Interactive Evolutionary Multi-Objective Optimization and Decision-Making using Reference Direction Method

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KanGAL Report Number 2007001

ABSTRACT

In this paper, we borrow the concept of reference direction approach from the multi-criterion decision-making literature and combine it with an EMO procedure to develop an algorithm for finding a single preferred solution in a multi-objective optimization scenario efficiently. EMO methodologies are adequately used to find a set of representative efficient solutions over the past decade. This study is timely in addressing the issue of optimizing and choosing a single solution using certain preference information. In this approach, the user supplies one or more reference directions in the objective space. The population approach of EMO methodologies is exploited to find a set of efficient solutions corresponding to a number of representative points along the reference direction. By using a utility function, a single solution is chosen for further analysis. This procedure is continued till no further improvement is possible. The working of the procedure is demonstrated on a set of test problems having two to ten objectives and on an engineering design problem. Results are verified with theoretically exact solutions on two-objective test problems. More such dual and hybrid methodologies involving an EMO and a multi-criterion decision-making tool must be tried for suggesting a complete solution to a multi-objective optimization problem.

Categories and Subject Descriptors

I.2.8 [**Computing Methodologies**]: Problem Solving, Control Methods, and Search

General Terms

Algorithms

Keywords

Multi-objective optimization, Reference direction method, Decision-making, Interactive EMO.

1. INTRODUCTION

EMO methodologies have amply demonstrated to find a set of trade-off optimal solutions in solving multi-objective optimization problems. Only recently, researchers have been putting efforts in combining decision-making techniques with an EMO to arrive at a single optimal solution. It is argued that the decision-making principles used in such studies are not possible to be converted as a single fixed goal before the optimization is performed, because in such a scenario the corresponding goal can be treated as a single objective of the optimization study and the resulting optimum can be found. Since the decision-making principles depend on the outcome of a previously performed optimization study, any methodology which will involve both optimization and decision-making must be an interactive one, involving a decision-maker and an optimizer.

The interactive EMO methods can hope to achieve the following tasks by exploiting the population aspect of EMO:

1. Instead of finding a single preferred solution on the Pareto-optimal front, an EMO can assist in finding a set of preferred solutions or a preferred region on the Pareto-optimal set. The advantage of finding a region of solutions instead of a single solution is that (i) the decision-maker can provide a tentative information about his/her preference (that is, the reference point need not be precisely chosen) (ii) a set of solutions near a preferred solution provides information about other solutions which are close to the preferred solution but may have interesting trade-off for the decision-maker to consider and (iii) the knowledge of more than one solution near the preferred point may help decipher common properties of such solutions, thereby providing salient information about desired solutions.

2. Instead of finding preferred solutions near a single portion of the Pareto-optimal frontier, an EMO can help find multiple preferred regions corresponding to differing preferences simultaneously. This task is particularly useful if the decision-maker is not sure whether to concentrate near a single preferred region or to explore multiple preferred regions simultaneously.

3. EMO can replace any repetitive application of single-objective optimizations which may be needed in a classical interactive multi-objective optimization and decision-making task.

The classical multi-objective optimization literature contains a plethora of multi-criteria decision-making (MCDM) principles [15]. In the context of evolutionary algorithms, Fonseca and Fleming [7] suggested a preference-based multi-objective genetic algorithm in optimizing a low-pressure spool-speed governor of a Pegasus gas turbine engine. Recently, Deb et al. [5] and Luque et al. [14] suggested reference point based EMO procedures which require the decision-maker to specify one or more reference points and the task of an EMO is, not to find the entire Pareto-optimal frontier, but to find a portion of the Pareto-optimal front which solves an achievement scalarizing function. In a loose sense, the target in such studies is to find a set of points close to the specified reference points.

In the remainder of this paper, we briefly discuss the reference direction method suggested by Korhonen and Laakso [12] as an aid to find a preferred solution. Thereafter, we describe how an EMO methodology can be embedded within the reference direction based MCDM procedure for speeding up the decision-making task. The proposed EMO methodology is a modification of NSGA-II procedure [1]. Finally, we show the working of a critical step of the proposed reference direction based NSGA-II or RD-NSGA-II on two to 10-objective test problems and the complete proposed RD-NSGA-II procedure on a couple of three-objective problems. These first results using a combined EMO-MCDM approach are promising and should motivate more such synergizes and hybrid procedures to be developed in the near future.

2. REFERENCE DIRECTION METHOD

Pekka and Laakso [12] suggested a reference direction based approach for multi-criterion optimization using the principle of solving achievement scalarizing functions repeatedly. For an optimization problem of the type:

$$\begin{array}{ll} \text{Minimize} & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x}))^T, \\ \text{subject to} & \mathbf{x} \in \mathcal{S}, \end{array} \quad (1)$$

the following iterative procedure was suggested:

Step 0: Choose an initial arbitrary point $\mathbf{q}^0$ in the objective space and let $k = 1$.

Step 1: Specify another vector $\mathbf{g}^k$ and determine the reference direction $\mathbf{d}^k = \mathbf{g}^k - \mathbf{q}^{k-1}$.

Step 2: Determine a set Q^k of efficient solutions $\mathbf{q}$ which solve the following achievement scalarizing function s:

$$\begin{array}{ll} \text{Minimize}_{\mathbf{z}} & s(\mathbf{z}, \mathbf{r}, \mathbf{w}) = \max_{i \in I}(z_i - r_i)/w_i, \\ \text{subject to} & \mathbf{r}(t) = \mathbf{q}^{k-1} + t\mathbf{d}^k. \end{array} \quad (2)$$

The parameter t is increased from zero to infinity, $\mathbf{w}$ is a weighting vector and I is the set of indices of objectives having a nonzero weight value.

