# A Hybrid Multi-Objective Optimization Procedure Using PCX Based NSGA-II and Sequential Quadratic Programming

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***Abstract*— Despite the existence of a number of procedures for multi-objective optimization using evolutionary algorithms, there is still the need for a systematic and unbiased comparison of different approaches on a carefully chosen set of test problems.**

**In this paper, a hybrid approach using PCX based NSGA-II and Sequential Quadratic Programming (SQP) is applied on 19 benchmark test problems consisting of two, three and five objectives. PCX-NSGA-II is used as a population based algorithm where SQP is used as a local search procedure. A population based approach helps in finding the non-dominated set of solutions with a good spread, whereas SQP improves the obtained set of non-dominated solutions locally. The results obtained by the present approach shows mixed performance on the chosen test problems.**

## I. INTRODUCTION

Due to increasing interest in solving real-world multi-objective optimization problems using evolutionary algorithms (EA), researchers have developed a number of evolutionary multi-objective algorithms (EMO) based on real parameters. In the same direction, a generic parent-centric based recombination (PCX) operator is used with EA [1]. PCX is a vector-wise recombination operator which uses more than two parents to create an offspring. It assigns more probability for an offspring to remain closer to the index parent than away from it. This recombination operator has shown an efficient way of solving real-parameter optimization problems. NSGA-II [2], [3] with PCX recombination operator was introduced and tested successfully on various multi-objective and epistatic test problems [4]. For locally improving the obtained set of non-dominated solutions of PCX-NSGA-II, a classical optimization method such as SQP is applied in this paper.

This paper is devoted to the special session for performance assessment and competition of different multi-objective optimization algorithms on a set of 19 test problems consisting of various properties in terms of number of objectives (separability and deception), uni-modality and multi-modality, convexity and concavity, and with complex geometry, and others [5]. A procedure which contains the properties of good convergence and diversity among the solutions can only solve such a wide variety of test problems with reasonable satisfaction. For this purpose, PCX-NSGA-II is a promising algorithm to find a diverse set of points. Thereafter, the non-dominated solutions of PCX-NSGA-II can be improved by a local search technique using a classical optimization method, such as Sequential Quadratic Programming (SQP). Most of the classical optimization methods are designed to solve single-objective optimization problems and for solving a multi-objective optimization problem, it needs to be scalarized to a single-objective problem. The $\epsilon$-constraint method is used for driving multi-objective to single-objective optimization problem [6]. In the proposed PCX-NSGA-II algorithm, no two solutions apart by less than $\varepsilon$ distance are preferred to achieve a uniform and diverse distribution [7]. In the following section, we describe the description of algorithm and the proposed procedure. In Section III and IV, we present the parameter setting and simulation results in tabular and graphical forms, respectively, and the paper is concluded in section V.

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## II. DESCRIPTION OF THE ALGORITHM

### A. Elitist Non-dominated Sorting Genetic Algorithm

Elitist Non-dominated Sorting Genetic Algorithm (NSGA-II) uses an elite-preserving strategy as well as an explicit diversity preserving mechanism [2]. PCX-NSGA-II differs form original NSGA-II in terms of using a parent-centric based recombination operator. Motivation behind using PCX operator is it's property that uses probability distributions around the parent solutions to create an offspring. It has been shown elsewhere [1], that PCX based NSGA-II can perform well even in the case of skewed initial population, which does not bracket the optimum.

PCX operator involves two parameters $\sigma_\zeta$ and $\sigma_\eta$ controlling the variance along the principal direction (centroid towards the index parent) and in each of the rest $(\mu - 1)$ directions, respectively (where $\mu$ is the number of selected parents). In PCX based NSGA-II procedure, we choose three solutions using the binary tournament selection operator from the parent population. Each of them is, in turn, used as the index parent and an offspring solution is created by applying PCX operator to three chosen solutions [4].

### B. Sequential Quadratic Programming

In *quadratic programming* (QP), a quadratic model for objective function and a linear model for constraints are used to solve a non-linear optimization problem. *Sequential Quadratic programming* (SQP) method solves QP in each iteration [8]. Forward difference technique is used to compute the gradients numerically, including $(n + 2)$ function evaluations ($n$ is the number of variables).

Test suite given in [5] comprises of multi-objective problems. For converting the same to single-objective problem, the $\epsilon$-constraint method is used, following which the SQP is applied to the single-objective optimization problem.

In the proposed procedure, PCX based NSGA-II is used as a population based algorithm which helps in finding the non-dominated set of solutions with good diversity and can act as a global optimizer whereas SQP is employed as a local search method. Steps of the proposed procedure are described below:

1) Population is initialized randomly.
2) PCX-NSGA-II is applied on the initial population.
3) After finding the diverse set of non-dominated solutions of PCX-NSGA-II, SQP is employed on these solutions.
4) Solutions obtained from SQP are again supplied to PCX-NSGA-II to run for a final iteration, so that adequate number of non-dominated solutions are created.

Test problems with $M = 2$ and $M = 3$ objectives are solved with the above procedure with only one final PCX-NSGA-II iteration in step 4. For $M = 5$ objectives, the obtained SQP solutions are supplied to PCX-NSGA-II and then, an archive of non-dominated solutions is maintained during the final three iterations of PCX-NSGA-II.

To obtain a uniform distribution of solutions of PCX-NSGA-II, no two solutions apart by less than $\varepsilon$ distance are preferred, where distance between any two solutions is calculated in the normalized objective space [7].

## III. Parameters Setting

### A. Test Suite

The performance of the hybrid algorithm is tested on a set of 19 benchmark problems [5], which include seven two-objective test problems, six three-objective, and six five-objective test problems.

