size of 100. A set of 25 samples of X_5 are generated from a normal distribution with a standard deviation of 0.03 mm. The results obtained show a trade-off between weight and reliability (shown in Figure 7) for a confidence value of 90%. For 25 samples, the maximum reliability that can be achieved with 90% confidence is 0.915247 which is also found to be the limit of the obtained trade-off front. A trade-off front having a sharp turn at an intermediate point is known to possess a 'knee' at the turned region. These problems are interesting from a decision-making point of view [10]. In such a front, usually a solution with the knee region is a preferred solution, as the trade-off for any point away from the knee region is usually not favorable. Investigating the knee region in this problem, we observe that despite having a wide range of trade-off between the weight and reliability, a solution near reliability value around 90% would be a preferred choice. We discuss this aspect more later.

Figure 8 shows the trade-off fronts obtained for different confidence levels when the number of samples of X_5 is fixed at 25. The same sample set is used to obtain the fronts. The front from a RBDO analysis is obtained using an earlier approach [11] and is also shown for comparison. It can be seen that as we try to have more confidence in the design for the same number of samples, we have to compromise more and more on the weight for the same reliability. The maximum reliability obtainable also decreases as the desired confidence level increases. The RBDO front is the one with complete information and thus is the best non-dominated front, while Bayesian analysis has less information due to epistemic uncertainty, leading to worse fronts than the RBDO fronts.

A comparison of fronts for varying number of samples of X_5 with a fixed level of confidence (90%) is done next and is depicted in Figure 9. The RBDO front is also shown for a comparison. comparison. In this case, we observe that the Bayesian RBDO front approaches the RBDO front as the number of samples increases. More samples add to the available information about the variable and the result approaches the case of complete information. It must be kept in mind that Bayesian RBDO results depend on the sample set (particularly when the sample size is small), and will change if the samples change, even if their number remains the same.

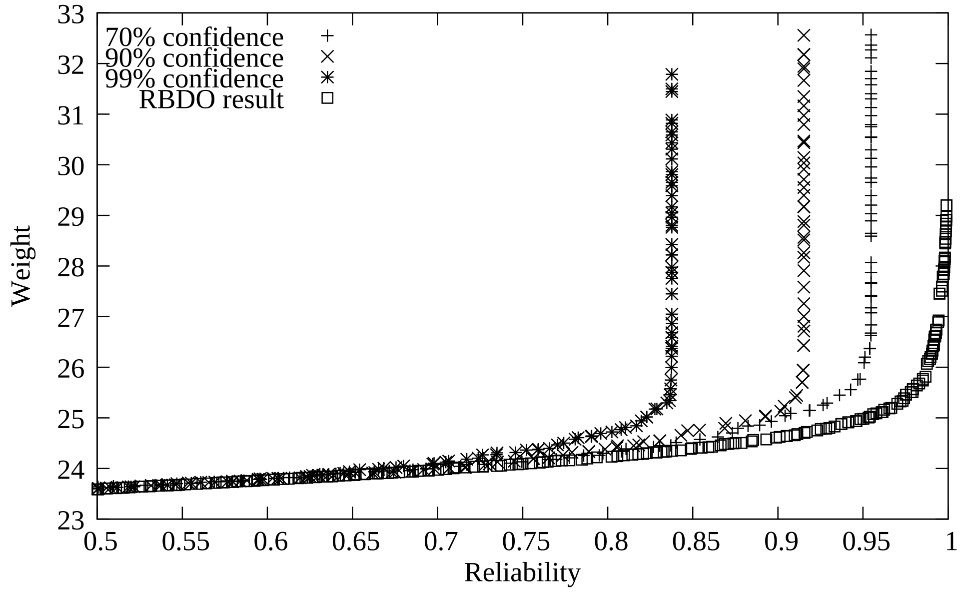

Figure 8: Trade-off fronts obtained using 25 samples of X_5 with different confidence levels (car side impact problem).