Step 3: Find the most preferred solution q^k in Q^k using a utility function or by other means.

Step 4: If $q^{k-1} \neq q^k$, set $k = k + 1$ and go to Step 1. Otherwise, check for optimality conditions (Kuhn-Tucker conditions [15] or other optimality conditions [12]) of the solution q^k. If q^k is optimal, terminate the procedure, else increment k and define a new reference direction and go to Step 2.

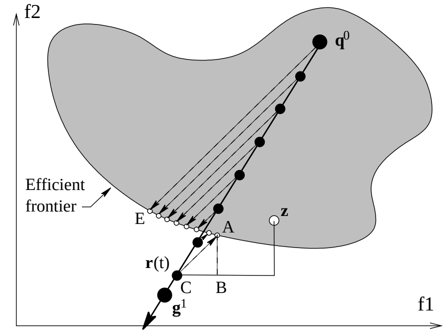

Figure 1: A sketch shows how a set of points on a reference direction can find a set of efficient solutions.

Figure 1 shows a sketch of Step 2 of the above procedure. For each point (say point 'C') marked on the reference direction (from q^0 towards g^1), an efficient solution (point 'A') is found by solving the achievement scalarizing problem given in equation 2. Step 2 of the above procedure involves multiple application of a single-objective optimization for different values of t, thereby finding a range of efficient solutions ('A' till 'E'). The idea of finding an efficient solution corresponding to a point on a reference direction is similar to the reference point approach of Wierzbicki [16]. Although the original study of reference direction approach and subsequent studies of Korhonen and his coauthors [13, 9, 11] concentrated on parametric solutions for multiple points on the reference direction, the principle can be used by forming multiple achievement scalarizing functions and solving them by a single-objective optimizer independently. An analytical hierarchy process was also used to determine the reference direction [8].

Interestingly, the reference direction approach of multiple efficient solutions corresponding points on a reference direction can be considered as a process of projecting the reference direction on the Pareto-optimal frontier [10].

3. REFERENCE DIRECTION BASED EMO

An evolutionary multi-objective optimization (EMO) procedure can be introduced to achieve Step 2 of the above reference direction procedure in finding multiple efficient solutions simultaneously. Since this task involves multiple independent optimizations, an EMO is an ideal choice for an efficient computational effort. Moreover, since the task in Step 2 involves finding efficient solutions which usually lies on a projection of the reference direction, the target in such an EMO is usually a one-dimensional efficient curve. Although EMO procedures have been found to be not suitable for finding the entire M-dimensional Pareto-optimal front in an M-objective optimization problem for $M \geq 5$ [4], the same study and other studies [2] has shown that NSGA-II procedure is able to find the entire Pareto-optimal front even in 20-objective optimization problems for which the Pareto-optimal front is one or two dimensional. Thus, even for handling problems having a large number of objectives using the reference direction based methodology, NSGA-II or other efficient EMO procedures is expected to have no difficulty in finding a set of efficient solutions corresponding to the achievement sclarizing functions formed from selected points along the reference direction.

3.1 Reference Direction Based NSGA-II (RD-NSGA-II)

One simple-minded approach would be to first apply the original NSGA-II to find a representative set of efficient solutions and then apply the above-mentioned achievement scalarizing function s from each point on the chosen reference direction to find the minimum s solution from the set. Although such an idea may work well for a two-objective optimization problem, for larger objective problems, such an idea is not adequate due to two reasons: (i) NSGA-II (or any EMO procedure) is shown to be not efficient in finding a set of well-distributed efficient points for problems having large number of objectives (such as five or more) [4] and (ii) NSGA-II must find a large number of efficient solutions to make a good set of optimal solutions corresponding to achievement scalarizing problems. We illustrate the second aspect by solving a three-objective DTLZ2 problem with the NSGA-II and then choosing the solutions corresponding to the minimum achievement scalarizing function values for each point on a chosen reference direction. Figure 2 shows 400 efficient solutions (the set P marked with small diamonds) obtained using the original NSGA-II procedure on a three-objective DTLZ2 test problem [6]. The efficient solutions lie on a non-convex front and NSGA-II with a clustering approach is able to find a good distribution over the entire efficient frontier. After the solutions are obtained by NSGA-II, we consider each point on the reference direction (shown by an arrow) one at a time and identify the corresponding point from the set P which minimizes the achievement scalarizing function. These points are marked with a bigger diamond and are joined to show the sequence of efficient solutions in the set of 400 NSGA-II solutions which corresponds to consecutive points on the reference direction. The true minimum solutions on the entire efficient frontiers are marked with shaded circles. It is clear that the two-step procedure does not find the desired solutions. Although the corresponding efficient points are supposed to lie a smooth curve on the Pareto-optimal frontier, the coarseness of obtained a finite set of NSGA-II solutions does not allow us to find the exact solutions. We now suggest a more efficient procedure.

