Monotonicity Analysis, Evolutionary Multi-Objective Optimization, and Discovery of Design Principles

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Abstract. Optimization algorithms are routinely used to find the minimum or maximum solution corresponding to one or more objective functions, subject to satisfying certain constraints. However, monotonicity analysis is a process which, in certain problems, can instantly bring out important properties among decision variables corresponding to optimal solutions. As the name suggests, the objective functions and constraints need be monotonic to the decision variables or the objective function must be free from one or more decision variables. Such limitations in their scope is probably the reason for their unpopularity among optimization researchers. In this paper, we suggest a generic two-step evolutionary multi-objective optimization procedure which can bring out important relationships among optimal decision variables and objectives to linear or non-linear optimization problems. Although this "innovization" (innovation through optimization) idea is already put forward by the authors elsewhere [6], this paper brings out the similarities of the outcome of the proposed innovization task with that of the monotonicity analysis and clearly demonstrates the advantages of the former method in handling generic optimization problems. The results of both methods are contrasted by applying them on a specific engineering design problem, for which the computation of exact optimal solutions can be achieved. Besides showing the niche of the proposed multi-objective optimization based procedure in such important design tasks, this paper also demonstrates the ability of evolutionary optimization algorithms in finding the exact optimal solutions.

1 Introduction

Optimization algorithms are used to find optimal designs characterized by one or more criteria, while satisfying requirements on performance measures or available resources. Typical engineering optimization examples include minimizing the weight as well as maximizing reliability against failures of a system while satisfying constraint on resources such as material strength and physical space limitations. The task of an optimization algorithm is then to start from one or more random (or known) solution(s) and iteratively progress towards the set of theoretical Pareto-optimal solutions.

Users of optimization algorithms are mostly happy with finding one or more optimal solutions. But since these solutions are special solutions with certain important local optimality properties, an investigation of what constitutes them to become optimal may reveal important insights about the underlying problem. Such a dual task of optimization followed by a post-optimality analysis of solutions can then be viewed as a learning aid and may serve the users in a much better way than simply finding the optimal solutions. Monotonicity analysis [9] is a pre-optimization technique which can also be applied to obtain certain relationships among the decision variables of the optimal solutions without even performing any optimization task in problems having monotonic objective and constraint functions. However, due these restrictions on the problem, these methods are not popularly used.

When only one objective is used, usually there is only one optimal solution. Although this optimal solution provides an idea of the variable combinations for it to become an optimal, not much information can be derived from one single solution. However, when more than one conflicting objectives are considered, theoretically there are multiple Pareto-optimal solutions, each corresponding to the optimal solution of a particular trade-off among the objectives. Since all these solutions are optimal, an investigation of the common relationships among variables across most Pareto-optimal solutions should provide a plethora of information about 'what really constitutes an optimal solution?'. Elsewhere [6, 11], authors have suggested a systematic 'innovization' procedure by first finding a set of Pareto-optimal solutions using an evolutionary multi-objective optimization (EMO) algorithm and then analyzing the solutions for discovering such important relationships.

This paper brings out clearly the similarities in the outcome of the monotonicity analysis and innovization task and also shows the advantages of the innovization procedure on more generic problem solving tasks. Further, this paper also demonstrates the ability of the evolutionary algorithms in arriving at close to the exact optimal solutions in the case studies considered here.

2 Monotonicity Analysis

A monotonic optimization method called monotonicity analysis was suggested by Papalambros and Wilde [9] as a pre-optimization technique to determine if an optimization problem is well-bounded prior to resorting to a numerical optimization task. The technique is based on the following two rules:

Rule 1: If the objective function is monotonic with respect to a variable, then there exists at least one active constraint which bounds the variable in the direction opposite to the objective. A constraint is active if it acts at its lower or upper bound.

Rule 2: If a variable is not contained in the objective function then it must be either bounded from both above and below by active constraints or not actively bounded at all (that is, any constraint monotonic with respect to that variable must be inactive or irrelevant).

We now take a simple case study to illustrate the working principle of the monotonicity analysis.

3 A Case Study: Pressure Vessel Design

The pressure-vessel design problem [2] involves finding four design variables – radius of vessel (R), length of vessel (L), thickness of cylindrical part of the vessel (T_s) and thickness of the hemispherical heads (T_h) – for minimizing cost of fabrication (f_1) and maximizing the storage capacity (f_2) of the vessel. The total cost comprises of the cost of the material and cost of forming and welding. All four variables are treated as continuous. Denoting the variable vector $\boldsymbol{x} = (T_s, T_h, R, L)$, we write the two-objective optimization problem as follows:

$$
\begin{array}{rl}
\text{Minimize} & f_1(\boldsymbol{x}) = 0.6224 T_s L R + 1.7781 T_h R^2 + 3.1661 T_s^2 L + 19.84 T_s^2 R, \\
\text{Minimize} & f_2(\boldsymbol{x}) = -\left(\pi R^2 L + 1.333\pi R^3\right), \\
\text{Subject to} & g_1(\boldsymbol{x}) = 0.0193R - T_s \le 0, \quad g_2(\boldsymbol{x}) = 0.00954R - T_h \le 0, \\
& g_3(\boldsymbol{x}) = 0.0625 - T_s \le 0, \quad g_4(\boldsymbol{x}) = T_s - 5 \le 0, \\
& g_5(\boldsymbol{x}) = 0.0625 - T_h \le 0, \quad g_6(\boldsymbol{x}) = T_h - 5 \le 0, \\
& g_7(\boldsymbol{x}) = 10 - R \le 0, \quad g_8(\boldsymbol{x}) = R - 200 \le 0, \\
& g_9(\boldsymbol{x}) = 10 - L \le 0, \quad g_{10}(\boldsymbol{x}) = L - 240 \le 0.
\end{array}
\tag{1}
$$

