DESIGN OF OPTIMUM CROSS-SECTIONS FOR LOAD-CARRYING MEMBERS USING MULTI-OBJECTIVE EVOLUTIONARY ALGORITHMS

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ABSTRACT

To reduce the induced undesired stresses, a power transmitting shaft or a load carrying beam should have its related area moment of inertia as large as possible. Similarly, to avoid early buckling and hence, to increase the capacity, a compressive load carrying strut should have a radius of gyration as large as possible. As the radius of gyration is directly proportional to the square root of area moment of inertia, a strut also should have its related area moment of inertia as large as possible. However, an increase in such moment of inertias comes with an increase in the transverse cross-sectional areas and hence, the weight of the members. Therefore, the maximization of moment of inertias should not take place at the cost of excessive weights of the members. Attempt has been made here to design optimum cross-sections for such load-carrying members, using a multi-objective evolutionary algorithm, for simultaneously maximizing moment of inertias and minimizing the cross-sectional areas. The success of the work has been shown through a few case studies.

Keywords

Multi-objective optimization, evolutionary algorithms, NSGA-II, moment of inertia, radius of gyration.

1. INTRODUCTION

Shafts used to transmit power or torque in machines, engines or any other rotating mechanism, suffer from the action of shear stresses (tangential stresses). Though these stresses are not desired, but can not be avoided. However, effect of such stresses can be reduced by properly designing a shaft. The induced shear stress ($\boldsymbol{\tau}$) is given as $\boldsymbol{\tau} = \boldsymbol{T}r/J$, where $\boldsymbol{T}$ is the mechanical torque to be transmitted, r is the radial distance from the neutral axis of the shaft to the fiber where the shear stress is $\boldsymbol{\tau}$, and J is the polar area moment of inertia of transverse cross-section of the shaft [1, 2]. Hence, it is seen that for given $\boldsymbol{T}$ and r, $\boldsymbol{\tau}$ can be minimized by maximizing J. Similar cases arise in case of beams used to carry loads or moments. Such a beam suffers from the action of undesired bending stress ($\boldsymbol{\sigma}$) that can be expressed as $\boldsymbol{\sigma} = \boldsymbol{M}y/I$, where $\boldsymbol{M}$ is the bending moment to be supported, y is the vertical distance from the neutral axis of the beam to the fiber where the bending stress is $\boldsymbol{\sigma}$ and I is the rectangular area moment of inertia of transverse cross-section of the beam [1, 2]. Here also, for given $\boldsymbol{M}$ and y, $\boldsymbol{\sigma}$ can be minimized by maximizing I. On the other hand, a strut used in different structures to support compressive loads, has a limited capacity and it starts buckling beyond that capacity. The capacity of a simply-hinged strut is given as $\boldsymbol{P} = \pi^2 EAk^2/l^2$, where E is the modulus of elasticity of the strut material, A is the transverse cross-sectional area, k is the radius of gyration and l is the length of the strut. Also, $Ak^2 = I$, where I is one of the rectangular area moment of inertias for which k is minimum [1, 2]. Hence, the capacity, $\boldsymbol{P}$, of the strut can be maximized by maximizing I. But, the cross-sectional area and hence, the weight of such a shaft, beam or strut will also increase as the moment of inertia is increased. Hence, it becomes a tedious job to get a cross-section that will have maximum moment of inertia and minimum or fixed area. Attempt has been made here to design such cross-sections. In few cases, a moment of inertia, either polar or a rectangular, is maximized and at the same time the cross-sectional area is minimized. These types of cross-sections are suitable for shafts or beams. As the capacity of a strut depends directly on the minimum rectangular moment of inertia, a few studies are made to maximize both the rectangular moment of inertias for fixed cross-sectional area. Hence, all the cases involve two objective functions of either maximizing one of the moment of inertias and minimizing area or maximizing both the rectangular moment of inertias for limited area. The multi-objective evolutionary algorithms are among the best methods for solving such problems [3]. One of such algorithms, called *Non-dominated Sorting Genetic Algorithm-II* (NSGA-II) [4], is used in the present work and a few obtained results are presented here.