### B. PC Configuration

- System: Mandrake Linux 10.1
- CPU: P-IV 2.8 GHz
- RAM: 1 GB
- Language: ANSI-C
- Compiler Used: GCC version-3.2.2

### C. Parameters Setting for PCX-NSGA-II

Population size ($N$)

1) For $M = 2$ objectives: $N = 100$
2) For $M = 3$ objectives: $N = 150$
3) For $M = 5$ objectives: $N = 300$

Probability of crossover $P_c = 0.9$
Probability of mutation $P_m = 0.033$
Distribution index for mutation $\eta_m = 15$
$\varepsilon = 0.001, 0.01$ and $0.05$ for two, three and five objective test problems respectively.

$$\sigma_\zeta = \begin{cases} 0.010 & \text{if PCX-NSGA-II generations } (\gamma) < 1{,}000 \\ 0.001 & \text{Otherwsie.} \end{cases}$$

$$\sigma_\eta = \begin{cases} 0.008 & \text{if PCX-NSGA-II generations } (\gamma) < 1{,}000 \\ 0.001 & \text{Otherwsie.} \end{cases}$$

### D. Parameters Setting and Termination Criteria of SQP

1) Norm of descent direction: $\| d \| \leq \epsilon$; where $\epsilon = 10^{-9}$ or
2) Maximum number of iterations allowed ($\tau$): $50$, $50$ and $20$ for two, three and five-objective problems respectively.

SQP terminates when any of the above criteria is satisfied first. The proposed procedure includes the FES of both PCX-NSGA-II and SQP. First, we calculate the function evaluations required by SQP which is as follows. To calculate gradients and objective function values, SQP requires an average of $(n + 2)$ number of function evaluations in each iteration for $n$ number of variables. For a maximum of $\tau$ iterations of SQP and $N$ non-dominated solutions, SQP takes a total of $\tau \times N(n+2)$ function evaluations. Therefore, FES left to PCX-NSGA-II is $(5(10^5) - \tau \times N(n + 2))$. Hence, $\gamma = (5(10^5) - \tau \times N(n + 2))/N$ number of generations are allowed to PCX-NSGA-II. For example, for S_ZDT1 problem with $n = 30$, $N = 100$, and $\tau = 50$, the number of generations allowed for PCX-NSGA-II procedure is $3,400$. The above calculations ensure that FES of the proposed procedure never exceeds the allowed function evaluations $(5(10^5))$ [5] and it is also clear that the results shown at $5(10^5)$ FES used SQP procedure.

As an initial parameter study, we have chosen four different values of $\sigma_\zeta$ (range: 0.001–1.0) and $\sigma_\eta$ (range: 0.001–1.0) and three different values of $\tau$ (range: 20–50) and $\varepsilon$ (range: 0.001–0.01). A parametric study on six of 19 test problems was performed to find good combinations of $\sigma_\zeta$, $\sigma_\eta$, $\tau$ and $\varepsilon$ parameters. We have chosen two test problems with 2, 3 and 5-objective test problems. A total of $(4 \times 4 \times 3 \times 3 \times 6)$ or $864$ runs were executed with $5(10^5)$ function evaluations (FES) for each run. Hence, a maximum of $4.32(10^8)$ FES were performed for tuning the parameters.

## IV. Simulation Results

### A. $R$, Hypervolume and Covered Sets Indicators

First, we show the $R$-indicator values, which compute the difference between the maximum value of the augmented Tchebycheff utility function of the supplied reference set and obtained solutions from the procedure. A negative $R$-indicator means a better obtained utility function value than that of the reference set. A value close to zero means almost similar utility function value between reference and obtained solutions.

The obtained results are presented in Tables I, II, III, IV, V, VI and VII. Best, median, worst, mean and standard deviation of 25 runs for each test problem are done for $R$ and $I_{\overline{H}}$ indicators.

First three tables show the values of $R$ indicator at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES. Tables I, II and III indicate the significance of local search using SQP while the procedure is already converged in $5(10^4)$ FES. A negative $R$-indicator value is obtained in seven out of 19 test problems after $5(10^5)$ FES, meaning that a better utility function value than the supplied reference set is found by the proposed procedure. $R$-indicator values close to zero is observed in rest of 12 problems, meaning that a similar utility function value to that of the reference set is found. On the basis of $R$-indicator values, the proposed procedure has performed well for the given test suite.

Tables V to VII show hypervolume indicator $I_{\overline{H}}$ at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES. A lower value of $I_{\overline{H}}$ indicator corresponds to a better approximated set. In six out of 19 test problems, the proposed procedure shows better $I_{\overline{H}}$ indicator values than the supplied reference set after $5(10^5)$ FES. Except both three and five-objective WFG1 problems, in 11 other problems, the $I_{\overline{H}}$ indicator value is close to zero.

Table IV shows the covered set indicator values for the SYMPART test problem only. Best, median, worst, mean and standard deviation values at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES are presented.

### B. Attainment Surface Plots

Attainment surface signifies a combination of both convergence and diversity of the obtained solutions. Figures 1, 2, 3 and 4 show the $0\%$, $50\%$ and $100\%$ attainment surfaces along with Pareto-front for two objectives test problems. For three objective problems, Figures 5, 6 and 7 show the $50\%$ attainment surface.

It can be observed from the plots that for two-objective test problems like OKA2, SYMPART, S_ZDT1 and S_ZDT2, procedure shows a good convergence and spread. In the remaining three, two-objective problems, the proposed procedure shows a steady progress towards the respective optima and the procedure is unable to solve within the specified number of FES. It also shows a good attainment surface for S_DTLZ2, WFG8 and WFG9 problems whereas it does not perform satisfactorily for R_DTLZ2 and S_DTLZ3. In case of WFG1 test problem, the attainment surface plot depicts the partial convergence and diversity.

Figure 8 shows pair-wise interaction among five-objective problems for WFG8 (above diagonal) and WFG9 (lower diagonal) problems. The function values are normalized in the range $[1, 2]$ using lower and upper bounds given in the reference data set [5]. Median approximation set with respect to R-indicator at $5(10^5)$ is used for plotting the same. Definite structures between objective pairs are visible from the plots.

In comparison to another study [9] using SBX based NSGA-II and SQP algorithm, the PCX-NSGA-II-SQP analysis performed better in R_ZDT4, S_DTLZ2_M3, WFG8_M3, R_DTLZ2_M5 and S_DTLZ3_M5 with respect to the R-indicator. Similarly, in R_ZDT4, R_DTLZ2_M3, WFG1_M3 and S_DTLZ3_M5 problems, PCX-based procedure showed better results with respect to $I_{\overline{H}}$ indicator. In the remaining 12 problems, the previous study with SBX operator performed better than the PCX based algorithm of this study.