3.1 Monotonicity Analysis for design principles

Monotonicity analysis is applied to reduce the number of variables, if possible, and simplify the problem for optimization. In this problem, the cost objective function increases monotonically with an increase in T_s and T_h, but the volume objective does not depend on these two variables. As T_s and T_h increases, constraint values g_1, g_2, g_3, and g_5 reduce. Hence, by Rule 1 of monotonicity analysis, g_1 or g_3 and g_2 or g_4 must be active at all Pareto-optimal solutions. Thus, the optimal solutions occur at $T_s = \max(0.0193R, 0.0625)$ and $T_h = \max(0.00954R, 0.0625)$. Since minimum possible R is 10, the optimal solutions must satisfy the following conditions:

$$T_s = 0.0193R, \quad T_h = 0.00954R.$$

Interestingly, such relationships about the *optimal* solutions of the two-objective problem can be obtained without even performing any optimization task.

Since f_1 increases monotonically with an increase in other two variables R and L, and f_2 decreases with an increase in them, the monotonicity rules cannot be applied to these variables. However, we can use the above relationships among T_s, T_h and R and reduce the original problem to the following two-variable optimization problem:

$$\begin{aligned}
\text{Minimize } & f_1(R,L) = 0.01319R^2L + 0.02435R^3, \\
\text{Minimize } & f_2(R,L) = -\left(\pi R^2 L + 1.333\pi R^3\right), \\
\text{Subject to } & g_1(R,L) = 10 - R \leq 0, \quad g_2(R,L) = R - 200 \leq 0, \\
& g_3(R,L) = 10 - L \leq 0, \quad g_4(R,L) = L - 240 \leq 0.
\end{aligned} \tag{2}$$

We construct a weighted-sum objective by multiplying f_1 with w_1 and f_2 with $(1 - w_1)$. Rearranging the terms, we have

$$\begin{aligned}
\text{Minimize } & f(R,L) = ((0.01319 + \pi)w_1 - \pi)R^2L + ((0.02435 + 1.333\pi)w_1 - 1.333\pi)R^3, \\
\text{Subject to } & g_1(R,L) = 10 - R \leq 0, \quad g_2(R,L) = R - 200 \leq 0, \\
& g_3(R,L) = 10 - L \leq 0, \quad g_4(R,L) = L - 240 \leq 0.
\end{aligned} \tag{3}$$

The above problem is solved by considering three different cases:

1. The function $f(R,L)$ increases monotonically with R and L, if the terms in brackets are positive. Simplifying, this implies $w_1 > 0.995819$. Hence, by Rule 1 of monotonicity analysis, R and L are bounded at their lower bounds. Hence $R = 10$ and $L = 10$ for $w_1 > 0.995819$.
2. The function $f(R,L)$ decreases monotonically with R and L, if the terms in brackets are negative, that is, when $w_1 < 0.994219$. Hence, by Rule 1 of monotonicity analysis, R and L are bounded at their upper bounds. Hence $R = 200$ and $L = 240$ for $w_1 < 0.994219$.
3. For $0.994219 \leq w_1 \leq 0.995819$, $f(R,L)$ is not monotonic with R but monotonically decreases with L. Hence, by Rule 1, L takes it's upper bound value or $L = 240$. To find the optimal R, KKT conditions can be applied to the single-variable (R) problem. The combined results are shown below:

w_1	L	R
0.0–0.994818	240	200
0.994818–0.995695	240	$160\frac{\pi-(\pi+0.01319)w_1}{(1.333\pi+0.02435)w_1-1.333\pi}$
0.995695–0.995819	240	10
0.995819–1.0	10	10

Substituting $L = 240$ in the expression for the cost and volume expressions in Equation 1, we obtain cost in terms of R at optimal configuration as below:

$$\text{Cost} = 0.02435R^3 + 3.1656R^2, \tag{4}$$

$$\text{Volume} = 1.333\pi R^3 + 240\pi R^2. \tag{5}$$

The cost-volume trade-off is shown in Figure 1 and the variation of R and L with w_1 are shown in Figure 2. The values of R and L for different w_1 are shown in Figure 2. The exact optimal R and L values are shown with the volume objective in Figure 3.

All solutions corresponding to $w_1 \geq 0.995819$ have the same values for R and L (Figure 2). These solutions correspond to a single point in Figure 1 and represent the minimum cost solution. Similarly, all solutions with $w_1 \leq 0.994818$ (corresponding to kink in R variation with w_1) also have the identical values (their upper limits) of R and L, and correspond to maximum volume solution in Figure 1. For optimal solutions with $0.994818 \leq w_1 \leq 0.995695$ (another transition kink in R), L takes a constant value of 240 and R varies from its lower bound to upper bound. Since L value suddenly jumps from 240 to 10, there is a discontinuity in the Pareto-optimal front which is clearly shown in Figure 1. However, there is no discontinuity in the objectives due to the parameter R, as is also clear from Figure 2.