In the proposed procedure, we modify the original NSGA-II procedure as follows. First, we mark a set of points ($\mathbf{r}(t)$, $t = 0, 1, \ldots$) on the given reference direction vector. For each point $\mathbf{r}(t)$ on the reference direction, we compute the achievement scalarizing function value $s(\mathbf{z}, \mathbf{r}, \mathbf{w})$ for a chosen weight vector $\mathbf{w}$ and for each population member $\mathbf{z}$ in a NSGA-II population. Thereafter, the population member $\bar{\mathbf{z}}$ having the smallest value of s is declared to lie on the first non-dominated front. This procedure is continued for each point $\mathbf{r}$ and corresponding population member for the minimum s is included in the first non-dominated front. Thereafter, these chosen population members are temporarily discounted from the population and the above procedure is repeated. The next set of minimum s solutions are then declared to be members of the second non-dominated front. This procedure is repeated till all population members are classified into a non-dominated frontier. Thereafter, the crowding distance procedure is repeated with the classified population members as usual [1].

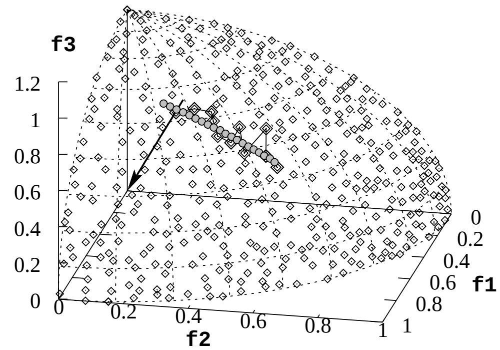

Figure 2: Two-step procedure of applying NSGA-II and then selecting solutions corresponding to the minimum achievement scalarizing functions is not an adequate procedure.

The above modification emphasizes minimum s solutions corresponding to each point on the reference direction equally, thereby finding the corresponding efficient solutions simultaneously. The hierarchy of importance introduced in the population by carefully assigning next-best s solutions to subsequent non-dominated fronts allows a systematic and efficient search towards the desired efficient solutions. A little thought will prevail that the population size in such a NSGA-II application should at least 2 or 3 times the number of points considered along the reference direction. The multiplicity is needed to ensure that the search is adequately guided towards the corresponding efficient point. However, it is not necessary that for every point on the reference direction there always exists a different efficient point. Since for multiple points on the reference direction a single efficient solution is likely to occur, there may not be N independent efficient points corresponding to N points on the reference direction.

It is also interesting to note that if multiple reference directions are to be considered simultaneously, the above procedure can be easily extended to handle such a case. First, multiple points can be found on each reference direction and then the above modified non-dominated sorting procedure can be extended for all such points.

4. SIMULATION RESULTS

In this section, we first consider test problems, for which the efficient frontier is known mathematically. For these problems, the range of efficient solutions corresponding to two extreme bounds on a reference direction can be found exactly by performing an analysis. We use these problems and show proof-of-principle results of Step 2 of RD-NSGA-II. Next, we have considered two problems (including one real-world problem) for which we perform the complete iterative RD-NSGA-II procedure till convergence to a single preferred solution.

In all simulation runs, we have used SBX recombination operator with probability 0.9 and distribution index of 10 and a polynomial mutation operator with probability of $1/n$ (where n is the number of variables) with a distribution index 20.

4.1 Two-Objective Test Problem ZDT2

First, we consider 30-variable, two-objective ZDT2 problem [6]. In this problem, the efficient frontier is non-convex. In this problem, the efficient frontier is known and is given by $f_2 = 1 - f_1^2$, as shown in Figure 3. We consider a reference direction between two points $q^0 = (\alpha_1, \beta_1)$ and $g^1 = (\alpha_2, \beta_2)$. Any point from q^0 till g^1 can be written as $r(t) = q^0 + t(g^1 - q^0)$. We define the negative of the slope of the reference direction as $\gamma = (\beta_1 - \beta_2)/(\alpha_1 - \alpha_2)$, such that $r(t) = q^0 - t(1, \gamma)^T$. For an equal weight to both objectives, the minimum of the achievement scalarizing function is that point on the Pareto-optimal front ($f_2 = 1 - f_1^2$) which makes the objective-wise difference between $(f_1, f_2)^T$ and reference point $r(t)$ equal, or, $|f_2 - (\beta_1 - t\gamma)| = |f_1 - (\alpha_1 - t)|$. In the context of Figure 1, for equal weights to each objective, point 'C' will find efficient point z=A, for which CB=BA. Solving and noting that for efficient solutions, f_1 lies within $[0, 1]$, we obtain the following region on the Pareto-optimal front as the result of the projection of the reference direction on the efficient frontier:

$$\begin{aligned}
&\text{For } \gamma > 1: \\
&\max\left\{0, \frac{-1+\sqrt{5-4(\beta_1-\alpha_1)}}{2}\right\} \leq f_1 \leq \min\left\{1, \frac{-1+\sqrt{5-4(\beta_2-\alpha_2)}}{2}\right\}, \\
&\text{For } \gamma < 1: \\
&\max\left\{0, \frac{-1+\sqrt{5-4(\beta_2-\alpha_2)}}{2}\right\} \leq f_1 \leq \min\left\{1, \frac{-1+\sqrt{5-4(\beta_1-\alpha_1)}}{2}\right\}, \\
&\text{For } \gamma = 1: \\
&f_1 = \frac{-1+\sqrt{5-4(\beta_1-\alpha_1)}}{2}.
\end{aligned} \tag{3}$$

In the case of a reference direction with a 45 degrees slope ($\gamma = 1$), all reference points result in a single efficient solution.