### C. Algorithm Complexity

Table VIII shows the complexity of the procedure. $T1 = (\sum_{i=1}^{m} t1_i)/m$, where $t1_i$ is the computing time for 10,000 function evaluation for problem $i$ and $m$ is the total number of test problems. Here $m = 19$.
$T2 = (\sum_{i=1}^{m} t2_i)/m$, where $t2_i$ is the computing time for the algorithm with 10,000 function evaluation for problem $i$. Time complexity of the hybrid procedure is $8.191$ seconds which depicts fast convergence capability.

## V. Conclusions

In this paper, we have presented a hybrid multi-objective optimization procedure consisting of evolutionary and classical algorithms. PCX based NSGA-II is used as a global optimizer and SQP as a local search method. The performance of the algorithm is tested on 19 test problems and assessment of performance have been done on the basis of R , $I_{\overline{H}}$, covered set indicators and attainment surfaces. Hybrid procedure has shown good performance for OKA2, SYMPART, S_ZDT1, S_ZDT2, S_DTLZ2, WFG8 and WFG9 test problems. For test functions S_ZDT4, S_ZDT6 and WFG1, procedure has exhibited fair performance whereas for R_ZDT4, R_DTLZ2 and S_DTLZ3 test problems, the performance has been reasonably well. Even in the case of higher number of objectives, the procedure has shown good convergence and diversity. Thus it can be concluded that the present procedure can be applied efficiently to a wide variety of multi-objective optimization problems. Time complexity of the procedure reveals its fast convergence. Thus, we have introduced a fast, efficient and hybrid procedure which solves a wide variety of multi-objective optimization problems with reasonable satisfaction.

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TABLE I

THE RESULTS FOR R INDICATOR ON TEST PROBLEMS 1-7 FOR M = 2 OBJECTIVES.

| FES | | 1.OKA2 | 2. SYMPART | 3. S_ZDT1 | 4. S_ZDT2 | 5. S_ZDT4 | 6. R_ZDT4 | 7. S_ZDT6 |
|---|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | -0.1002e-2 | 0.3265e-3 | 0.5994e-1 | 0.1164e-0 | 0.5944e-1 | 0.1609e-2 | 0.1351e-0 |
| | Median | 0.4221e-2 | 0.5181e-3 | 0.8170e-1 | 0.1307e-0 | 0.8526e-1 | 0.3413e-2 | 0.1478e-0 |
| | Worst | 0.4878e-1 | 0.6480e-3 | 0.9949e-1 | 0.1536e-0 | 0.1031e-0 | 0.3321e-2 | 0.1524e-0 |
| | Mean | 0.9064e-2 | 0.5142e-3 | 0.7929e-1 | 0.1321e-0 | 0.8586e-1 | 0.3475e-2 | 0.1446e-0 |
| | Std | 0.1145e-1 | 0.6464e-4 | 0.9340e-2 | 0.1070e-1 | 0.1146e-1 | 0.1247e-2 | 0.6041e-2 |
| $5 \times 10^4$ | Best | -0.8510e-3 | 0.2133e-4 | 0.2771e-2 | 0.6766e-2 | 0.4288e-2 | 0.3125e-3 | 0.5480e-1 |
| | Median | 0.2349e-2 | 0.3465e-4 | 0.4591e-2 | 0.8032e-2 | 0.1085e-1 | 0.9415e-3 | 0.6143e-1 |
| | Worst | 0.2322e-1 | 0.4838e-4 | 0.6587e-2 | 0.2463e-1 | 0.2029e-1 | 0.2333e-2 | 0.6572e-1 |
| | Mean | 0.4922e-2 | 0.3478e-4 | 0.4710e-2 | 0.8911e-2 | 0.1142e-1 | 0.1035e-2 | 0.6071e-1 |
| | Std | 0.6770e-2 | 0.6665e-5 | 0.1017e-3 | 0.3373e-2 | 0.3938e-2 | 0.4565e-3 | 0.3115e-2 |
| $5 \times 10^5$ | Best | -0.1065e-2 | 0.3692e-4 | 0.1956e-3 | 0.2165e-3 | 0.2485e-6 | 0.2720e-3 | 0.1547e-1 |
| | Median | -0.3036e-3 | 0.6584e-4 | 0.7055e-3 | 0.8663e-3 | 0.2021e-5 | 0.7894e-3 | 0.1865e-1 |
| | Worst | 0.1797e-3 | 0.8667e-4 | 0.1950e-2 | 0.3140e-2 | 0.3672e-4 | 0.1963e-2 | 0.2040e-1 |
| | Mean | -0.1644e-2 | 0.6401e-4 | 0.7982e-3 | 0.9618e-3 | 0.5350e-5 | 0.8636e-3 | 0.1859e-1 |
| | Std | 0.7360e-3 | 0.1561e-4 | 0.5025e-3 | 0.6136e-3 | 0.7894e-5 | 0.3779e-3 | 0.1316e-2 |

TABLE II

THE RESULTS FOR R INDICATOR ON TEST PROBLEMS 8-13 FOR M = 3 OBJECTIVES.