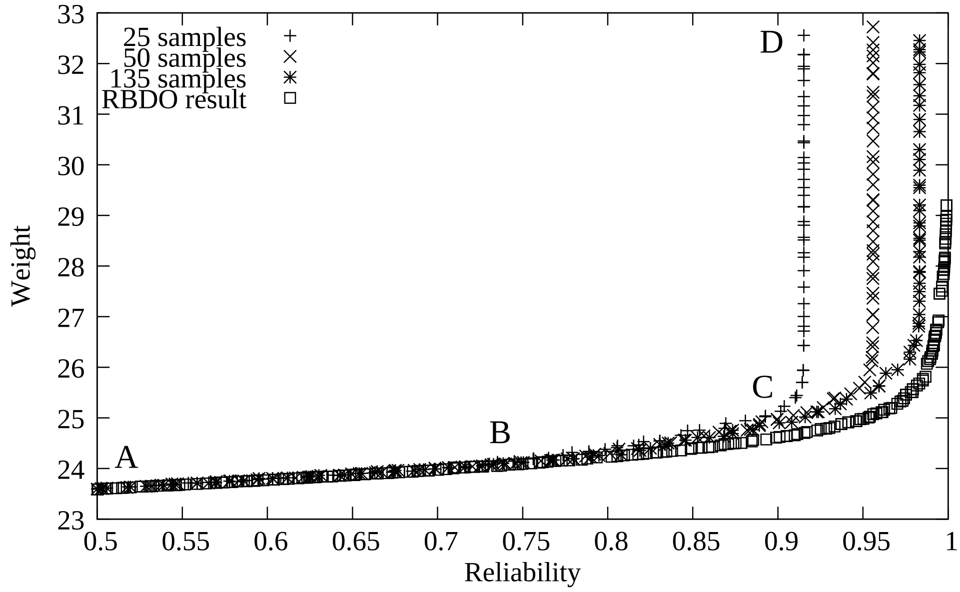

Figure 9: Trade-off fronts obtained using different number of samples of X_5 with 90% confidence (car side impact problem).

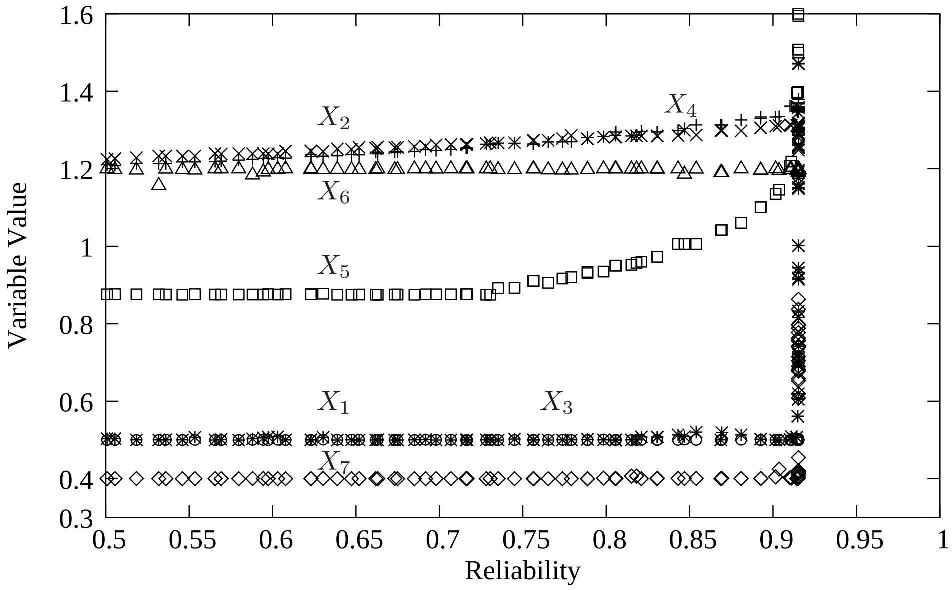

Figure 10: Variation of design variables for the car side impact problem (25 samples of X_5, 90% confidence).