3.2 Innovative Design Principles

We summarize all the innovative design principles that have been obtained for this problem:

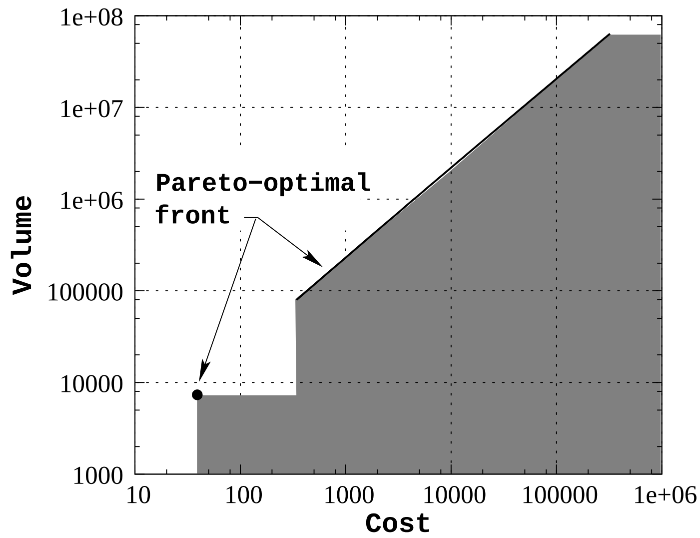

Fig. 1. Pareto-optimal front for the pressure vessel design problem.

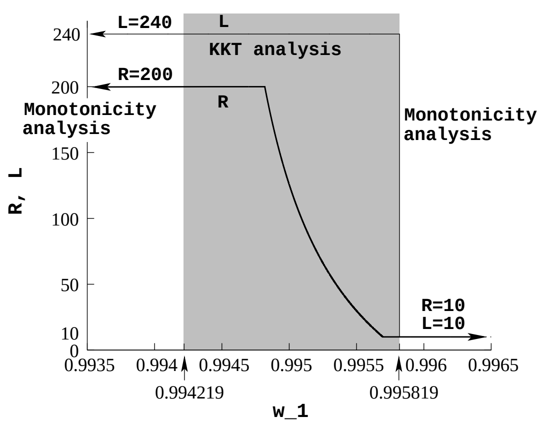

Fig. 2. Different cases of optimality with w_1.

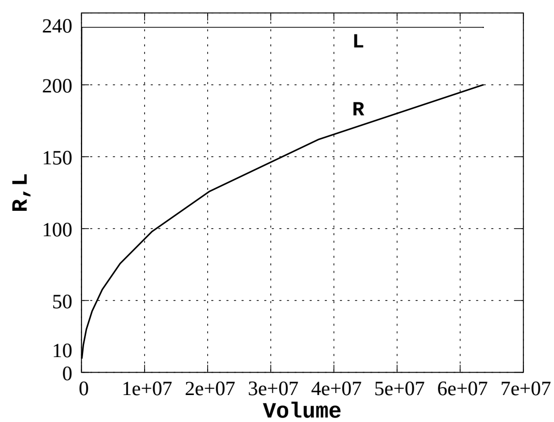

Fig. 3. Optimal R and L values with volume.

1. The thickness of shell, T_s is proportional to R for all solutions and varies as $T_s = 0.0193R$.
2. The thickness of head, T_h is also proportional to R for all solutions and varies as $T_h = 0.00954R$.
3. The optimal value for the length of the pressure vessel (L) can only be either 10 (for minimum cost solution) or 240 (for any other trade-off between cost and volume). Thus, in general, a recipe to have an optimal design is to fix the length of pressure vessel to 240 (upper bound).
4. Radius of the pressure vessel R takes the value 10 for minimum cost solution, 240 for maximum volume solution and must be increased with for a larger cost design in a particular manner, as given by Equation 4 (also shown later in Figure 8).

Since this pressure vessel design problem involves differentiable functions, the monotonicity analysis followed by a mathematical optimality consideration is possible to be applied easily. This problem has been simplified to a large extent by the monotonicity analysis which has finally reduced the problem to one of a single variable. Monotonicity analysis thus simplifies a problem (if Rule 1 or Rule 2 or both are applicable) and at the same time can be used to obtain partial design principles in some cases without performing any optimization. Since it is a prior optimization technique, it is computationally efficient procedure.

4 Modified Pressure Vessel Design

We now re-formulate the above problem in a slightly different manner to construct a problem which may not be possible to be solved by the monotonicity analysis. The pressure vessel problem is slightly modified by making the unit cost of welding and the unit cost of forming depending on R and L, respectively. This makes the problem more pragmatic, as the overall welding cost comprising of weld material and labor costs actually depends on length of weld, thereby involving an additional dimension (length and radius) in third and fourth terms of f_1, respectively. The modified problem is stated below:

$$\begin{array}{rl}
\text{Minimize} & f_1(\boldsymbol{x}) = 0.6224T_sLR + 1.7781T_hR^2 + (3.1661/2)T_s^2L^2 + (19.84/2)T_s^2R^2, \\
\text{Minimize} & f_2(\boldsymbol{x}) = -\left(\pi R^2 L + 1.333\pi R^3\right), \\
\text{Subject to} & g_1(\boldsymbol{x}) = 0.0193R - T_s \le 0, \quad g_2(\boldsymbol{x}) = 0.00954R - T_h \le 0, \\
& g_3(\boldsymbol{x}) = 0.0625 - T_s \le 0, \quad g_4(\boldsymbol{x}) = T_s - 5 \le 0, \\
& g_5(\boldsymbol{x}) = 0.0625 - T_h \le 0, \quad g_6(\boldsymbol{x}) = T_h - 5 \le 0, \\
& g_7(\boldsymbol{x}) = 10 - R \le 0, \quad g_8(\boldsymbol{x}) = R - 200 \le 0, \\
& g_9(\boldsymbol{x}) = 10 - L \le 0, \quad g_{10}(\boldsymbol{x}) = L - 240 \le 0.
\end{array} \tag{6}$$

The weighted combination of the above two objectives is still monotonic with respect to T_s and T_h. The problem can be thus reduced to a two-variable problem, as in Equation 7:

$$\begin{array}{rl}
\text{Minimize} & f(R,L) = ((0.012012+\pi)w_1 - \pi)R^2L + ((0.016963 + 1.333\pi)w_1 - 1.333\pi)R^3 \\
& +w_1(0.0005897R^2L^2 + 0.0036951R^4), \\
\text{Subject to} & g_1(R,L) = 10 - R \le 0, \quad g_2(R,L) = R - 200 \le 0, \\
& g_3(R,L) = 10 - L \le 0, \quad g_4(R,L) = L - 240 \le 0.
\end{array} \tag{7}$$

However, the monotonicity analysis cannot be applied to find the Pareto-optimal solutions to this problem as the function is no more monotonic with L for all values of w_1. Although the quadratic term for L is always positive, the linear term can be negative for some values of w_1. Thus, monotonicity analysis has been rendered inapplicable and a KKT-approach must be used to the above two-variable problem for finding the optimal solutions. We show later that this modified pressure vessel problem has also possess important and innovative design principles which are not possible to be found by the monotonicity analysis. In real world problems involving a large number of variables and objectives, it is unlikely that the monotonicity analysis would be applicable. Thus, there is a need for a more generalized procedure which would find the optimal design principles for all kinds of problems. The next section presents a newly suggested *innovization* procedure [7, 11] as a generalized procedure for discovering important design principles. However, we argue here that if a problem allows the monotonicity analysis to be applied, by all means it should be used to reduce the problem complexity as much as possible. The remaining problem can then be handled by using the suggested innovization task, which we briefly describe next.

5 Innovization Procedure

Innovization [6] − innovation through optimization − is a task which attempts to first find a set of trade-off optimal solutions in a problem and then unveils important properties which lies common to these multiple optimal solutions. The analysis of the optimized solutions will result in worthwhile design principles, if only the trade-off solutions are close to the true Pareto-optimal solutions. Since for engineering and complex scientific problem-solving, we need to use a numerical optimization procedure and since in such problems, the exact optimum is not known a priori, adequate experimentation and verification must have to be done first to gain confidence about the closeness of the obtained solutions to the actual Pareto-optimal front. For this purpose, we first use the well-known elitist non-dominated sorting genetic algorithm or NSGA-II [5] as the multi-objective optimization tool. NSGA-II begins its search with a random population of solutions and iteratively progresses towards the Pareto-optimal front so that at the end of a simulation run, multiple trade-off optimal solutions are obtained simultaneously. Due to its simplicity and efficacy, NSGA-II is adopted in a number of commercial optimization softwares and has been extensively applied to various multi-objective optimization problems in the past few years. For a detail procedure of NSGA-II, readers are referred to the original study [5]. The NSGA-II solutions are then clustered to identify a few well-distributed solutions. The clustered NSGA-II solutions are then modified by using a local search procedure (we have used Benson's method [1,4] here) to obtain the exact (or close to exact) Pareto-optimal front. Further the obtained NSGA-II-cum-local-search solutions are verified by different single-objective procedures one at a time. We present the proposed innovization procedure here:

Step 1: Find individual optimum solution for each of the objectives by using a single-objective GA (or sometimes using NSGA-II by specifying only one objective) or by a classical method. Thereafter, note down the *ideal* point.
Step 2: Find the optimized multi-objective front by NSGA-II.
Step 3: Normalize all objectives using ideal and nadir points and cluster a few solutions $Z^{(k)}$ ($k = 1, 2, \ldots, 10$), preferably in the area of interest to the designer or uniformly along the obtained front.
Step 4: Apply a local search (Benson's method [1] is used here) and obtain the modified optimized front.
Step 5: Perform the normal constraint method (NCM) [8] starting at a few locations to verify the obtained optimized front. These solutions constitute a reasonably confident optimized front.
Step 6: Analyze the solutions for any commonality principles as plausible innovized relationships.

In all simulations here, we have used a population size of 2,000 and run NSGA-II for 2,000 generations. These rather large values are used to achieve near-optimal solutions. The crossover and mutation probabilities of 0.9 and 0.25 are used. The distribution indices of SBX and polynomial operators [4] are 3 and 80, respectively.

5.1 Original Pressure Vessel Design

The innovization procedure is illustrated for the original pressure vessel problem discussed in Section 3. First, we find the individual minimum of each objective. The extreme solutions for the above two objectives are found using two single-objective optimization of each objective by NSGA-II and the results are shown in Table 1. Both these extreme solutions match with the extreme solutions computed

Table 1. The individual optimal solutions for the original pressure vessel design problem obtained using innovization procedure.