To illustrate the working of Step 2 of the proposed RD-NSGA-II on ZDT2, we consider the following points: $q^0 = (\alpha_1, \beta_1)^T = (1.6, 1.8)^T$ and $g^1 = (\alpha_2, \beta_2)^T = (0.4, 0)^T$. The corresponding γ is 1.5, which is greater than one. By using the first condition given in equation 3, we obtain the following range of f_1 values for which the the reference direction gets projected on the efficient frontier:

$$0.525 \leq f_1 \leq 0.785.$$

We apply the RD-NSGA-II using the following parameters: population size = 100 and maximum number of generations = 500. We consider 15 equi-spaced points between the two extreme points on the reference direction and obtained a set of points shown in Figure 3. By investigating the extreme values of these solutions we also observe that the bounds match with the above theoretical bounds, thereby confirming the accuracy of our RD-NSGA-II procedure on the ZDT2 problem.

To show the performance with more than one reference directions simultaneously, we consider two reference directions R_1 and R_2:

	(α_1, β_1)	(α_2, β_2)
R_1	$(0.75, 1.50)^T$	$(0.10, 0.20)^T$
R_2	$(1.10, 0.50)^T$	$(0.20, 0.00)^T$

Using the above theory, we find that for R_1, the resulting efficient solutions will lie in $f_1 \in [0.207, 0.572]$. For the second direction R_2, $\gamma = 0.556$, which is less than one. By using the second condition in equation 3, we obtain $f_1 \in [0.704, 0.860]$. The RD-NSGA-II with the same parameter values find an identical set of ranges on f_1, as shown in Figure 4. 15 points are considered in each case.

4.2 Two-Objective Test Problem ZDT1

This problem involves 30 variables and two objectives [6] but the efficient frontier is convex: $f_2 = 1 - \sqrt{f_1}$. A careful thought about the relationship between ZDT2 and ZDT1 will reveal the range of f_1 as a projection of a reference direction between $q^0 = (\alpha_1, \beta_1)$ and $g^1 = (\alpha_2, \beta_2)$. The lower and upper bounds, presented in equation 3 for ZDT2 must now get squared for ZDT1. To illustrate, we consider the same two reference directions as considered on the ZDT2 test problem with identical parameter settings and obtain the efficient points shown in Figure 5. The theoretical results above predict that the ranges on f_1 will be as follows: [0.043, 0.328] for R_1 and [0.496, 0.740] for R_2. The Figure 5 confirms these ranges for the chosen reference directions.

4.3 Three-Objective Test Problem DTLZ2

Next, we apply Step 2 of the proposed RD-NSGA-II procedure to three-objective DTLZ2 test problem [6] having one reference direction. Figure 6 shows the obtained efficient solutions for 20 points along the reference direction. RD-NSGA-II is run with 100 population members and for 400 generations. The figure shows how a set of efficient solutions can be found for a set of reference points simultaneously.

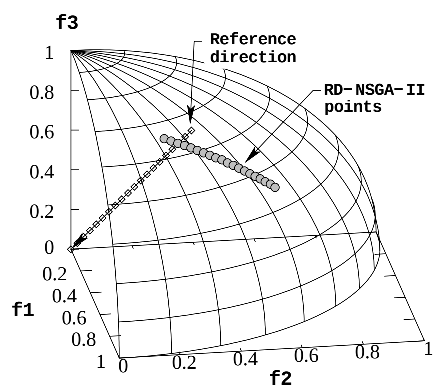

Figure 6: Step 2 of proposed algorithm in DTLZ2 with one reference direction.

To test the efficacy of the proposed procedure, next we consider three reference directions simultaneously in a single simulation. For 15 points in each direction, RD-NSGA-II working with 100 population members and run for 400 generations is able to find efficient solutions corresponding to all three reference directions simultaneously.

4.4 Five-Objective Test Problem DTLZ2

To test the proposed procedure on more than two objectives, we solve the five-objective DTLZ2 problem for three reference directions, shown below:

	q^0	g^1
R_1 :	$(0.8, 0.2, 0, 2, 0.2, 0.2)^T$	$(0, 0, 0, 0, 0)^T$
R_2 :	$(0.8, 0.8, 0.8, 0.2, 0.2)^T$	$(0, 0, 0, 0, 0)^T$
R_3 :	$(0.2, 0.2, 0.2, 0.2, 0.8)^T$	$(0, 0, 0, 0, 0)^T$

15 equi-spaced points along q^0 to g^1 are chosen along each

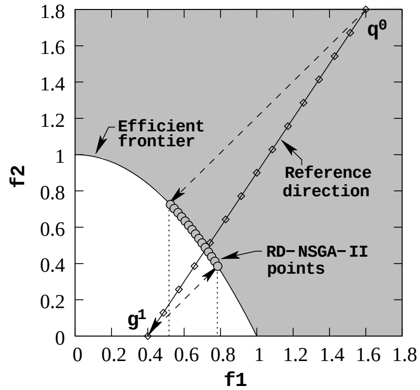

Figure 3: Step 2 of proposed algorithm in ZDT2 with one reference direction.

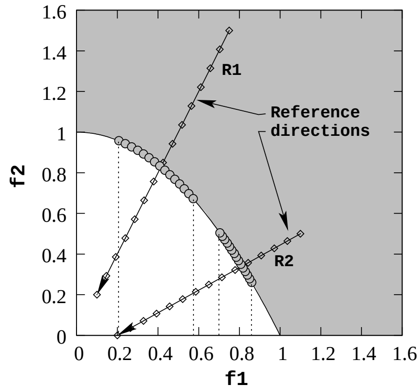

Figure 4: Step 2 of proposed algorithm in ZDT2 with two reference directions.