This is a consequence of having epistemic uncertainty in the design variables. Recall that a similar observation was made in Figure 4 in comparing the fronts obtained by NSGA-II and an earlier approach [17]. The behavior of the fronts is also interesting in that points around C seem to be *preferred* points. To the right of C, one needs to make a large sacrifice on weight in order to gain a small value in reliability while towards the left of C, a small improvement in weight is obtained for a large compromise on reliability. Such points are called knee points [10]. A comparison of knee points in Tables 2 and 3 shows that although the fronts for different number of samples or different confidence levels change, the preferred solutions are quite close in terms of objective function value (weight). It is also notable that values of X_4 and X_5 are the only ones which differentiate the knee points of the Bayesian RBDO analysis from the RBDO analysis.

The variation of the variable values with reliability is shown in Figure 10. We also note in Figures 9 and 10 that in the region from A to B, only the variables X_2 and X_4 change, while B onwards, X_5 begins to contribute to the change in the front. The location of B is found to be a function of the number of samples of X_5 used. It is also interesting to note that although different reliability solutions are expected to depend on X_5 due to it being an epistemic variable in our problem formulation, it is now observed that the optimal trade-off between objective function and reliability also depends on the changes in X_2 and X_4. Higher the values in these two variables, better is the reliability of the design. Such a direct relationship of a few specific variables on the reliability of a design is interesting and remains as a valuable knowledge for the designers.

6.3 Car Side-Impact Problem with Three Epistemic Variables and Parameters

Next, for the same problem, we further examine the effect of epistemic uncertainties by considering the material property parameters X_8 and X_9 to be epistemic as well, besides X_5. Thus, there are three epistemic variables in the new case. Material properties in general are good

Table 2: Knee points for different confidence levels using 25 samples (car side impact problem). Only X_5 has epistemic uncertainty.

Conf.	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Weight	Reliability
70%	0.550	1.306	0.500	1.383	1.140	1.199	0.404	25.558	0.943
90%	0.500	1.310	0.500	1.334	1.135	1.199	0.405	25.129	0.902
99%	0.564	1.262	0.505	1.308	1.299	1.197	0.402	25.342	0.836
RBDO	0.509	1.341	0.500	1.406	0.875	1.199	0.400	25.197	0.967

Table 3: Knee points for different number of samples and with a confidence level of 90% (car side impact problem). Only X_5 has epistemic uncertainty.

Samples	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Weight	Reliability
25	0.500	1.309	0.500	1.334	1.135	1.199	0.405	25.130	0.902
50	0.500	1.333	0.500	1.388	1.178	1.196	0.405	25.724	0.944
135	0.518	1.348	0.500	1.477	1.198	1.199	0.404	26.159	0.978
RBDO	0.509	1.341	0.500	1.406	0.875	1.199	0.400	25.197	0.967

candidates for being considered as epistemic since they tend to be unpredictable in general depending on the quality of the material. The fronts for varying confidence levels and number of samples are shown in Figures 11 and 12, respectively. It can be seen that the overall behavior of the front does not change substantially, which encourages one to not require further sampling of the material properties. The knee points for both comparative studies are presented in Tables 4 and 5. It can be seen after a comparison of knee points for the same confidence level fronts that the added epistemic uncertainty makes the points with higher weight and lower reliability preferred. Similarly on comparison of knee points for fronts with different number of samples, it can be seen that for a particular sample set size the reliability for the preferred solution is lower due to the added epistemic uncertainties, although for larger number of samples, the weight corresponding to the preferred point is smaller.

Table 4: Knee points for different confidence levels using 25 samples (modified car side impact problem). X_5, X_8, and X_9 have epistemic uncertainty.

Conf.	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Weight	Reliability
70%	0.566	1.343	0.500	1.375	1.105	1.199	0.401	25.779	0.941
90%	0.553	1.349	0.500	1.324	1.086	1.196	0.404	25.527	0.895
99%	0.563	1.349	0.500	1.279	1.111	1.199	0.400	25.432	0.820
RBDO	0.509	1.341	0.500	1.406	0.875	1.199	0.400	25.197	0.967

6.4 Results on a Spring Design Problem

In order to demonstrate the approach on another problem with a little information about the uncertainty, we choose the problem of designing a helical spring for minimum volume. There are three decision variables ($\mathbf{X}$) in the problem formulation: the number of turns N, the wire diameter d, and the mean coil diameter D. For our analysis, we assume all variables to be continuous. The two parameters $\mathbf{P}$ – material properties G and S – are assumed to vary in a