| FES | | 8. S_DTLZ2 | 9. R_DTLZ2 | 10. S_DTLZ3 | 11. WFG1 | 12. WFG8 | 13. WFG9 |
|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | 0.2488e-3 | 0.3789e-3 | 0.3750e-3 | 0.5591e-1 | -0.1282e-1 | -0.8500e-2 |
| | Median | 0.5923e-3 | 0.5286e-3 | 0.4402e-3 | 0.5665e-1 | -0.8907e-2 | -0.5319e-2 |
| | Worst | 0.1126e-2 | 0.6297e-3 | 0.5731e-3 | 0.5751e-1 | -0.6129e-2 | -0.2414e-3 |
| | Mean | 0.5972e-3 | 0.5268e-3 | 0.4487e-3 | 0.5666e-1 | -0.9257e-2 | -0.5136e-2 |
| | Std | 0.2023e-3 | 0.5143e-4 | 0.5525e-4 | 0.4234e-3 | 0.2014e-2 | 0.2420e-2 |
| $5 \times 10^4$ | Best | 0.1376e-3 | 0.4608e-4 | 0.5768e-4 | 0.5551e-1 | -0.2169e-1 | -0.1042e-1 |
| | Median | 0.2936e-3 | 0.1945e-3 | 0.8245e-4 | 0.5641e-1 | -0.1777e-1 | -0.6941e-2 |
| | Worst | 0.6957e-3 | 0.2487e-3 | 0.1303e-3 | 0.5766e-1 | -0.1298e-1 | -0.5218e-2 |
| | Mean | 0.3128e-3 | 0.1769e-3 | 0.8456e-4 | 0.5648e-1 | -0.1738e-1 | -0.6983e-2 |
| | Std | 0.1253e-3 | 0.5926e-4 | 0.1629e-4 | 0.5139e-3 | 0.2179e-2 | 0.1114e-2 |
| $5 \times 10^5$ | Best | 0.6266e-4 | 0.1484e-4 | 0.2259e-4 | 0.4984e-1 | -0.2925e-1 | -0.1232e-1 |
| | Median | 0.8778e-4 | 0.3731e-4 | 0.3079e-4 | 0.5202e-1 | -0.2722e-1 | -0.8973e-2 |
| | Worst | 0.2945e-3 | 0.6610e-4 | 0.4006e-4 | 0.5393e-1 | -0.2405e-1 | -0.7119e-2 |
| | Mean | 0.1010e-3 | 0.3688e-4 | 0.3028e-4 | 0.5211e-1 | -0.2700e-1 | -0.9422e-2 |
| | Std | 0.5030e-4 | 0.1131e-4 | 0.4621e-5 | 0.1122e-2 | 0.1393e-2 | 0.1691e-2 |

TABLE III

THE RESULTS FOR R INDICATOR ON TEST PROBLEMS 8-13 FOR M = 5 OBJECTIVES.

| FES | | 8. S_DTLZ2 | 9. R_DTLZ2 | 10. S_DTLZ3 | 11. WFG1 | 12. WFG8 | 13. WFG9 |
|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | -0.9820e-5 | 0.1468e-4 | -0.1473e-7 | 0.4705e-1 | 0.5753e-3 | -0.2153e-2 |
| | Median | 0.1703e-3 | 0.5957e-4 | 0.1019e-4 | 0.4778e-1 | 0.3191e-2 | -0.1144e-2 |
| | Worst | 0.5041e-3 | 0.2013e-3 | 0.3281e-3 | 0.4812e-1 | 0.5143e-2 | 0.2422e-2 |
| | Mean | 0.2002e-3 | 0.7245e-4 | 0.4553e-4 | 0.4777e-1 | 0.3001e-2 | -0.7903e-3 |
| | Std | 0.1858e-3 | 0.4652e-4 | 0.9115e-4 | 0.2339e-3 | 0.1109e-2 | 0.1190e-2 |
| $5 \times 10^4$ | Best | 0.1177e-3 | -0.5195e-5 | -0.1473e-7 | 0.4626e-1 | -0.4038e-2 | -0.2094e-2 |
| | Median | 0.2530e-3 | 0.6128e-4 | 0.3461e-3 | 0.4731e-1 | -0.5922e-3 | -0.1030e-2 |
| | Worst | 0.3749e-3 | 0.2836e-3 | 0.6230e-3 | 0.4806e-1 | 0.2346e-2 | 0.6813e-3 |
| | Mean | 0.2587e-3 | 0.6924e-4 | 0.2876e-3 | 0.4731e-1 | -0.6877e-3 | -0.9847e-3 |
| | Std | 0.5098e-4 | 0.6150e-4 | 0.2179e-3 | 0.4762e-3 | 0.1625e-2 | 0.7598e-3 |
| $5 \times 10^5$ | Best | 0.6832e-5 | -0.9719e-5 | -0.1473e-8 | 0.4325e-1 | -0.7767e-2 | -0.2118e-2 |
| | Median | 0.1269e-3 | 0.6145e-4 | 0.4258e-3 | 0.4447e-1 | -0.1112e-2 | -0.9683e-3 |
| | Worst | 0.1883e-3 | 0.2323e-3 | 0.6501e-3 | 0.4672e-1 | 0.2057e-2 | 0.8521e-3 |
| | Mean | 0.1250e-3 | 0.7799e-4 | 0.3908e-3 | 0.4462e-1 | -0.1655e-2 | -0.8432e-3 |
| | Std | 0.4623e-4 | 0.6743e-4 | 0.1705e-3 | 0.9060e-3 | 0.2567e-2 | 0.7226e-3 |

TABLE IV

THE RESULTS FOR COVERED SETS FOR TEST PROBLEM SYMPART.

| FES | $5 \times 10^3$ | $5 \times 10^4$ | $5 \times 10^5$ |
|---|---|---|---|
| Best | 1.0000e-0 | 1.0000e-0 | 1.0000e-0 |
| Median | 1.0000e-0 | 1.0000e-0 | 1.0000e-0 |
| Worst | 1.0000e-0 | 1.0000e-0 | 1.0000e-0 |
| Mean | 1.0000e-0 | 1.0000e-0 | 1.0000e-0 |
| Std | 0.0 | 0.0 | 0.0 |

TABLE V

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 1-7 FOR M = 2 OBJECTIVES.