Solution	T_s	T_h	R	L	f_1	f_2
Min. Cost	0.2	0.0973	10	10	38.982	7329.341
Max. Volume	3.871	1.910	200	240	$3.223(10^5)$	$6.366(10^7)$

using the monotonicity and mathematical optimality conditions earlier.

Next, we find a set of Pareto-optimal solutions using NSGA-II. The Pareto-optimal front obtained through the use of NSGA-II is shown in the Figure 4.

Next, we verify the near-optimality of this front by using several single-objective optimizations. We find a number of intermediate optimal solutions using NCM and obtained solutions are plotted in Figure 4. We observe that the NCM solutions lie on the Pareto-optimal front obtained by NSGA-II. This gives us confidence about the optimality of the obtained NSGA-II Pareto-optimal front.

Now we are ready to analyze the NSGA-II solutions for any common principles and we make the following observations.

1. We observe that for all solutions the constraints g_1 and g_2 are active, thereby making following relationships among decision parameters as follows:

$$T_s = 0.0193R, \tag{8}$$
$$T_h = 0.00954R. \tag{9}$$

Recall that the monotonicity analysis also found the above relationships by using Rule 1 on T_s and T_h. Here, we discover the same properties by analyzing the NSGA-II solutions.

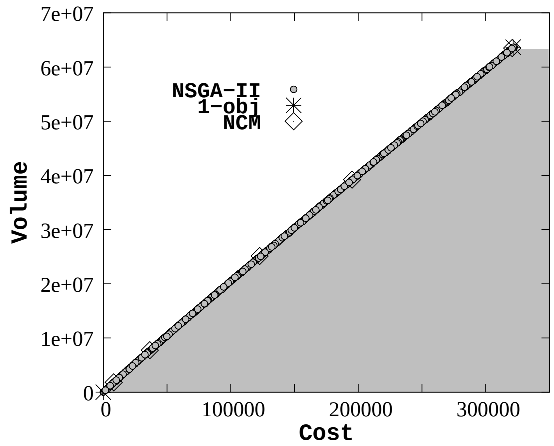

Fig. 4. Pareto-optimal front obtained using NSGA-II.

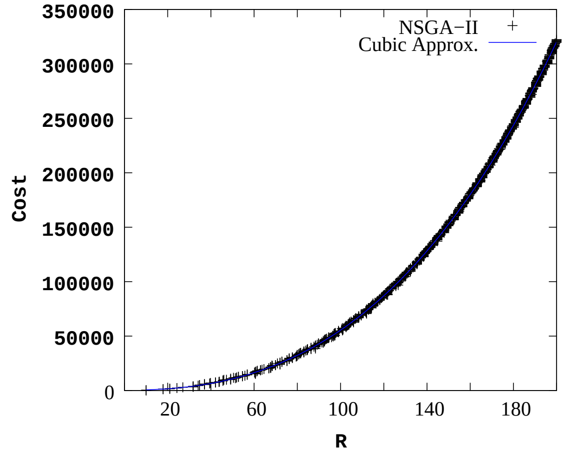

Fig. 5. Cost versus R, NSGA-II solutions and it's cubic approximation.

2. L takes the value 240 for all solutions except for the minimum cost solution where $L = 10$.
3. The variation of R with cost is approximated with a polynomial and we find that a third-degree polynomial fits well the relationship. The approximate polynomial and the actual points from NSGA-II are shown in Figure 5. The variation between R and cost obtained by our cubic approximation and the one obtained by exact analysis (Section 3) are shown below:

$$\text{Cost} = 0.02439R^3 + 3.2683R^2 - 17.74R + 473.87 \quad \text{(Innovized relation)}, \tag{10}$$
$$\text{Cost} = 0.02435R^3 + 3.1656R^2 \quad \text{(Exact relation)}. \tag{11}$$

On comparison, we find that the above results match closely with the ones obtained from the exact analysis shown earlier.

5.2 Modified Pressure Vessel Design

Next, we apply the innovization procedure to the modified pressure vessel problem (Section 4). As discussed earlier, the monotonicity analysis is not applicable to this problem.

As per the steps involved in the innovization procedure, we first find the individual minima using NSGA-II and the results are shown in Table 2. Next, the Pareto-optimal front is obtained using NSGA-II and is shown in Figure 6 along with the extreme solutions obtained before. These optimized solutions are then verified using the Normal Constraint Method (NCM) and the NCM results are also plotted in the Figure 6. The agreement of all such solutions with each other gives us confidence in the optimality of obtained solutions.

Table 2. The extreme solutions for the modified pressure vessel design problem using NSGA-II.

Solution	T_s	T_h	R	L	f_1	f_2
Min. Cost	0.193	0.0964	10	10	72.092	7329.34
Max. Volume	3.863	1.909	200	240	$7.533(10^6)$	$6.366(10^7)$

The Pareto-optimal solutions are analyzed for any common principles and the outcomes are summarized below:

1. The T_s and T_h relationships with R are the same as before. The Pareto-optimal front extends over a larger range of cost value, as compared to that for the original problem.