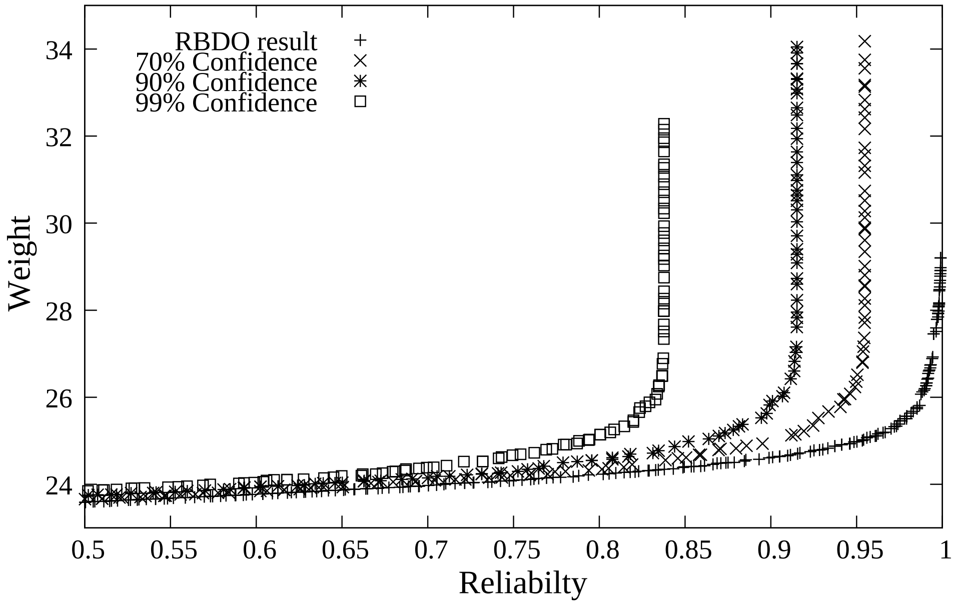

Figure 11: Trade-off fronts obtained using 25 samples of X_5, X_8 and X_9 with different confidence levels (modified car side impact problem).

Table 5: Knee points for different number of samples (modified car side impact problem) and with a confidence level of 90%. X_5, X_8, and X_9 have epistemic uncertainty.

Samples	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Weight	Reliability
25	0.553	1.349	0.500	1.324	1.086	1.196	0.404	25.527	0.895
50	0.516	1.348	0.500	1.349	1.077	1.199	0.400	25.409	0.921
135	0.555	1.349	0.500	1.423	1.125	1.199	0.405	26.008	0.966
RBDO	0.509	1.341	0.500	1.406	0.875	1.199	0.400	25.197	0.967

continuous manner. The deterministic formulation is as follows:

$$
\begin{aligned}
\underset{\mathbf{X}}{\text{minimize}} \quad & f(\mathbf{X}, \mathbf{P}) = 0.25\pi^2 d^2 D(N+2), \\
\text{subject to:} \quad & g_1(\mathbf{X}, \mathbf{P}) = l_{max} - \frac{P_{max}}{k} - 1.05(N+2)d \geq 0, \\
& g_2(\mathbf{X}, \mathbf{P}) = d - d_{min} \geq 0, \\
& g_3(\mathbf{X}, \mathbf{P}) = D_{max} - (d+D) \geq 0, \\
& g_4(\mathbf{X}, \mathbf{P}) = C - 3 \geq 0, \\
& g_5(\mathbf{X}, \mathbf{P}) = \delta_{pm} - \delta_p \geq 0, \\
& g_6(\mathbf{X}, \mathbf{P}) = (P_{max} - P)/k - \delta_w \geq 0, \\
& g_7(\mathbf{X}, \mathbf{P}) = S - \frac{8KP_{max}D}{\pi d^3} \geq 0, \\
& g_8(\mathbf{X}, \mathbf{P}) = V_{max} - 0.25\pi^2 d^2 D(N+2) \geq 0, \\
& 1 \leq N \leq 32, \quad 0.005 \leq d \leq 0.5, \quad 1 \leq D \leq 30,
\end{aligned} \tag{18}
$$