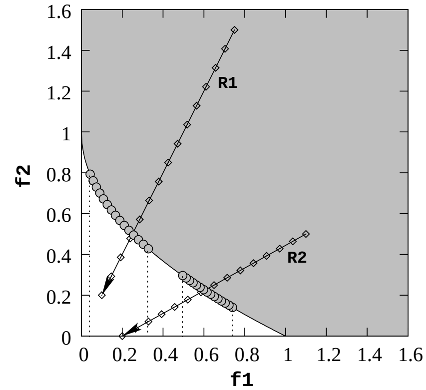

Figure 5: Step 2 of proposed algorithm in ZDT1 with two reference directions.

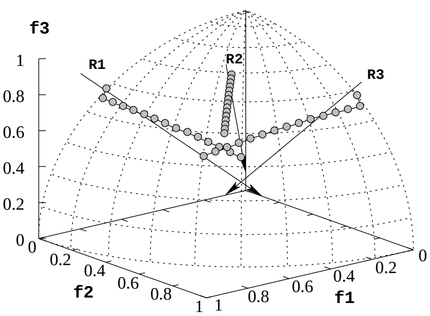

Figure 7: Step 2 of proposed algorithm in three-objective DTLZ2 with three reference directions.

direction. Figure 8 shows the value-path diagram of efficient solutions obtained for each of the three reference directions. RD-NSGA-II is used with 100 population members and is run for 500 generations. The figure shows that obtained solutions are different from each other and in each case solutions follow a similar pattern of values as they are in the $(g^1 - q^0)$ vector.

4.5 10-Objective Test Problem DTLZ2

We extend our study to 10-objective DTLZ2 problem and consider following two reference directions simultaneously:

$$R_1: \begin{cases} q_i^0 = 0.4, g_i^1 = 0, & i = 1, \ldots, 5, \\ q_i^0 = 0.1, g_i^1 = 0, & i = 6, \ldots, 10, \end{cases}$$
$$R_2: \begin{cases} q_i^0 = 0.1, g_i^1 = 0, & i = 1, \ldots, 5, \\ q_i^0 = 0.4, g_i^1 = 0, & i = 6, \ldots, 10, \end{cases}$$

Figure 9 shows the corresponding points obtained using RD-NSGA-II using 400 population size and run for 800 generations. Although the original NSGA-II is shown to have difficulties in converging and maintaining a well-distributed set of solutions to the entire efficient frontier [4], since the focus here is to find only a small set of efficient solutions (lying on a Pareto-optimal curve), the task is not difficult.

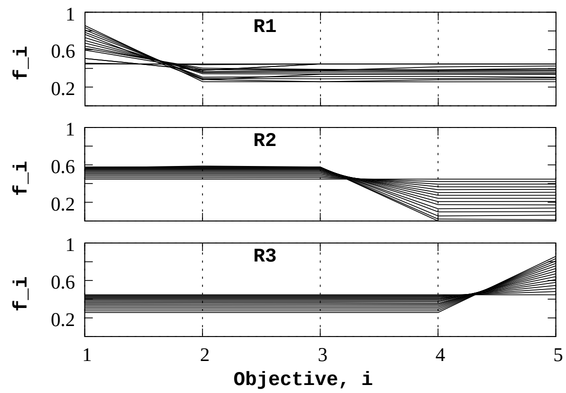

Figure 8: Step 2 of proposed algorithm in five-objective DTLZ2 with two reference directions.

4.6 Pekka and Laakso's Three-Objective Problem

Pekka and Lakso [12] illustrated their reference direction procedure on a three-variable, three-objective, four-constraint optimization problem as follows:

$$\begin{array}{rl} \text{Maximize} & f_1(\mathbf{x}) = x_1, \\ \text{Maximize} & f_2(\mathbf{x}) = x_2, \\ \text{Maximize} & f_3(\mathbf{x}) = x_3, \\ \text{subject to} & g_1(\mathbf{x}) = 3x_1 + 2x_2 + 3x_3 \leq 18, \\ & g_2(\mathbf{x}) = x_1 + 2x_2 + x_3 \leq 10, \\ & g_3(\mathbf{x}) = 9x_1 + 20x_2 + 7x_3 \leq 96, \\ & g_4(\mathbf{x}) = 7x_1 + 20x_2 + 9x_3 \leq 96, \\ & x_1, x_2, x_3 \geq 0. \end{array} \quad (4)$$

In our study here, we set a upper bound on all variables: $x_i \leq 6$ for $i = 1, 2, 3$. The original study [12] considered the following q^0 and g^1 vectors:

$$q^0 = (0.706, 4.240, 0.706)^T,$$
$$g^1 = (1, 6, 1)^T.$$

The study also used a utility function (which was maximized): $U = \min(3f_1, 5f_2, 3f_3)$. Since the constraints and

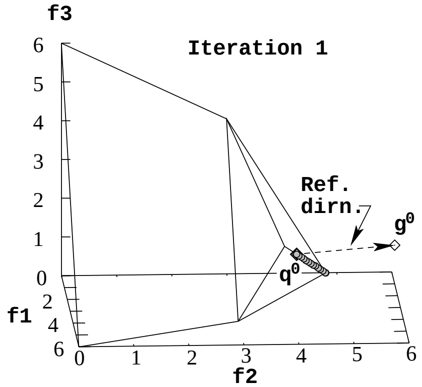

Figure 10: First iteration of RD-NSGA-II procedure on the Pekka and Laakso problem.