2. PROBLEM FORMULATION

A multi-objective optimization problem for maximizing a moment of inertia (I) and minimizing the area (A) of the transverse cross-section of a load carrying member can be expressed in brief as:

$$\left.\begin{array}{rl} \text{Maximize} & I, \\ \text{Minimize} & A, \\ \text{Subject to} & A^{(l)} \leq A \leq A^{(u)}, \end{array}\right\} \quad (1)$$

where $A^{(l)}$ and $A^{(u)}$ are, respectively, lower and upper limits on the area A.

Similarly, the problem of maximizing the rectangular moment of inertias (I_x and I_y) for fixed area (A) can be expressed as:

$$\left.\begin{array}{rl} \text{Maximize} & I_x, \\ \text{Maximize} & I_y, \\ \text{Subject to} & A^{(l)} \leq A \leq A^{(u)}. \end{array}\right\} \quad (2)$$

The problem in hand is a topology optimization problem. A rectangular block is taken as a quadrant of a symmetric rectangular cross-section of a load-carrying member.

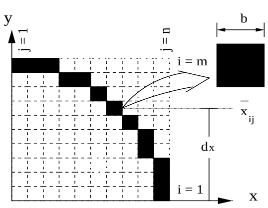

Figure 1: Quadrant of a symmetric rectangular cross-section

As shown in Figure 1, the block is then discretized into a number of small rectangular elements having equal dimensions. A filled element indicates having material and an empty element indicates a void in the cross-section [5]. A small element of horizontal breadth b and vertical depth d, can be represented by x_{ij}, where the suffices i and j are the row and column numbers of the element from the x- and y-axes respectively. x_{ij} also represents the material status in the element as:

$$x_{ij} = \begin{cases} 1 & \text{if the element is filled with material,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Now, the rectangular moment of inertia (dI_x) of the elemental area of i-th row and j-th column, about the x-axis can be given as:

$$dI_x = \left[\frac{1}{12}bd^3 + ad_x^2\right] x_{ij} = bd^3\left[\frac{1}{12} + \left(i - \frac{1}{2}\right)^2\right] x_{ij}, \quad (4)$$

where a = area of each small element = bd,
d_x= Distance from the x-axis to the centroidal x-axis ($\overline{x}_{ij}$) of the element,
$= (i - \frac{1}{2})d$.

Hence, the rectangular moment of inertia (I_x) of the whole block about the x-axis can be expressed as:

$$I_x = bd^3 \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{1}{12} + \left(i - \frac{1}{2}\right)^2\right] x_{ij}, \quad (5)$$

where m and n are, respectively, total number of rows and columns of small elements.

Similarly, the rectangular moment of inertia (I_y) of the whole block about the y-axis can be expressed as:

$$I_y = b^3 d \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{1}{12} + \left(j - \frac{1}{2}\right)^2\right] x_{ij}. \quad (6)$$

As the polar moment of inertia (J_z) is the summation of rectangular moment of inertias (I_x and I_y), it can be expressed as:

$$J_z = I_x + I_y, \quad (7)$$

where I_x and I_y are given by Eqns. (5) and (6) respectively.

Now, the area (A) of the whole block can be given as:

$$A = bd \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}. \quad (8)$$

Hence, I in Eqn.(1) can be any of I_x, I_y or J_z, given by Eqn.(5), (6) or (7) respectively. The design variables for the optimization problem are x_{ij}, $i = 1, 2, ...m$ and $j = 1, 2, ..n$. Since x_{ij}'s are allowed to take any of the two values of 0 or 1 only, and Eqns.(5) to (8) are linear in x_{ij}, the optimization problem in hand becomes an integer linear multi-objective programming problem.