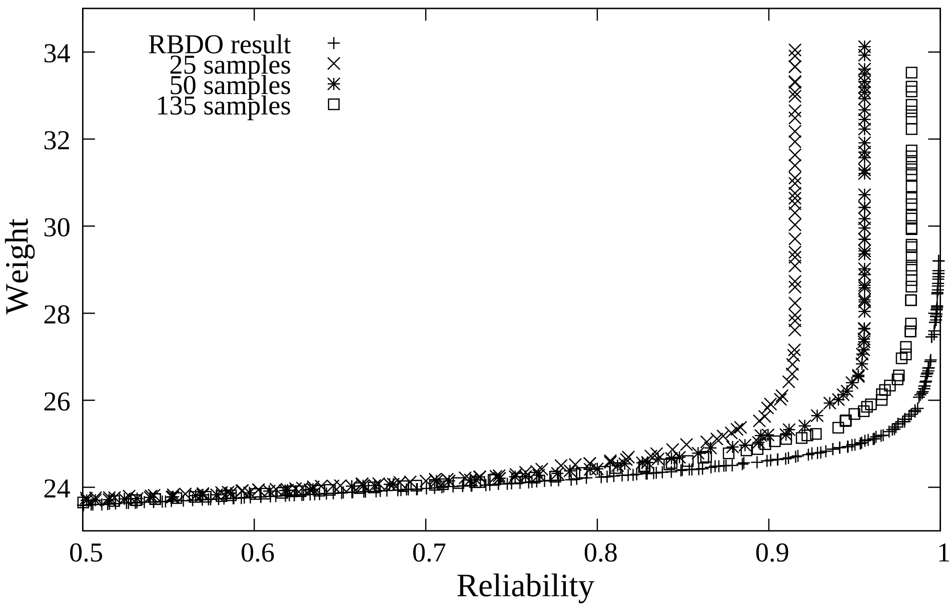

Figure 12: Trade-off fronts obtained using different number of samples of X_5, X_8 and X_9 with 90% confidence (modified car side impact problem).

with the defined parameters used above being:

$$\begin{array}{llll} K = \dfrac{4C-1}{4C-4} + \dfrac{0.615d}{D}, & P = 300\text{lb}, & D_{max} = 3\text{in}, \\ k = \dfrac{Gd^4}{8ND^3}, & P_{max} = 1000\text{lb}, & \delta_w = 1.25\text{in}, \\ \delta_p = P/k, & l_{max} = 14\text{in}, & \delta_{pm} = 6\text{in}, \\ S = 189000\text{psi}, & d_{min} = 0.2\text{in}, & C = D/d. \end{array} \tag{19}$$

As before, we partition the variables and parameters into variables with aleatory uncertainty $\mathbf{X_t} = \{N, D\}$, variable with epistemic uncertainty $\mathbf{X_s} = \{d\}$, and parameters with epistemic uncertainty $\mathbf{P_s} = \{G, S\}$. Thus, there are three epistemic variable and parameters for this problem. The variables N and D are normally distributed about their means with standard deviations equal to 0.1 and 0.2, and the mean values of G and S are 115×10^5 and 189×10^3, respectively. All quantities are expressed in FPS units.

The Bayesian RBDO procedure using NSGA-II is applied to this problem with a population size of 100 evolving over 150 generations. For our simulations, 25 samples are taken for the epistemic quantities d, G, and S from normal distributions given by $\mathcal{N}(0.1, 0.001)$, $\mathcal{N}(11500000, 500)$ and $\mathcal{N}(189000, 500)$, respectively. The resulting trade-off fronts obtained for different values of confidence level and various number of samples are shown in Figures 13 and 14, along with the RBDO front (assuming all quantities to be aleatory). In this problem, it can be seen that the epistemic nature of the uncertainties causes the fronts to be substantially away from the RBDO front. This is in contrast with the results for the car side impact problem in the previous section. As more and more variables become epistemic, appropriate information about true uncertainties is less and more conservative design choices are available. Unlike in the car side-impact problem, here, a sharp knee region is absent. However, we use the knee location procedures suggested in another study [10]. An analysis of knee points for different confidence levels (Table 6) and different sample sizes (Table 7) as in the car side impact study shows that N is the most sensitive variable in terms of affecting the preferable points as the level of confidence changes or more samples become available. Thus, it would be a worthy investment to exercise tighter control over realizing the intended value of N to ensure a high reliability in design. Such valuable information about importance of a particular variable in