| FES | | 1.OKA2 | 2. SYMPART | 3. S_ZDT1 | 4. S_ZDT2 | 5. S_ZDT4 | 6. R_ZDT4 | 7. S_ZDT6 |
|---|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | 0.1365e-2 | 0.9514e-3 | 0.2047e-0 | 0.2995e-0 | 0.1821e-0 | 0.5738e-2 | 0.3438e-0 |
| | Median | 0.1166e-1 | 0.1513e-2 | 0.2676e-0 | 0.3547e-0 | 0.2736e-0 | 0.1066e-1 | 0.3780e-0 |
| | Worst | 0.6194e-1 | 0.1888e-2 | 0.3392e-0 | 0.4221e-0 | 0.3185e-0 | 0.1993e-1 | 0.3895e-0 |
| | Mean | 0.1640e-1 | 0.1499e-2 | 0.2676e-0 | 0.3569e-0 | 0.2674e-0 | 0.1147e-1 | 0.3690e-0 |
| | Std | 0.1471e-1 | 0.1887e-3 | 0.3131e-1 | 0.3695e-1 | 0.3605e-1 | 0.3318e-2 | 0.1612e-1 |
| $5 \times 10^4$ | Best | -0.5200e-3 | 0.6332e-4 | 0.1275e-1 | 0.1333e-1 | 0.1282e-1 | 0.1090e-2 | 0.1294e-0 |
| | Median | 0.2988e-2 | 0.1023e-3 | 0.1651e-1 | 0.1828e-1 | 0.3249e-1 | 0.3368e-2 | 0.1447e-0 |
| | Worst | 0.3632e-1 | 0.1417e-3 | 0.2225e-1 | 0.2855e-1 | 0.5964e-1 | 0.7452e-2 | 0.1555e-0 |
| | Mean | 0.7395e-2 | 0.1026e-3 | 0.1706e-1 | 0.1894e-1 | 0.3387e-1 | 0.3538e-2 | 0.1440e-0 |
| | Std | 0.9413e-2 | 0.1955e-4 | 0.2453e-2 | 0.3251e-2 | 0.1173e-1 | 0.1405e-2 | 0.7805e-2 |
| $5 \times 10^5$ | Best | 0.3933e-2 | 0.1128e-3 | 0.1406e-2 | 0.1182e-2 | 0.3100e-5 | 0.8473e-3 | 0.3411e-1 |
| | Median | 0.8539e-2 | 0.1981e-3 | 0.1650e-2 | 0.1490e-2 | 0.1083e-4 | 0.2868e-2 | 0.4099e-1 |
| | Worst | 0.1387e-1 | 0.2598e-3 | 0.2357e-2 | 0.2204e-2 | 0.1742e-3 | 0.6117e-2 | 0.4511e-1 |
| | Mean | 0.8563e-2 | 0.1927e-3 | 0.1693e-2 | 0.1532e-2 | 0.3475e-4 | 0.2965e-2 | 0.4132e-1 |
| | Std | 0.2643e-2 | 0.4626e-4 | 0.2094e-3 | 0.2494e-3 | 0.4806e-4 | 0.1209e-2 | 0.2801e-2 |

TABLE VI

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 8-13 FOR M = 3 OBJECTIVES.

| FES | | 8. S_DTLZ2 | 9. R_DTLZ2 | 10. S_DTLZ3 | 11. WFG1 | 12. WFG8 | 13. WFG9 |
|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | 0.1373e-1 | 0.8822e-2 | 0.8449e-2 | 0.2886e-0 | -0.6884e-1 | -0.4472e-1 |
| | Median | 0.2071e-1 | 0.2385e-1 | 0.1203e-1 | 0.2919e-0 | -0.5549e-1 | -0.2968e-2 |
| | Worst | 0.4505e-1 | 0.3077e-1 | 0.2211e-1 | 0.2960e-0 | -0.2743e-1 | 0.2064e-1 |
| | Mean | 0.2404e-1 | 0.2206e-1 | 0.1254e-1 | 0.2919e-0 | -0.5312e-1 | -0.8135e-2 |
| | Std | 0.8692e-2 | 0.5646e-2 | 0.2762e-2 | 0.2136e-2 | 0.1174e-2 | 0.1853e-1 |
| $5 \times 10^4$ | Best | 0.2037e-2 | 0.4433e-4 | 0.1278e-3 | 0.2852e-0 | -0.1286e-0 | -0.4628e-1 |
| | Median | 0.3931e-2 | 0.1754e-2 | 0.2226e-3 | 0.2895e-0 | -0.1078e-0 | -0.2631e-1 |
| | Worst | 0.7329e-2 | 0.3043e-2 | 0.6266e-3 | 0.2960e-0 | -0.8351e-1 | -0.8911e-2 |
| | Mean | 0.4069e-2 | 0.1600e-2 | 0.2395e-3 | 0.2898e-0 | -0.1040e-0 | -0.2692e-1 |
| | Std | 0.1317e-2 | 0.9047e-3 | 0.1062e-3 | 0.2509e-2 | 0.1256e-1 | 0.9393e-2 |
| $5 \times 10^5$ | Best | 0.4475e-3 | 0.3109e-5 | 0.5943e-5 | 0.2592e-0 | -0.1741e-0 | -0.7253e-1 |
| | Median | 0.1783e-2 | 0.3947e-4 | 0.1421e-4 | 0.2692e-0 | -0.1578e-0 | -0.4731e-1 |
| | Worst | 0.2955e-2 | 0.1306e-3 | 0.2489e-4 | 0.2795e-0 | -0.1359e-0 | -0.2844e-1 |
| | Mean | 0.1790e-2 | 0.3998e-4 | 0.1425e-4 | 0.2694e-0 | -0.1579e-0 | -0.4815e-1 |
| | Std | 0.6700e-3 | 0.2982e-4 | 0.5933e-5 | 0.5202e-2 | 0.1104e-1 | 0.1301e-1 |

TABLE VII

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 8-13 FOR M = 5 OBJECTIVES.