3. METHODOLOGY

There exist a number of algorithms for handling integer programming problems [6] as described in Section 2. However, many of them suffer from either computational complexity or more than one objective functions. Among them, evolutionary algorithm is the best which can easily handle multiple objective functions as well as integer variables [3]. Such a multi-objective evolutionary algorithm, *Non-dominated Sorting Genetic Algorithm-II* (NSGA-II), is chosen for solving the optimization problem as given by either Eqn.(1) or Eqn.(2), which is described in details through Eqns.(3) to (8). *Tournament selection operator* is used for creating mating pool for crossover [7, 8]. Constraints are handled using *constraint-domination principle* outlined in [3]. The *mutation operator* is used as a local search operator and for maintaining diversity among solutions [9]. However, the following modifications have been made in the use of NSGA-II:

1. Boolean design variables (x_{ij}) are used to represent material status in a cross-section,

2. The largest interconnected cross-section is used for evaluation purpose, and

3. A two-dimensional binary crossover operator is used to recombine two cross-sections.

3.1 Handling of Design Variables

Design variables (x_{ij}'s) are used here to represent the positions as well as the material status of small elements of the cross-section. Variable x_{ij} represents the element of i-th row and j-th column with the numerical values specified in Eqn.(3). Since x_{ij}'s are allowed to take any of the two values of 0 and 1 only, they have been used as single bit binary variables.

3.2 Clustering a Solution

A solution of the optimization problem, given by either Eqn.(1) or Eqn.(2), represents a quadrant of a symmetric cross-section of a load carrying member. Since a small element of the cross-section, represented by design variable x_{ij}, may be a void or filled with material, it is essential that all the material-filled elements must be connected with each other to represent a valid cross-section as shown in Figure 1. However, an intermediate solution in NSGA-II may not be such a solution, but one where material-filled elements are disconnected from one another. In that case, a solution is searched among different groups of connected elements that are filled with material, and the biggest one of such groups is accepted as the searched solution [10]. Moreover, since a symmetric cross-section is considered and only a quadrant of it is taken for optimization, the final cross-section will be a disconnected one if the optimized quadrant does not touch the middle edges of the cross-section (x- and y-axes as shown in Figure 1). In that case, the accepted group of elements is forced to meet the edges. This is done by first searching the row or column which contains a material-filled element nearest to an edge, and then filling with materials the remaining empty elements of that row or column until the edge is reached.

3.3 Two-Dimensional Binary Crossover Operator

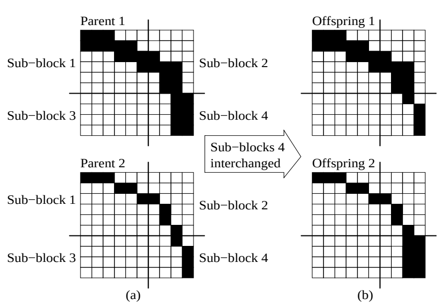

Figure 2: Two-Dimensional Binary Crossover

A two-dimensional binary crossover operator has been designed here to crossover between two two-dimensional rectangular blocks, representing two quadrants of two symmetric rectangular cross-sections. In this operator, two random integer numbers are generated in the range of (0, row) and (0, col) to get the crossover sites along row and column respectively. Here row and col represent the total number of rows and columns of small elements in a block. These two random numbers will divide a block into four sub-blocks as shown in Figure 2(a). Then a third random integer number, in the range of [1, 4], is generated to decide which sub-block is to be interchanged (Figure 2(b)).

4. RESULTS AND DISCUSSIONS

For the optimization problem given by either Eqn.(1) or Eqn.(2), which is described in details through Eqns.(3) to (8), the modified NSGA-II, as mentioned in Section-3 above, has been simulated for different cases using the following genetic algorithm (GA) parameters:

Population size	: 100
Number of generations	: 400
Type of crossover	: Two-dimensional binary crossover
Crossover probability	: 0.8
Mutation probability	: 0.01

Case-1: Maximization of Rectangular Moment of Inertia and Minimization of Area of a Symmetric Square Cross Section: A quadrant of a symmetric square cross-section of area 20x20 unit^2 is taken for study. The quadrant is discretized into 10x10 small square elements each of 1 unit^2 area and unconstrained optimization is performed for two objective functions: maximization of rectangular moment of inertia about x-axis (centroidal x-axis of the entire cross-section) and minimization of total area of the quadrant. The Pareto-optimal front and a few selective optimized cross-sections are shown in Figure 3 and Figure 4, respectively.