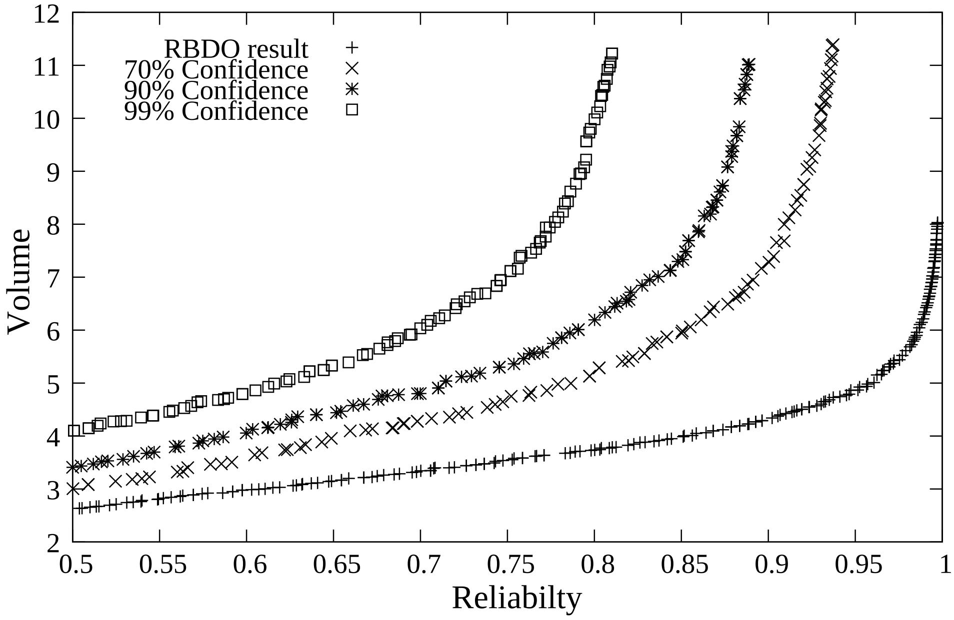

Figure 13: Trade-off fronts obtained using 25 samples of d, G and S for different confidence levels (spring design problem).

assuring a high reliability design is important to designers and remains as a hallmark aspect of this study.

Table 6: Knee points for different confidence levels using 25 samples (spring design problem).

Confidence	N	d	D	Volume	Reliability
70%	5.968	0.348	2.286	5.451	0.895
80%	5.848	0.348	2.289	5.376	0.871
90%	6.187	0.349	2.306	5.698	0.855
RBDO	5.236	0.349	2.295	5.008	0.961

7 Parallelization of BRBDO Approach

As stated before, one of the primary advantages of using Bayesian inference to directly obtain the probability of constraint feasibility is that a direct relationship between the amount of available information and the reliability is ensured. When a large number of samples for the uncertain design variables and parameters are available, an analysis without the need to fit available information to standard probability distributions would be preferred. However, for large sample sets and having a large number of non-linear constraints, a large number of feasibility checks can make the reliability calculation significantly computationally expensive for practical use. Since the constraint feasibility checks may require significant computations too, the additional overhead will further slow down the overall optimization. To counter this