| FES | | 8. S_DTLZ2 | 9. R_DTLZ2 | 10. S_DTLZ3 | 11. WFG1 | 12. WFG8 | 13. WFG9 |
|---|---|---|---|---|---|---|---|
| $5 \times 10^3$ | Best | 0.6815e-5 | 0.1739e-3 | -0.4440e-15 | 0.5304e-0 | -0.2356e-0 | -0.2126e-0 |
| | Median | 0.2029e-2 | 0.5988e-3 | 0.6980e-4 | 0.5374e-0 | -0.1984e-0 | -0.1982e-0 |
| | Worst | 0.1835e-1 | 0.3249e-2 | 0.1870e-1 | 0.5404e-0 | -0.1823e-0 | -0.1326e-0 |
| | Mean | 0.4126e-2 | 0.8342e-3 | 0.1364e-2 | 0.5371e-0 | -0.2010e-0 | -0.1918e-0 |
| | Std | 0.5749e-2 | 0.7579e-3 | 0.4010e-2 | 0.2169e-2 | 0.1260e-1 | 0.1945e-1 |
| $5 \times 10^4$ | Best | 0.9943e-3 | -0.8872e-4 | -0.6661e-15 | 0.5223e-0 | -0.3121e-0 | -0.2119e-0 |
| | Median | 0.4046e-2 | 0.6357e-3 | 0.2538e-1 | 0.5329e-0 | -0.2622e-0 | -0.1964e-0 |
| | Worst | 0.1094e-1 | 0.7423e-2 | 0.6506e-1 | 0.5398e-0 | -0.2070e-0 | -0.1674e-0 |
| | Mean | 0.4610e-2 | 0.8917e-3 | 0.2518e-1 | 0.5327e-0 | -0.2592e-0 | -0.1940e-0 |
| | Std | 0.2273e-2 | 0.1447e-2 | 0.2141e-1 | 0.4513e-2 | 0.2648e-1 | 0.1207e-1 |
| $5 \times 10^5$ | Best | 0.5137e-4 | -0.1517e-3 | -0.1332e-14 | 0.4934e-0 | -0.3623e-0 | -0.2127e-1 |
| | Median | 0.8622e-3 | 0.6235e-3 | 0.4432e-1 | 0.5054e-0 | -0.2552e-0 | -0.1939e-1 |
| | Worst | 0.1758e-2 | 0.4338e-2 | 0.1212e-0 | 0.5268e-0 | -0.2108e-0 | -0.1683e-1 |
| | Mean | 0.9661e-3 | 0.8260e-3 | 0.4353e-1 | 0.5068e-0 | -0.2677e-0 | -0.1919e-1 |
| | Std | 0.4749e-3 | 0.9611e-3 | 0.2515e-1 | 0.8733e-2 | 0.4144e-1 | 0.1189e-2 |

TABLE VIII

COMPUTATIONAL COMPLEXITY (TIME IN SECONDS)

| $T1$ | $T2$ | $(T2-T1)/T1$ |
|---|---|---|
| 0.24 | 2.206 | 8.191 |

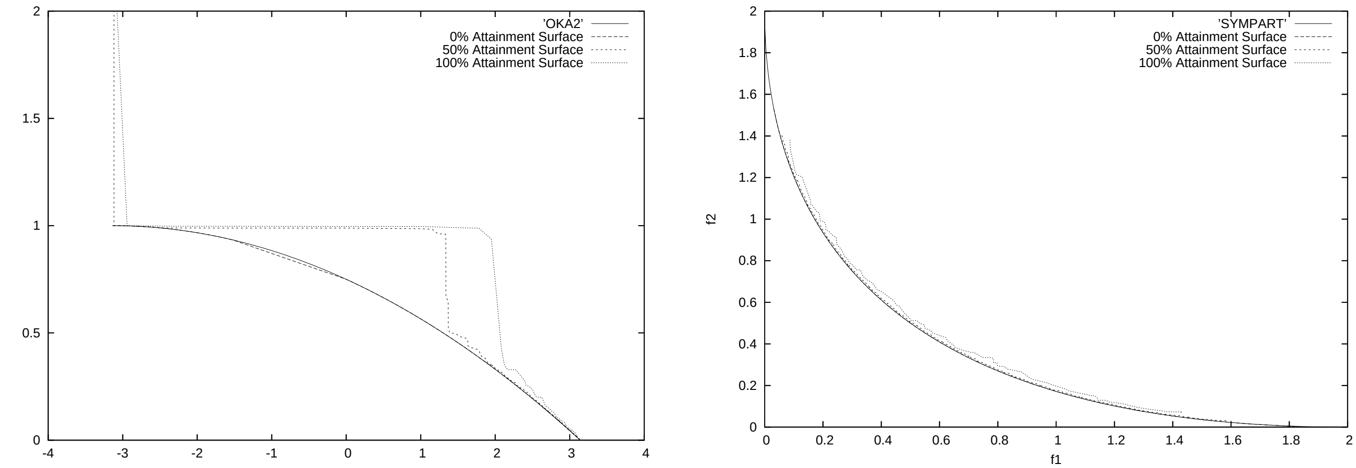


Fig. 1. Attainment plots of OKA2 and SYMPART.

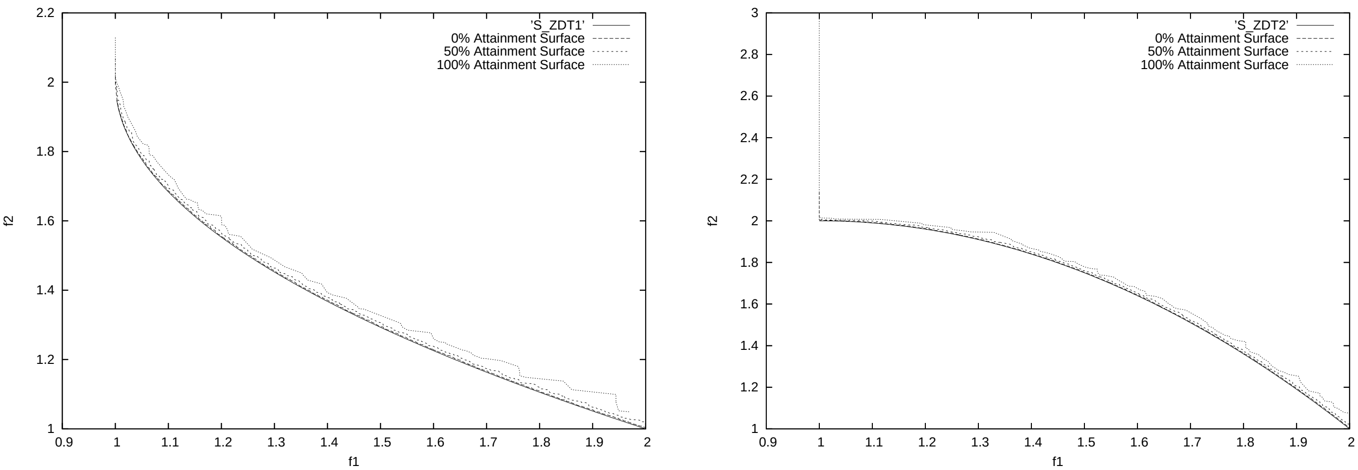


Fig. 2. Attainment plots of S_ZDT1 and S_ZDT2.