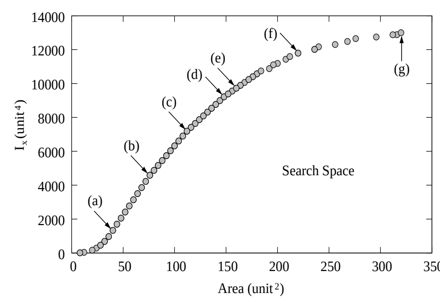

Figure 3: Pareto-optimal front for Case-1

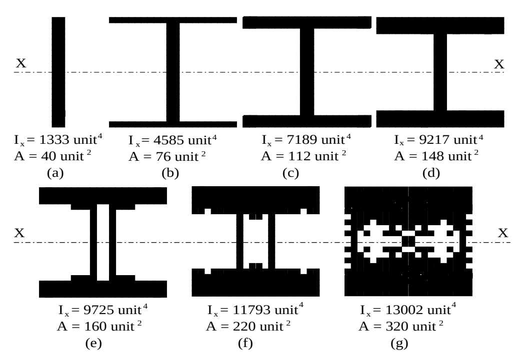

Figure 4: Optimum cross-sections for Case-1

It is seen that an optimized cross-section for maximum rectangular moment of inertia and minimum area should be I-shaped (hollow or solid). Figure 4(a) shows the cross-section just as a vertical strip which may be due to the fact that the allowed quantity of material was not sufficient to produce an I-shaped cross-section. Figures 4(b)-4(g) indicate that for maximum moment of inertia, material should be distributed as far as possible from the axis about which the moment of inertia is obtained. Although the inner shapes of the cross-sections in Figures 4(e)-4(g) are not uniform, still they are almost of I-shaped. Moreover, since the moment of inertia about x-axis is independent of horizontal distances (x-coordinates) to elements of the cross-section, it does not matter where a small element lies along x-axis. Hence, these cross-sections can be transformed to a better one, without altering the values of moment of inertias, simply by sliding the irregular elements along x-axes, toward the vertical central axes of the cross-sections. However, these can possibly be modified by refining the assumed GA parameters also.

Figure 4 also depicts an interesting property of multi-objective optimization. It has been repeatedly shown in the literature ([11, 12]) that the Pareto-optimal solutions usually follow some common properties. Here, we observe that all Pareto-optimal solutions take the shape of an I-beam and the trade-off in them appears by the width of the top and bottom flanges of the I-shaped sections. Although I-beams are optimal outcome for problems with a design emphasis on moment of inertia and area, this fact is clearly emerges from Figures 3 and 4.

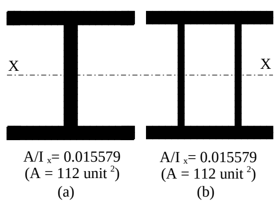

Figure 5: Optimum cross-sections for Case-1, obtained from single objective minimization of A/I_x)

Figure 5 shows two optimal cross-sections from an unconstrained single-objective optimization where the objective function is the minimization of A/I_x. These solutions are also obtained from NSGA-II simulation. The cross-section of Figure 5(a) is identical with that of Figure 4(c). However, the cross-section of Figure 5(b) can also be transformed to a similar one, without altering the value of moment of inertia, just by sliding the vertical flanges along x-axis, toward the vertical central axis of the cross-section.