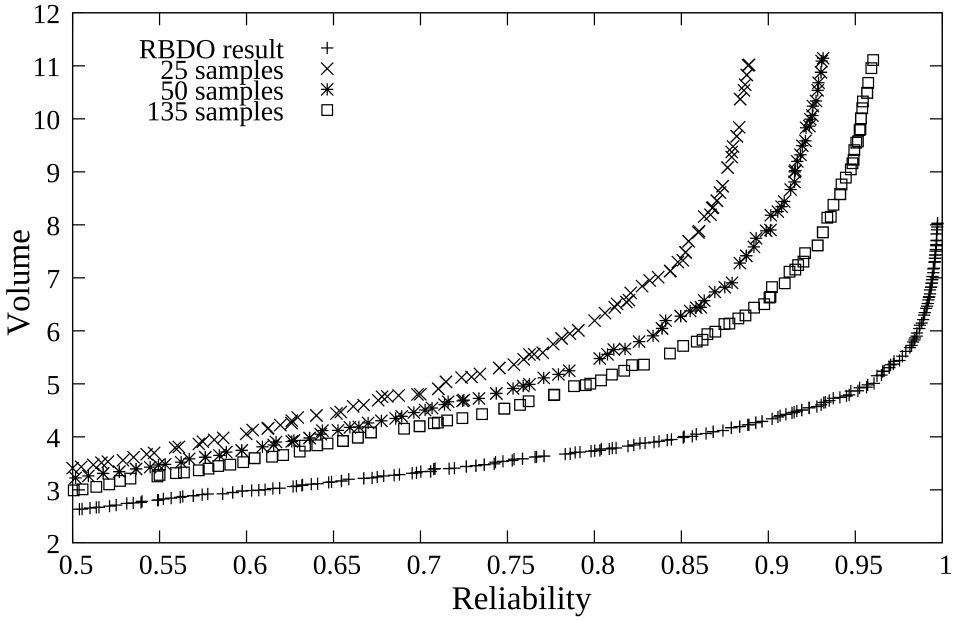

Figure 14: Trade-off fronts obtained using different number of samples of d, G and S for 90% confidence (spring design problem).

Table 7: Knee points for different number of samples and with a confidence level of 90% (spring design problem).

Samples	N	d	D	Volume	Reliability
25	6.187	0.349	2.306	5.698	0.855
50	7.221	0.365	2.276	6.906	0.879
135	6.596	0.365	2.302	6.504	0.898
RBDO	5.236	0.349	2.295	5.008	0.961

effect, we propose GPU (Graphical Processing Unit) based parallelization of the method in this paper.

7.1 GPU and CUDA Programming

In recent years, interest has steadily developed in the area of using GPUs for general-purpose computing (GPGPU). A graphics processing unit (GPU) is designed for doing image processing tasks facilitating the rendering of computer generated imagery. Since, the majority of image processing tasks are *data-parallel*[1], the GPUs have traditionally favored multi-threaded many-core architectures, whereby more transistors are devoted to ALUs (Arithmetic Logic Unit) compared to a CPU core at the expense of having fewer cache and control-flow units. Today, software developers are able to efficiently use the computational powers of GPUs with the help of easy to learn programming APIs, the most popular being Nvidia's CUDA [25].

The CUDA programming model has been specifically designed to allow control over the parallel processing capabilities of the GPUs for general purpose applications, and to take GPU computing to the mass. CUDA provides an API which extends the industry standard program-

[1]The same set of instructions are to be performed for different pieces of distributed data.

ming languages like C and Fortran, providing constructs for parallel *threads*, shared memory and barrier synchronization. This ensures that a CUDA programmer can start utilizing the parallel architecture readily after familiarization with these key abstractions without needing to understand the graphics rendering process of a standard Nvidia GPU.

To do computations on the GPU, the user first needs to copy the relevant data to the GPU device. The GPU device has its own DRAM (Dynamic Random Access Memory) for the storage of large data, required throughout the execution of a program. The user then needs to specify the layout and the number of threads to create, and invoke a GPU method (called as *kernel*). A *kernel* contains the instruction set which is to be simultaneously executed by all the threads, albeit depending on their unique indices. The layout specification includes defining a grid of thread-blocks, wherein each thread-block contains a given number of threads. The arrangement of threads in a thread-block and the arrangement of thread-blocks in a grid, both can be specified to be in a 1D, 2D or 3D lattice (with some restrictions), depending on which three-dimensional unique indices are computed.