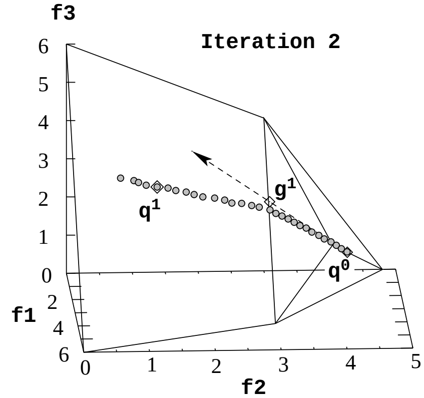

Figure 11: Second iteration of RD-NSGA-II procedure on the Pekka and Laakso problem.

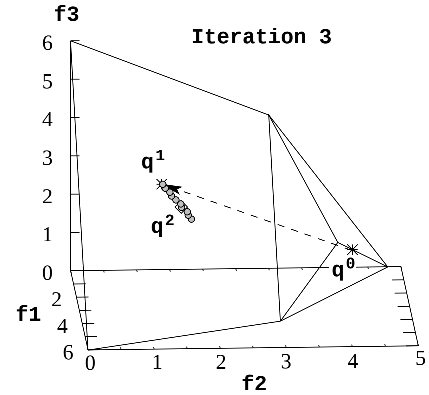

Figure 12: Third iteration of RD-NSGA-II procedure on the Pekka and Laakso problem.

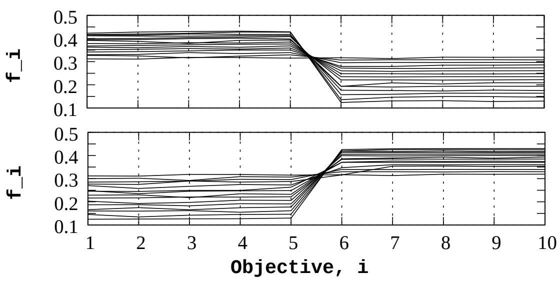

Figure 9: Step 2 of proposed algorithm in 10-objective DTLZ2 with two reference directions.

objective functions are all linear, the original study used a parametric LP formulation for finding a sequence of solutions of the achievement scalarizing function corresponding to any reference point $r(t)$ along q^0 and g^1. However, for a non-linear optimization problem, such a parametric optimization may not be possible, but our approach of using the concept by choosing a finite set of points along the reference directions and then applying a single-objective optimizer to find the optimal solution of the corresponding achievement scalarizing function for each point simultaneously can still be used. The best solution q^1 corresponding to the utility function was reported to be $q^1 = (0.706, 4.24, 0.706)^T$.

Here, we apply the RD-NSGA-II with the following parameter values: population size = 60, crossover probability = 0.9, mutation probability = 1/3, SBX distribution index = 20, polynomial mutation distribution index = 50, and maximum generations = 500. We choose 15 equi-spaced points between q^0 and g^1 (given above) and obtain all 15 points shown in Figure 10 in a single simulation run. The corresponding solution which maximizes the utility function given above is shown below:

$$q^1 = (0.703, 4.236, 0.703)^T.$$

This solution is close to that obtained in the original study. It is interesting that the use of an EMO procedure can be used to find multiple solutions corresponding to multiple reference points in a single simulation run, thereby making the RD-NSGA-II procedure an efficient computational procedure.

The original study considered a new reference direction to improve the first and third objectives: $g^2 = (2, 3, 2.5)^T$. We also choose the same g^2 vector and consider 30 points in the direction from q^1 towards g^2 and go beyond till twice the distance of $|g^2 - q^1|$. The corresponding solutions are shown in Figure 11. Interestingly, all these solutions lie on the Pareto-optimal frontier. The solution corresponding to the maximum utility function is $q^2 = (2.164, 1.285, 2.978)^T$. The point reported in the original study was $(2.18, 1.30, 2.96)^T$, which is close to our obtained vector.

In the third iteration, $g^3 = (2.4, 1.5, 2.6)^T$ was used. We use the reference direction q^2 to g^3 and extend till two times the distance between q^2 to g^3. Figure 12 shows 10 RD-NSGA-II points obtained for 10 equi-spaced points along the reference direction. The corresponding best solution ($q^3 = (2.473, 1.568, 2.482)^T$) for the chosen utility function is also marked in the figure. The reported solution in the original study was $(2.48, 1.57, 2.48)^T$, which is again close to our obtained solution. Although the problem is linear and a parametric LP procedure is ideal for solving this problem, the RD-NSGA-II approach is shown to find solutions corresponding to multiple reference points simultaneously in a single simulation run. For non-linear problems, such a technique will be more efficient than the repetitive application of single-objective optimizations. We illustrate such an application in the following subsection.