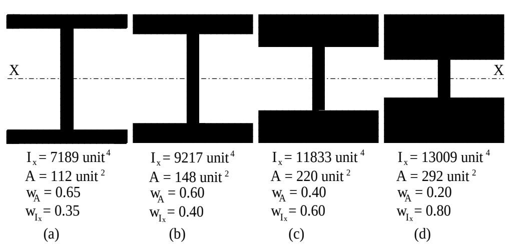

Figure 6: Optimum cross-sections for Case-1, obtained from single objective minimization of *weighted-sum* of $w_1 A/A_{max}$-$w_2 I_x/I_{x,max}$

NSGA-II is again used to solve another single-objective function. This time a *weighted-sum* objective function of $w_1 A/A_{max}$ - $w_2 I_x/I_{x,max}$ is taken for minimization. The maximum allowable area and rectangular moment of inertia for the quadrant of the symmetric cross-section in hand, are taken as $A_{max} = 100$ unit^2 and $I_{x,max} = 3333.33$ unit^4, respectively. The parameters w_1 and w_2 are, respectively, the weightage of area and moment of inertia. The results for different w_1 and w_2 are shown in Figure 6. The cross-sections of Figures 6(a) and 6(b) are identical with those of Figures 4(c) and 4(d), respectively, obtained by multi-objective optimization. The cross-section of Figure 6(c) is almost similar with that of Figure 4(f) (it is to be noted that the cross-section of Figure 4(f) is not of regular I-shaped). However, the result, shown in Figure 6(d), was not obtained in the Pareto-optimal front of multi-objective optimization, as shown in Figure 3.

Case-2: Maximization of Polar Moment of Inertia and Minimization of Area of Symmetric Square Cross Section: The problem of Case-1 is now optimized for polar area moment of inertia. Two objective functions in this case are: maximization of polar moment of inertia and minimization of total area of the quadrant. The Pareto-optimal front and a few selective optimized cross-sections are shown in Figure 7 and Figure 8, respectively.

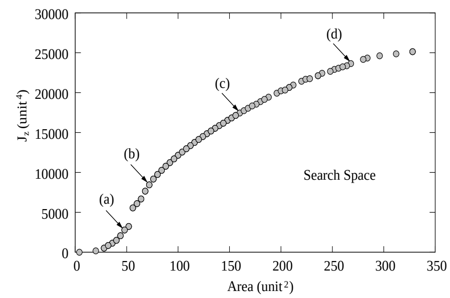

Figure 7: Pareto-optimal front for Case-2

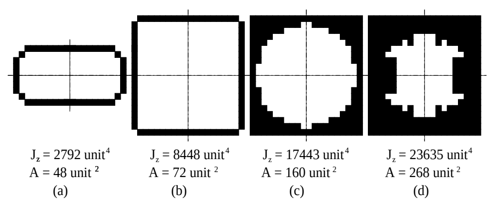

Figure 8: Optimum cross-sections for Case-2

The obtained results suggest that a cross-section with a circular or elliptical central hole maximizes the polar moment of inertia and minimizes the area. Though the inner hole of the cross-section of Figure 8(d) is not uniform, the appearance of an inner hole is obvious. It is observed that the outer surface can either be circular or elliptical depending on the cross-sectional area. As the area increases, the outer surface becomes rectangular and more material is placed in the four corners of the cross-section by making a central hole.

Case-3: Maximization of Polar Moment of Inertia and Minimization of Area of a Symmetric Circular Cross Section: 83 small square elements, each of 1 unit^2 area, are taken to roughly form a quadrant of a circular cross-section in such a way that there are 10 elements along the middle edges (x- and y-axes as shown in Figure 1). The quadrant is then optimized for two objective functions:

maximization of polar moment of inertia and minimization of total area of the quadrant. No constraint is imposed in the problem. The Pareto-optimal front and a few selective optimized cross-sections are shown in Figure 9 and Figure 10, respectively.