During compilation, each thread is allotted the required number of 32-bit memory registers, depending on the *kernel* code. While different thread-blocks function independently, threads inside a thread-block can synchronize their execution and are also allowed to share data using a device-specific limited shared memory. Thus, the user is provided with flexibility of two levels of parallelization, which should be efficiently exploited. After computations, the result data needs to be copied back to the CPU memory to be used further. Detailed descriptions and illustrations of the CUDA architecture are available in [25].

7.2 Parallel Implementation of Proposed Algorithm

To parallelize the EA-based Bayesian RBDO algorithm, we distribute the population evaluation among the different thread blocks since the evaluation of each population member is independent of the others. The shared memory available to each thread-block is utilized to store the corresponding individual specific information. Since EAs use a population of solutions in each generation, the use of a parallel computing platform to implement an individual level parallelization becomes beneficial. This aspect also addresses a common criticism against EA's computational expense to evaluate a population of solutions vis-a-vis a single solution evaluation in point-by-point approaches.

A number of parallel threads can also be used for the evaluation of each individual – the constraint qualification checks for all the samples in our case, which are independent for each constraint and each sample. The threads must be synchronized before the reliability value is computed since all samples must be checked before the Beta distribution of constraint feasibility is considered. The computation of the reliability value from the Beta distribution is performed on the CPU using CDFLIB [6]. Thus we parallelize the population evaluation at two levels – individual level and sample level.

For the parallelization study, we use a Tesla C1060 Nvidia processor with 30 SMs for our simulations. The codes are run on a AMD Phenom(tm) II X4 945 processor system using only one CPU core. Although 1000 samples is a modest size of the sample set, and the CPU only takes a few seconds to complete an optimization run, the speed-up due to parallelization already begins to show up. We note that due to the use of simple analytical expressions of the constraints and objective function, the overall time required for their evaluation is still small compared to the overheads introduced by the EA steps, the reliability calculation step, etc. In practice, such analytical expressions are seldom available and we posit that the speed-ups obtained will be much larger when each constraint evaluation has to be performed using additional simulations such as a more involved finite element or CFD analysis. We demonstrate this scenario by introducing small delays of 10^{-5}, 10^{-4}, and 10^{-3} seconds in the evaluation of

Table 8: Running times for the CPU and GPU runs of the proposed algorithm for the car side-impact problem having a single epistemic variable (all times in seconds).

Delay	CPU	GPU	Speed-up
0	3.25	1.33	2.44
10^{-5}	255.49	3.39	75.37
10^{-4}	2519.70	23.48	107.31
10^{-3}	25135.48	224.29	112.07

each constraint, and show the results obtained in Table 8. It can be seen that the speed-up offered by GPU parallelization rapidly rises to about 100 with a solution evaluation time of 10^{-4} seconds or more.

8 Conclusions

In this paper, we have discussed a technique of using samples of uncertain variables and parameters to obtain an estimate of design reliability. The technique, based on a Bayesian approach is a better way of handling uncertainties when samples of the uncertain quantities are available than the method of fitting probability distributions to the samples and then performing traditional reliability based design optimization. Using an evolutionary approach, a well-distributed trade-off front has been obtained between design objective and reliability (with a fixed confidence) which can be extremely useful for a designer. The Bayesian inference technique also provides the possibility of updating the reliability estimates as more samples become available. We have also suggested a combined approach in which both aleatory and epistemic variables can be handled. In our simulations, we have considered as many as three epistemic variables and 10 aleatory variables and parameters, which make our applications as one of the most practical studies among other Bayesian based design optimization studies to handle epistemic uncertainty. As the time required for such computations may be high in some cases, GPU-based parallelization has been shown to significantly lower the computation time (as many as 100 times) while providing the benefits of a suitable and complete analysis using a bi-objective approach.

Results for numerical and engineering design problems are consistent with expected variations when different confidence levels or different numbers of samples are used. This is an important feature of this method as the relationship between the amount of information about the uncertainties and the design reliability is more direct than conventional approaches. Overall, with the fixed confidence approach, the integrated Bayesian RBDO algorithm, and the use of an evolutionary algorithm on a parallel computing platform, we have demonstrated a practical approach for design optimization in the presence of epistemic uncertainties.

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