4.7 Car Side Impact Problem

The final problem we consider is a three-objective car side impact problem having seven real-parameter design variables and 10 constraints [3].

$$
\begin{array}{ll}
\text{Min.} & f_1(\mathbf{x}) = \text{Weight}, W, \\
\text{Min.} & f_2(\mathbf{x}) = \text{Pubic force}, F, \\
\text{Min.} & f_3(\mathbf{x}) = \text{Avg. vel. of V-Pillar}, 0.5*(V_{MBP}+V_{FD}), \\
\text{s.t.} & g_1(\mathbf{x}) \equiv \text{Abdomen load} \le 1 \text{ kN}, \\
& g_2(\mathbf{x}) \equiv V*C_u \le 0.32 \text{ m/s}, \\
& g_3(\mathbf{x}) \equiv V*C_m \le 0.32 \text{ m/s}, \\
& g_4(\mathbf{x}) \equiv V*C_l \le 0.32 \text{ m/s}, \\
& g_5(\mathbf{x}) \equiv D_{ur} \text{ upper rib deflection} \le 32 \text{ mm}, \\
& g_6(\mathbf{x}) \equiv D_{mr} \text{ middle rib deflection} \le 32 \text{ mm}, \\
& g_7(\mathbf{x}) \equiv D_{lr} \text{ lower rib deflection} \le 32 \text{ mm}, \\
& g_8(\mathbf{x}) \equiv F \text{ Pubic force} \le 4 \text{ kN}, \\
& g_9(\mathbf{x}) \equiv V_{MBP} \text{ Vel. of V-Pillar at mid-pt.} \le 9.9 \text{ m/s}, \\
& g_{10}(\mathbf{x}) \equiv V_{FD} \text{ Vel. of front door at V-Pillar} \le 15.7 \text{ m/s}, \\
& 0.5 \le x_1 \le 1.5, \quad 0.45 \le x_2 \le 1.35, \quad 0.5 \le x_3 \le 1.5, \\
& 0.5 \le x_4 \le 1.5, \quad 0.875 \le x_5 \le 2.625, \quad 0.4 \le x_6 \le 1.2, \\
& 0.4 \le x_7 \le 1.2.
\end{array} \tag{5}
$$

The expressions for all the above functions are given in the appendix. It is somewhat intuitive that if the weight of the car is small, the pubic force experienced by a passenger and the average velocity of the V-Pillar responsible for withstanding the impact load will be large. It is not so obvious but if a design manages to reduce the pubic force due to a side impact, it is probably due to the large share of load absorbed by the V-Pillar, thereby causing a large deflection of the pillar. Thus, these three objectives are supposed to produce a trade-off optimal frontier, if all three are to be minimized in a multi-objective optimization sense.

We first apply the original NSGA-II on the above constrained three-objective optimization problem using the following GA parameters: population size = 100 and maximum generation = 500. Figure 13 shows the obtained three-dimensional frontier with small diamonds. The three-way trade-off among the objectives is clear from the figure.

Now, we apply the reference direction based NSGA-II on the same problem by first finding the ideal and nadir points:

$$
\begin{array}{rl}
\text{Ideal point:} & \mathbf{z}^* = (24.368, 3.585, 10.611)^T \\
\text{Nadir point:} & \mathbf{z}^{\text{nad}} = (42.686, 3.997, 12.440)^T
\end{array}
$$

The ideal point is found by minimizing each of three objectives independently and the nadir point is found by a NSGA-II based procedure developed elsewhere [2]. The set of NSGA-II solutions shown in Figure 13 also agree with these points. The first reference direction is chosen using $q^0 = \mathbf{z}^{\text{nad}}$ and $g^1 = \mathbf{z}^*$. 25 equi-spaced points are chosen between q^0 and g^1 and the corresponding optimal solutions to the achievement scalarizing function is found simultaneously using RD-NSGA-II. These points are shown in Figure 13 with circles. It is interesting to note that these points lie on the obtained NSGA-II frontier. To choose a single solution from the RD-NSGA-II solutions, we use the following utility function:

$$U(f, g) = (g_1 + f_2)/2.$$

This function is an average of the loads experienced at the abdomen and at the pubic area. We choose the solution which corresponds to the minimum of the above utility function. The corresponding point is given as follows and is also marked in the figure:

$$q^1 = (35.946, 3.585, 11.531)^T.$$

This completes one iteration of the reference direction based NSGA-II procedure.

From this point, we may decide to continue in a direction in which the average velocity on the V-Pillar (objective f_3) is smaller. To improve the third objective (avg. velocity) at the expense of other two objectives, we choose the following g point: $g^2 = (42.686, 3.997, 10.611)^T$ and choose 15 equi-spaced points from q^1 till g^2. Applying the RD-NSGA-II procedure with identical parameter setting, we obtain 15 solutions, shown in Figure 14. Thereafter, we choose a single solution which makes the above utility function the smallest. This solution is given as follows:

$$q^2 = (40.976, 3.809, 10.611)^T.$$

It is interesting to note that f_3 is reduced from q^1 at the expense of increasing both f_1 and f_2 values.

Next, we may want to reduce both f_1 and f_2 from this solution and try the following g vector: $g^3 = (24.368, 3.585, 12.440)^T$. We obtain 30 points using RD-NSGA-II for this reference direction and the points are shown in Figure 15. The chosen utility function is smallest for the following RD-NSGA-II solution:

$$q^3 = (40.916, 3.813, 10.613)^T.$$

For brevity, we may consider this solution close to q^2 and terminate the simulation. The corresponding solution is given as follows:

$$\mathbf{x} = (1.496, 1.350, 1.500, 1.045, 2.625, 1.200, 1.198)^T.$$

Most values are close to their upper bounds except that of x_4. The corresponding abdomen load is 0.504 kN and the pubic force is 1.35 kN, which are well within their allowable constraint values. Starting with a multi-objective optimization problem, the above procedure shows how a single preferred efficient solution can be found interactively by the use a reference directions and an EMO procedure.