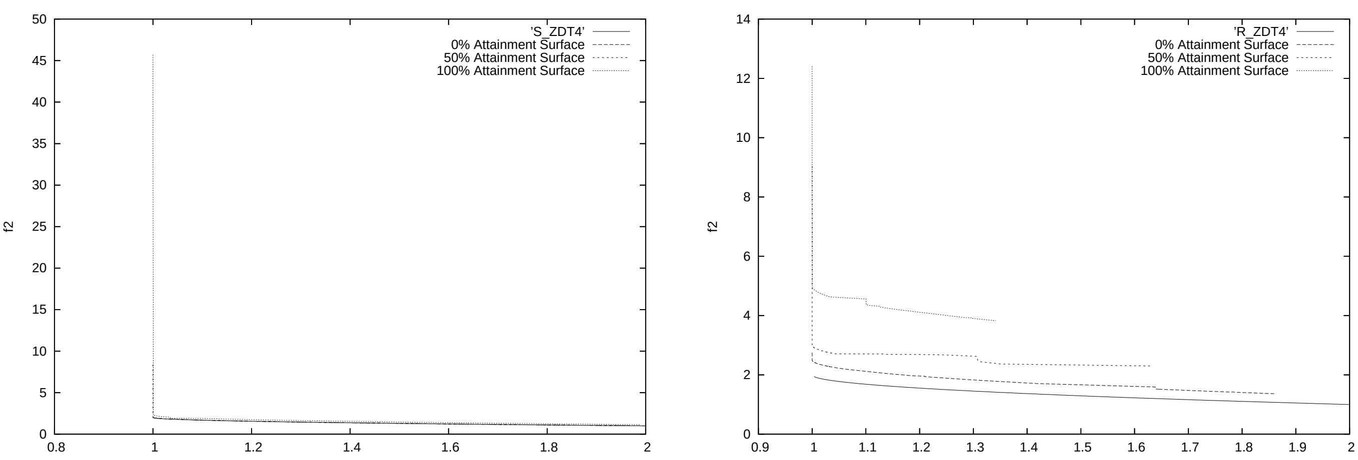


Fig. 3. Attainment plots of S_ZDT4 and R_ZDT4.

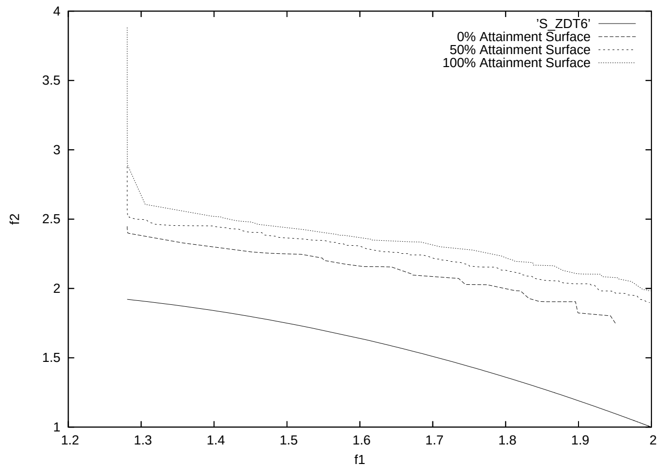

Fig. 4. Attainment plots of S_ZDT6.

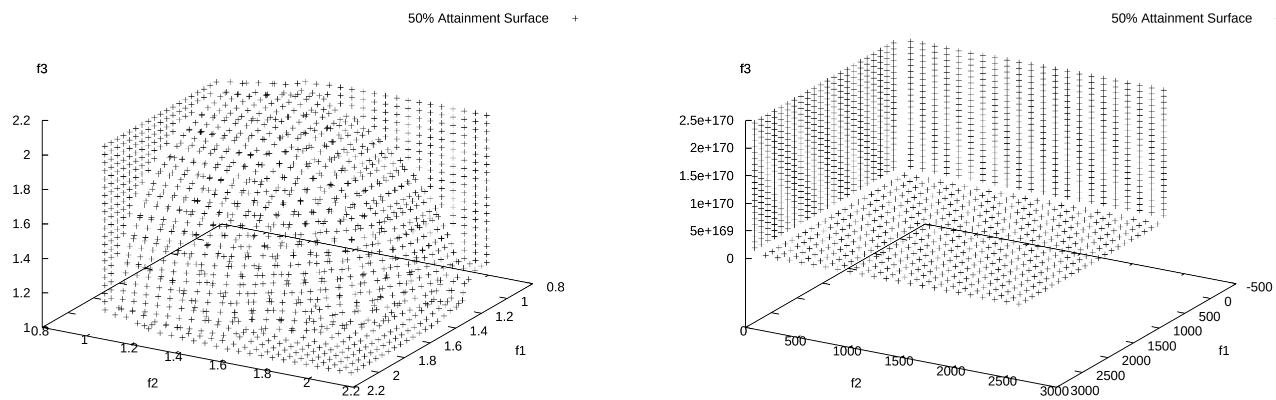

Fig. 5. Attainment surfaces of S_DTLZ2_M3 and R_DTLZ2_M3.