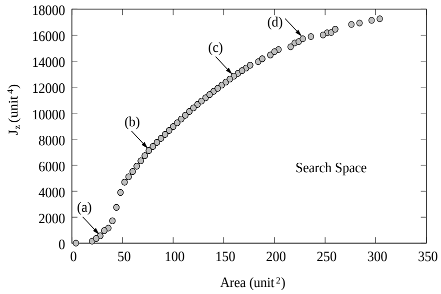

Figure 9: Pareto-optimal front for Case-3

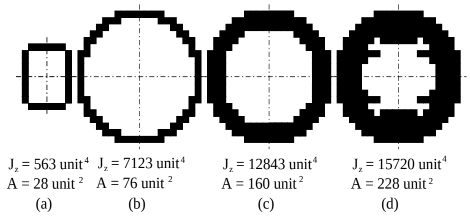

Figure 10: Optimum cross-sections for Case-3

From this study, it is observed that a circular cross-section with a circular central hole or an elliptical cross-section with an elliptical central hole maximizes the polar moment of inertia and minimizes the area. Such results are intuitive and can be found in texts [1, 2].

Case-4: Maximization of Rectangular Moment of Inertias of a Symmetric Square Cross-Section: The problem of Case-1 is considered once again. This time it is optimized for both the rectangular moment of inertias, i.e., maximization of both of I_x and I_y. The area of the cross-section is constrained not to be more than 112 unit^2 (limiting to 28 unit^2 area in the quadrant under consideration). GA parameters are kept the same as before. The Pareto-optimal front is shown in Figure 11. Two extreme and one intermediate cross-sections are shown in Figure 12. The corresponding minimum radius of gyration for each case are also shown.

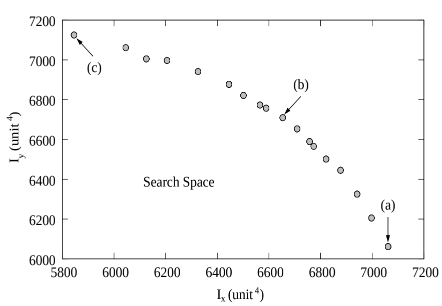

Figure 11: Pareto-optimal front for Case-4

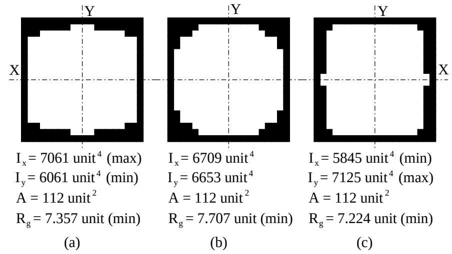

Figure 12: Optimum cross-sections for Case-4

Figures 12(a) and 12(c) show extreme cross-sections with maximum I_x and maximum I_y, respectively. Figure 12(b) shows an intermediate cross-section. Circular or square-type central holes are visible in all the three cross-sections. Possibly uniform holes could be obtained if GA parameters were tuned properly. However, the central hole of the cross-section of Figure 12(b) is more circular compared with those of the extreme cross-sections. Moreover, it has larger value of the minimum radius of gyration (which is the requirement for increasing the capacity of a strut in carrying compressive axial loads). Hence, we may conclude that a cross-section with a circular central hole has larger value of the minimum radius of gyration of the cross-section.

Though we have already obtained I-shaped cross-sections by maximizing a rectangular moment of inertia and minimizing cross-sectional area (Case-1), no such result is obtained during the maximization of both the rectangular moment of inertias. Hence, a study has been made with the current problem by imposing two I-shaped solutions in the initial population of GA. One of these solutions is that of Figure 4(c) with $I_x = 7189$ unit^4 and $A = 112$ unit^2. A similar solution with $I_y = 7189$ unit^4 and $A = 112$ unit^2, is taken as the second solution (this was obtained by rotating the earlier one through 90^0 about the z-axis). The Pareto-optimal front and three cross-sections with minimum radii of gyration are shown in Figure 13 and Figure 14, respectively. As earlier, two of the shown cross-sections are extreme solutions and the other one is an intermediate solution.