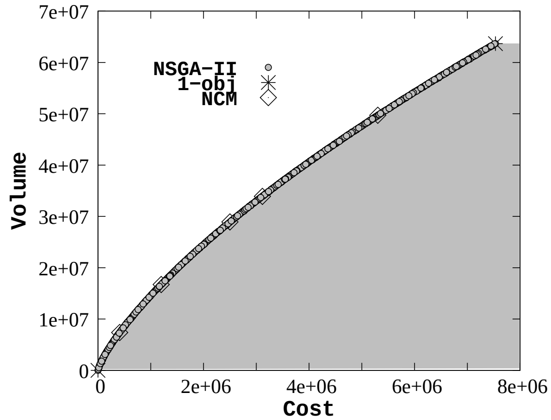

Fig. 6. Pareto-optimal front obtained using NSGA-II for the modified problem.

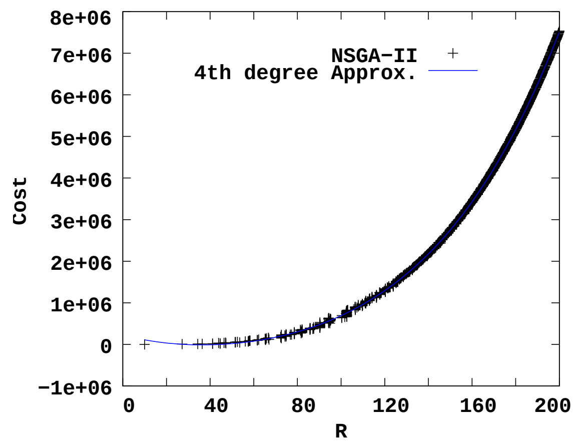

Fig. 7. Cost versus radius for optimal trade-off solutions for the modified problem.

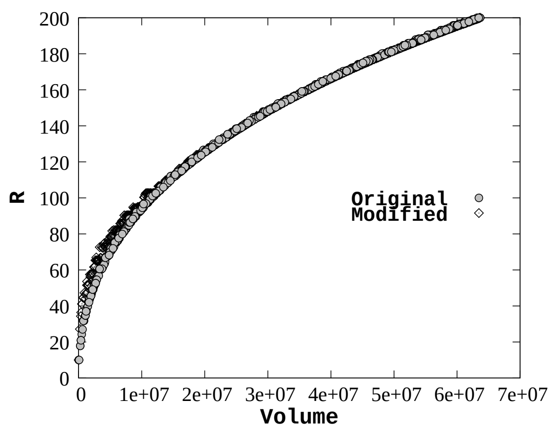

Fig. 8. Radius versus volume relationships for all trade-off optimal solutions for the original and modified problems.

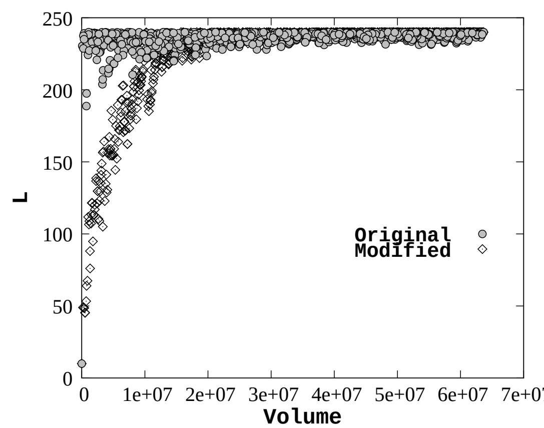

Fig. 9. Length versus volume relationships for all trade-off optimal solutions for the original and modified problems.

2. Cost varies with radius (R) of the vessel as a polynomial function, as shown in Figure 7. This function is different from the one obtained in the original pressure vessel design problem in that the new relationship is a fourth-degree polynomial:

$$\text{Cost} = 0.005475R^4 - 0.9674R^3 + 225.056R^2 - 13621R + 223202, \tag{12}$$
$$\text{Volume} = 2.79396R^3 + 1219R^2 - 35996.9R - 461463. \tag{13}$$

The variation of R with volume is slightly different in the modified problem, as shown in Figure 8. The variation of R for the original problem is similar to that shown exactly in Figure 3.

3. The most significant way the solutions vary between the two problems is in the way L varies. Instead of remaining constant to $L = 240$ with desired volume, the length of the pressure vessel in the modified problem gradually needs to be changed from its lower bound to its upper bound before remaining constant at its upper bound of $L = 240$ for rest of the Pareto-optimal solutions. This variation is depicted in Figure 9. In the original problem, there was only one solution with L value at its lower bound ($L = 10$ in the extreme cost solution) and L was fixed near to its upper bound of 240 for all other solutions in the Pareto-optimal front. The variation of L is similar to that found exactly in Figure 3.

We observe that the suggested innovization procedure is quite generic and solves the modified pressure vessel problem which could not be solved by the monotonicity analysis.

6 Exact Optima with Evolutionary Algorithms

Evolutionary algorithms with local search can lead to exact optima for most problems, although there does not exist any such mathematical proof for any EA and for a limited number of evaluations. However, a hybrid methodology of using an EA and then following with a local search can lead to the true optimum or near to it with a limited number of evaluations. For multi-objective optimization, in addition to the convergence to the true Pareto-optimal front, the spread of solutions over the entire true Pareto-optimal front must also be verified.