5. CONCLUSIONS

An EMO methodology has been embedded in a classical multi-criterion optimization and decision-making task in which the decision-maker works with a reference direction in each iteration. In a multi-objective problem, only those efficient solutions which are optimum solutions to achievement scalarizing functions formed at different points along the reference direction are found using an EMO procedure. The NSGA-II procedure has been modified to find a subset of efficient solutions from the entire efficient frontier. On two to 10 objective problems, a critical step of the proposed procedure has been shown to find theoretically correct solutions to some test problems. On a couple of problems including an engineering design problem, the complete procedure has demonstrated a viable approach of a combination of optimization and decision-making towards finding a single preferred efficient solution, a matter which is important yet has not been pursued enough in the past.

6. ACKNOWLEDGMENTS

This study is supported by a research grant from GM R&D, Bangalore.

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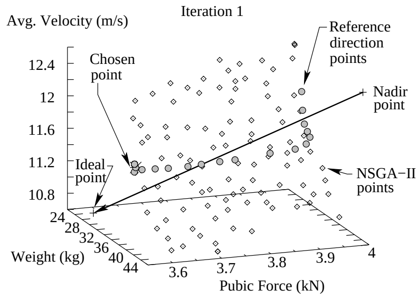

Figure 13: First iteration of RD-NSGA-II procedure on the car side impact problem.

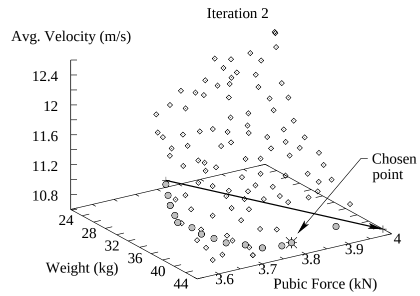

Figure 14: Second iteration of RD-NSGA-II procedure on the car side impact problem.

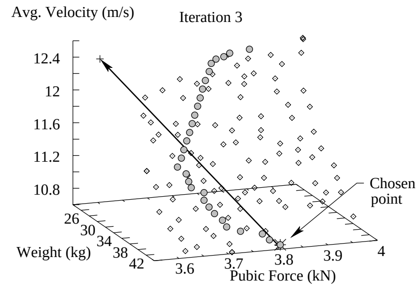

Figure 15: Third iteration of RD-NSGA-II procedure on the car side impact problem.

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APPENDIX

A. FUNCTIONS FOR THE CAR SIDE IMPACT PROBLEM

$W(\mathbf{x}) = 1.98 + 4.9x_1 + 6.67x_2 + 6.98x_3 + 4.01x_4 + 1.78x_5$
$+0.00001x_6 + 2.73x_7,$
$g_1(\mathbf{x}) = 1.16 - 0.3717x_2x_4 - 0.00931x_2x_{10} - 0.484x_3x_9$
$+0.01343x_6x_{10},$
$g_2(\mathbf{x}) = 0.261 - 0.0159x_1x_2 - 0.188x_1x_8 - 0.019x_2x_7 + 0.0144x_3$
$x_5 + 0.87570.001x_5x_{10} + 0.08045x_6x_9 + 0.00139x_8x_{11}$
$+1.575(10^{-6})x_{10}x_{11},$
$g_3(\mathbf{x}) = 0.214 + 0.00817x_5 - 0.131x_1x_8 - 0.0704x_1x_9 + 0.03099x_2x_6$
$-0.018x_2x_7 + 0.0208x_3x_8 + 0.121x_3x_9 - 0.00364x_5x_6$
$+0.0007715x_5x_{10} - 0.0005354x_6x_{10} + 0.00121x_8x_{11}$
$+0.00184x_9x_{10} - 0.018x_2x_2,$
$g_4(\mathbf{x}) = 0.74 - 0.61x_2 - 0.163x_3x_8 + 0.001232x_3x_{10} - 0.166x_7x_9$
$+0.227x_2x_2,$
$g_5(\mathbf{x}) = 28.98 + 3.818x_3 - 4.2x_1x_2 + 0.0207x_5x_{10} + 6.63x_6x_9$
$-7.77x_7x_8 + 0.32x_9x_{10},$
$g_6(\mathbf{x}) = 33.86 + 2.95x_3 + 0.1792x_{10} - 5.057x_1x_2 - 11x_2x_8$
$-0.0215x_5x_{10} - 9.98x_7x_8 + 22x_8x_9,$
$g_7(\mathbf{x}) = 46.36 - 9.9x_2 - 12.9x_1x_8 + 0.1107x_3x_{10},$
$g_8(\mathbf{x}) = 4.72 - 0.5x_4 - 0.19x_2x_3 - 0.0122x_4x_{10} + 0.009325x_6x_{10}$
$+0.000191x_{11}x_{11},$
$g_9(\mathbf{x}) = 10.58 - 0.674x_1x_2 - 1.95x_2x_8 + 0.02054x_3x_{10}$
$-0.0198x_4x_{10} + 0.028x_6x_{10},$
$g_{10}(\mathbf{x}) = 16.45 - 0.489x_3x_7 - 0.843x_5x_6 + 0.0432x_9x_{10}$
$-0.0556x_9x_{11} - 0.000786x_{11}x_{11}.$