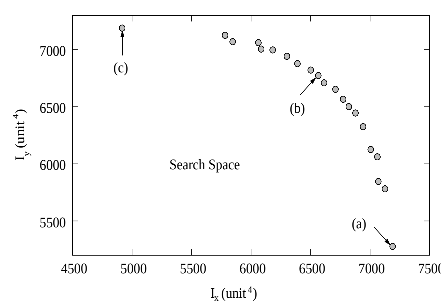

Figure 13: Pareto-optimal front for Case-4 with two forced I-shaped initial solutions

It is observed from Figure 14 that none of the imposed initial I-shaped solutions is retained in the final solutions. Though I_x in Figure 14(a) is retained (extreme value of I_x), but its corresponding I_y is increased. This happens due to the fact that one of the objective functions of the optimization was to increase I_y as well. Hence, the imposed I-shape of Figure 4(c) is changed to that of Figure 14(a). Similar

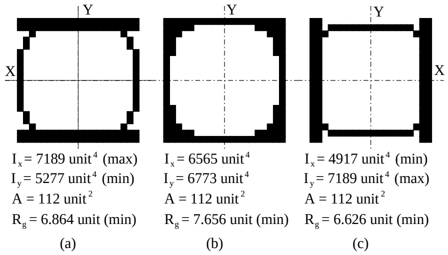

Figure 14: Optimum cross-sections for Case-4 with two forced I-shaped initial solutions

situation occurs with the second extreme solution also, as shown Figure 14(c). Though I_y is retained (extreme value of I_y), but its corresponding I_x is increased due to the importance of maximizing I_x. However, no significant change is observed in intermediate solutions.

Case-5: Maximization of Rectangular Moment of Inertias of a Symmetric Circular Cross-Section: As Case-4, the problem of Case-3 is considered for maximizing the rectangular moment of inertias (I_x and I_y) while restricting the area not to be more than 112 unit^2. The same GA parameters as in Case-4 are used here. The Pareto-optimal front is shown in Figure 15. Two extreme and one intermediate cross-sections, along with the corresponding minimum radius of gyration in each case, are shown in Figure 16.

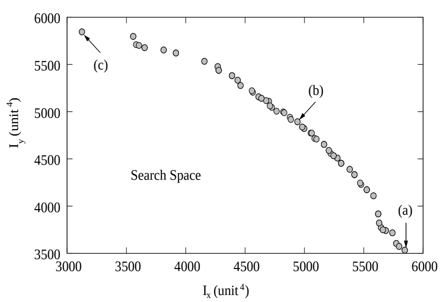

Figure 15: Pareto-optimal front for Case-5

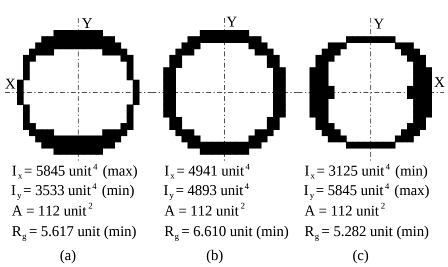

Figure 16: Optimum cross-sections for Case-5

Figures 16(a) and 16(c) show extreme cross-sections with maximum I_x and maximum I_y, respectively. Figure 16(b) shows an intermediate cross-section. In all the three cross-sections, circular-type central holes are visible, which could possibly be made more smooth by a tuning of the GA parameters. It is observed from Figure 16(b) that, like Case-4, here also the central hole of an intermediate solution is more circular compared to that of the extreme cross-sections. Moreover, it has a larger value of the minimum radius of gyration. Hence, we may draw a similar conclusion that a cross-section with a circular central hole has larger value of the minimum radius of gyration.

5. CONCLUSIONS

Different transverse cross-sections for load-carrying members have been studied for either maximizing an area moment of inertia and minimizing area, or maximizing both the rectangular area moment of inertias for limited area. All the studies have produced intuitive and well-engineered solutions. NSGA-II has been found to find different engineered solutions for different objective considerations. Thus, it can be argued that NSGA-II is an adequate multi-objective optimization strategy for solving engineering design problems. More such applications must be attempted to understand the underlying problems better and to decipher important information for a solution to become optimal. For simplicity, only symmetrical cross-sections were considered here. However, the study can be extended for other unsymmetrical cross-sections as well.

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