As mentioned earlier, we used Benson's method as a local search procedure in this study. The solutions obtained after local search with the NSGA-II solutions are also verified by two independent procedures:

1. The extreme Pareto-optimal solutions are verified by running a single-objective optimization procedure (an evolutionary algorithm is used here) independently on each objective function, subjected to satisfying given constraints.
2. Some intermediate Pareto-optimal solutions are verified by using the normal constraint method (NCM) [8] starting at different locations on the hyper-plane constructed using the individual best solutions obtained from the previous step.

Solutions obtained through such a systematic evaluation procedure are very close to the exact Pareto-optimal solutions and thus give us enough confidence on any further analysis (in this paper the discovery of design principles) carried with these solutions. We illustrate here with the help of an engineering example that the solutions obtained by such a systematic study actually satisfy mathematical conditions of optimality, namely Karush-Kuhn-Tucker (KKT) conditions and hence are truly optimal solutions.

6.1 Welded Beam Design

The welded beam design problem is well studied in the context of single-objective optimization [10]. A beam needs to be welded on another beam and must carry a certain load F. It is desired to find four design parameters (thickness of the beam, b, width of the beam t, length of weld ℓ, and weld thickness h) for which the cost of the beam is minimum and simultaneously the vertical deflection at the end of the beam is minimum. The overhang portion of the beam has a length of 14 in and $F = 6,000$ lb force is applied at the end of the beam. It is intuitive that a design which is optimal from the cost consideration is not optimal from rigidity consideration (or end-deflection) and vice versa. Such conflicting objectives lead to interesting Pareto-optimal solutions. In the following, we present the mathematical formulation of the two-objective optimization problem of minimizing cost and the end deflection [3]:

$$
\begin{array}{rl}
\text{Minimize} & f_1(\boldsymbol{x}) = 1.10471h^2\ell + 0.04811tb(14.0+\ell),\\
\text{Minimize} & f_2(\boldsymbol{x}) = \frac{2.1952}{t^3 b},\\
\text{Subject to} & g_1(\boldsymbol{x}) \equiv 13,600 - \tau(\boldsymbol{x}) \geq 0,\\
& g_2(\boldsymbol{x}) \equiv 30,000 - \sigma(\boldsymbol{x}) \geq 0,\\
& g_3(\boldsymbol{x}) \equiv b - h \geq 0,\\
& g_4(\boldsymbol{x}) \equiv P_c(\boldsymbol{x}) - 6,000 \geq 0,\\
& 0.125 \leq h, b \leq 5.0,\\
& 0.1 \leq \ell, t \leq 10.0.
\end{array}
\tag{14}
$$

There are four constraints. The first constraint makes sure that the shear stress developed at the support location of the beam is smaller than the allowable shear strength of the material (13,600 psi). The second constraint makes sure that normal stress developed at the support location of the beam is smaller than the allowable yield strength of the material (30,000 psi). The third constraint makes sure that thickness of the beam is not smaller than the weld thickness from a practical standpoint. The fourth constraint makes sure that the allowable buckling load (along t direction) of the beam is more than the applied load F. A violation of any of the above four constraints will make the design unacceptable. The stress and buckling terms are highly non-linear to design variables and are given as follows [10]:

$$\begin{aligned}
\tau(\boldsymbol{x}) &= \sqrt{(\tau')^2 + (\tau'')^2 + (\ell\tau'\tau'')/\sqrt{0.25(\ell^2 + (h+t)^2)}},\\
\tau' &= \frac{6,000}{\sqrt{2}h\ell},\\
\tau'' &= \frac{6,000(14 + 0.5\ell)\sqrt{0.25(\ell^2 + (h+t)^2)}}{2\left\{0.707h\ell(\ell^2/12 + 0.25(h+t)^2)\right\}},\\
\sigma(\boldsymbol{x}) &= \frac{504,000}{t^2 b},\\
P_c(\boldsymbol{x}) &= 64,746.022(1 - 0.0282346t)tb^3.
\end{aligned}$$

Table 3 presents the two extreme solutions obtained by the single-objective genetic algorithm (GA) and also by NSGA-II. Figure 10 shows these two extreme solutions and a set of Pareto-optimal solutions obtained using NSGA-II. The obtained front is verified by finding a number of Pareto-optimal solutions

Table 3. The extreme solutions for the welded-beam design problem.

Solution	x_1 (h) (in)	x_2 (ℓ) (in)	x_3 (t) (in)	x_4 (b) (in)	f_1	f_2 (in)
Min. Cost	0.2443	6.2151	8.2986	0.2443	2.3815	0.0157
Min. Deflection	1.5574	0.5434	10.0000	5.0000	36.4403	$4.3904(10^{-4})$