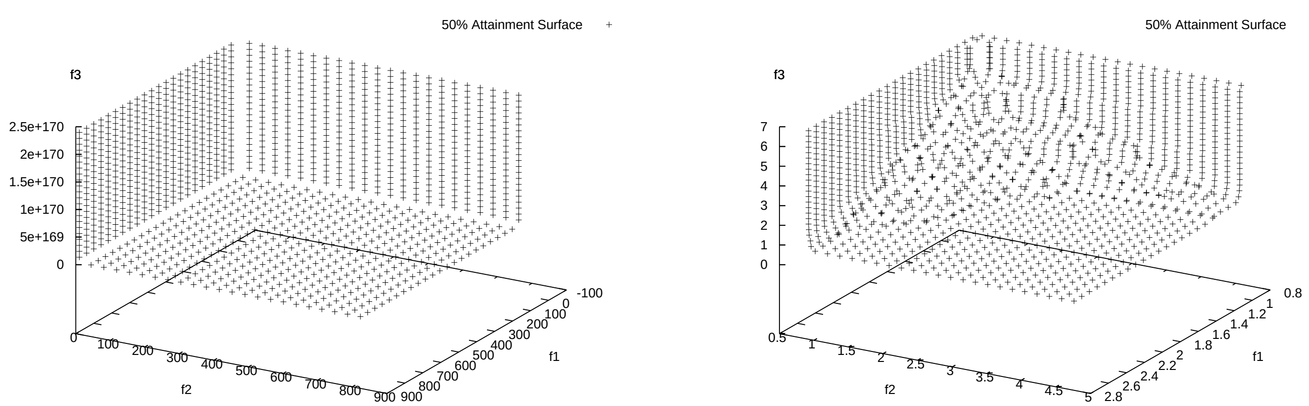

Fig. 6. Attainment surfaces of S_DTLZ3_M3 and WFG1_M3.

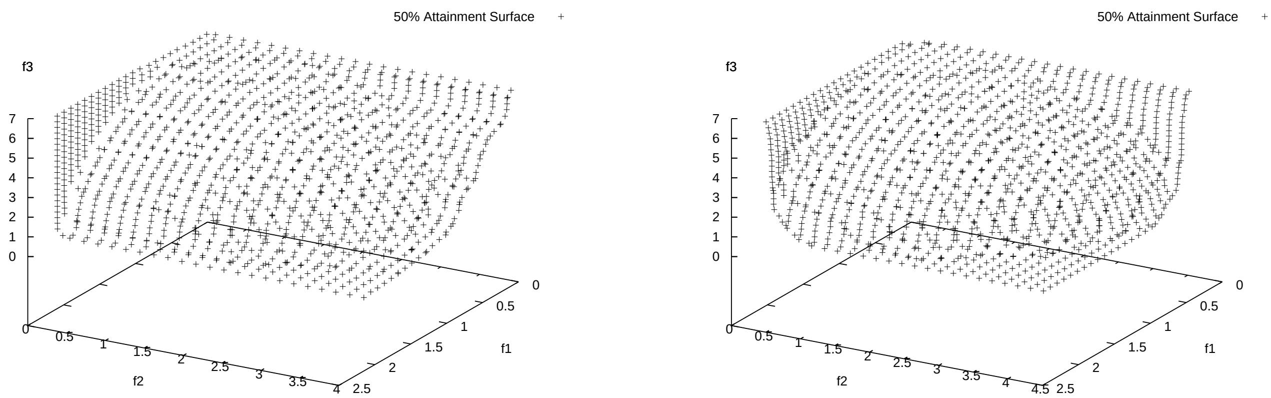


Fig. 7. Attainment surfaces of WFG8_M3 and WFG9_M3.

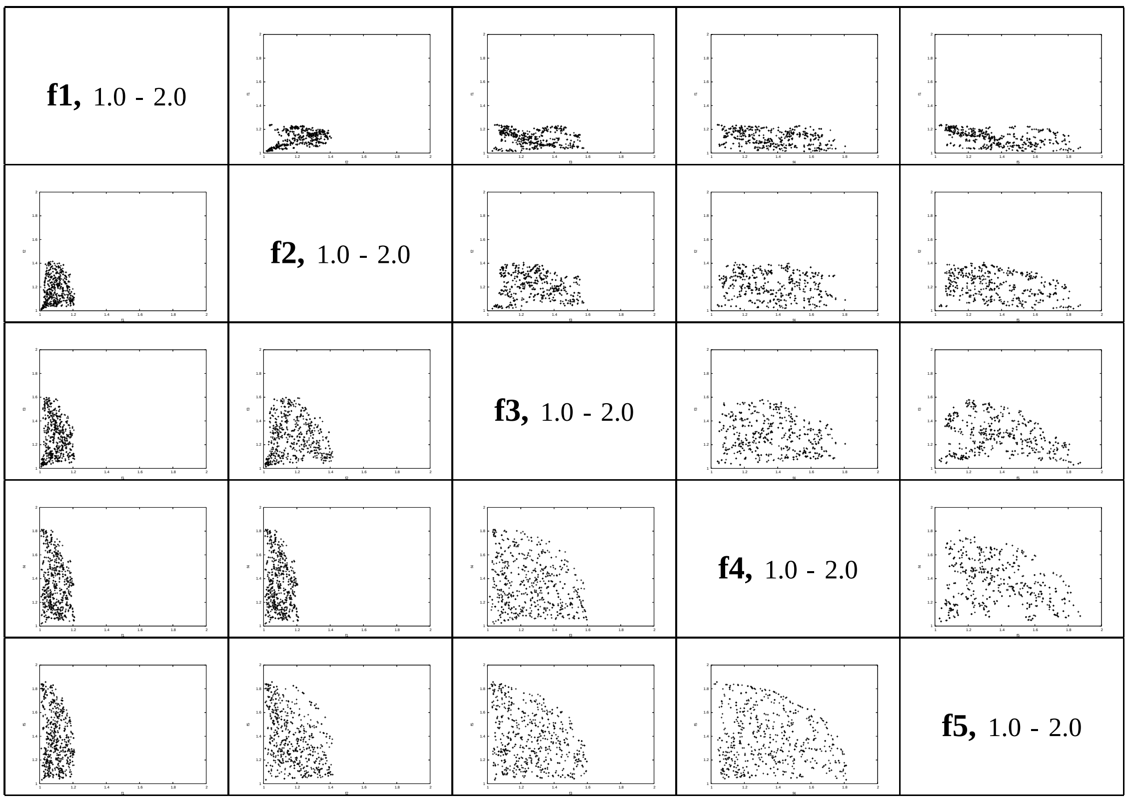


Fig. 8. Upper diagonal plots for WFG8 ($M = 5$) and lower diagonal plots are for WFG9 ($M = 5$) with respect to the median approximation set of R-indicator at $5(10^5)$ FES.


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