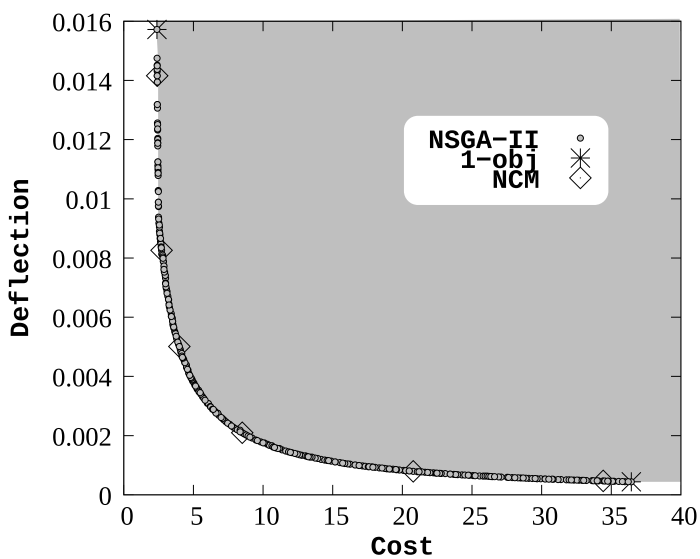

Fig. 10. NSGA-II solutions are shown for the welded-beam design problem.

using the NC method. We see that the NC solutions match with the obtained Pareto-optimal solutions. We further perform mathematical verification of the Pareto-optimal solutions by testing a few solutions for Karush-Kuhn-Tucker (KKT) conditions. These conditions require a vector norm to be equal to zero.

The KKT conditions are shown below for the welded beam problem for a set of weights w_1 and w_2.

$$\left.\begin{array}{r l}
& w_1\nabla f_1(\boldsymbol{x}) + w_2\nabla f_2(\boldsymbol{x}) - \sum_{j=1}^{12} u_j \nabla g_j(\boldsymbol{x}) = 0, \\
& u_j g_j(\boldsymbol{x}) = 0, u_j \geq 0, \text{ for all } j = 1, 2..., 12. \\
\text{where} & f_1(\boldsymbol{x}) = 1.10471h^2\ell + 0.04811tb(14.0+\ell), \\
& f_2(\boldsymbol{x}) = \frac{2.1952}{t^3 b}, \\
& g_1(\boldsymbol{x}) \equiv 13,600 - \tau(\boldsymbol{x}) \geq 0, \\
& g_2(\boldsymbol{x}) \equiv 30,000 - \sigma(\boldsymbol{x}) \geq 0, \\
& g_3(\boldsymbol{x}) \equiv b - h \geq 0, \\
& g_4(\boldsymbol{x}) \equiv P_c(\boldsymbol{x}) - 6,000 \geq 0, \\
& g_5(\boldsymbol{x}) \equiv h - 0.125 \geq 0, \\
& g_6(\boldsymbol{x}) \equiv 5 - h \geq 0, \\
& g_7(\boldsymbol{x}) \equiv b - 0.125 \geq 0, \\
& g_8(\boldsymbol{x}) \equiv 5 - b \geq 0, \\
& g_9(\boldsymbol{x}) \equiv \ell - 0.1 \geq 0, \\
& g_{10}(\boldsymbol{x}) \equiv 10 - \ell \geq 0, \\
& g_{11}(\boldsymbol{x}) \equiv t - 0.1 \geq 0, \\
& g_{12}(\boldsymbol{x}) \equiv 10 - t \geq 0.
\end{array}\right\} \quad (15)$$

We randomly choose two points from the Pareto-optimal front and check for the above conditions and the values of different parameters are shown in Table 4. We also perform the above check for the two extreme solutions. We observe from the table that all u_j are positive. Also, the norm of the left term of the first KKT condition is close to zero for all four points. All four solutions satisfy KKT conditions and hence give enough confidence on the optimality of the obtained Pareto-optimal solutions. Thus, we argue that the solutions obtained are truly optimal or near-optimal.

Table 4. KKT analysis of four NSGA-II points for the welded-beam design problem.

Parameter	Point 1	Point 2	Minimum Cost sol.	Minimum Deflection sol.
$f_1(\boldsymbol{x})$	6.4298	14.2321	2.3815	36.4403
$f_2(\boldsymbol{x})$	0.0028	0.0012	0.0157	$4.3904(10^{-4})$
Norm	$1.4816(10^{-5})$	0.0023	$2.4506(10^{-7})$	$3.2393(10^{-6})$
u_1	0.0008	0.0062	1.3780	0
u_2	0	0	0.2080	0.0196
u_3	0	0	0.5966	0
u_4	0	0	0.3892	0
u_5	0	0	0	0
u_6	0	0	0	0
u_7	0	0	0	0
u_8	0	0	0	0
u_9	0	0	0	0.00022
u_{10}	0	0	0	0
u_{11}	0	0	0	0
u_{12}	0.0011	0.0005	0	0

7 Conclusions

This paper clearly presents the similarities between the innovization procedure and the monotonicity analysis and also brings out the generality of the former through the help of an engineering example. However, since monotonicity analysis is a computationally efficient procedure, it must always be preferred wherever found applicable. The innovization procedure and monotonicity analysis can be combined to obtain an efficient procedure for discovering innovative design principles. Further, this paper illustrates that evolutionary algorithms coupled with a hybrid search procedure (the procedure adopted here for the innovization analysis) can be used to find true Pareto-optimal solutions. This is shown by solving an engineering design problem having nonlinear and non-convex but differentiable objective and constraint functions. In our opinion, this paper makes impact in two ways. On one hand, it suggests and demonstrates a generic procedure for deciphering important properties among optimal solutions compared to the monotonicity analysis. On the other hand, this paper should also bring evolutionary algorithms closer to mathematical optimization community in showing exactness of single and multi-objective evolutionary algorithms in finding true optimal solutions in problems allowing to find the exact optimal solutions by mathematical optimality conditions.

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