Approximate Solutions in Multiobjective Optimization

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Abstract

In this paper we study various concepts of approximate solutions in multiobjective optimization. We introduce various new definitions of ε-proper efficiency, relate them with existing ones, study various related concepts and develop very general necessary optimality conditions for a few of them. Some special results for multiobjective linear programming in terms of ε-proper efficiency are derived.

Key Words Multiobjective optimization, ε-efficiency, optimality conditions, evolutionary algorithms.

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1 Introduction

The theory of multiobjective optimization is one of the most rapidly growing areas of modern optimization theory, see for example Luc [18], Miettinen [19], Ehrgott [10], Jahn [14] and the references therein. Since there are multiple solution concepts in multiobjective optimization it often becomes a challenging issue both in theory and practice. Though the notion of Pareto optimal solution play a central role in multiobjective optimization other solution concepts like weak Pareto optimal solutions, strict Pareto solutions, etc are equally important (see for example Ehrgott [10]). The set of all efficient points as is well known lies in the boundary of the objective space and is thus referred to as the efficient frontier. However all points on the frontier need not have equally nice properties which a decision maker may desire and thus one needs to filter out the bad Pareto points and keep the good ones. Such nice Pareto points are referred to in the literature as Proper Pareto solutions. The study of proper Pareto solutions was first carried out by Kuhn and Tucker and then followed by Geoffrion, Benson and Henig. Thus the need for potentially good solutions has always been one of the primary aims in multiobjective optimization. Good solutions can be thought of as "knee-points" on the efficient frontier or that are good in some sense like that of Geoffrion or Kuhn-Tucker etc. However in most of the practical and large scale problems, the user may not get the exact efficient frontier and thus he has to be content with approximate solutions. This usually happens if one uses population based approaches like *Multiobjective Evolutionary Algorithms* or any other practical algorithm using classical methods.

Approximate solutions are also referred to as ε-efficient solutions where ε refers to the precision parameter. Recently several authors have studied ε-efficiency in multiobjective optimization see for example [16], [8], [22]. The concept of ε-efficiency is practically useful from the fact that to a decision maker good approximate solutions are very practical and helpful in decision making. However like Pareto points or efficient points there are also ε-Pareto points with undesirable preperties. Thus even in the approximate case we need to filter out the bad ones and keep the so called ε-proper Pareto solutions.

The paper has been organized in 7 sections of which this is the first. Existing and new definitions of ε-proper efficiency are presented and relevant results are developed in section 2. Benson's method is modified for finding approximate solutions of multiobjective problems in section 3. Some new definitions of proper Pareto optimality are proposed and discussed in sections 4. The Kuhn Tucker type optimality conditions are developed in section 5. Results concerning linear multiobjective problems are shown in section 6. Finally conclusions and the implications of these theoretical results for *Multiobjective Evolutionary Algorithms* are discussed in the last section.

2 Preliminaries and Basic Results

Consider the following vector optimization problem (VP):

Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$

subject to $x \in X$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$. In what follows we will consider $\varepsilon \in \mathbb{R}^m_+$, i.e. $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i \geq 0$ for all i. In some cases we will set $\varepsilon_i = \varepsilon'$, for all i and then $\varepsilon = \varepsilon' e$ where $e = (1, \ldots, 1) \in \mathbb{R}^m_+$. The closure of a set A is denoted as clA. The cone generated by a set A is denoted by $coneA$ and interior is denoted as $intA$.

Definition 2.1 ε-Pareto optimal *Let $\varepsilon \in \mathbb{R}^m_+$ be given then a point $x^* \in X$ is said to be an ε-Pareto optimal of (VP) if there exists no $x \in X$ such that,*

$$f_i(x) \leq f_i(x^*) - \varepsilon_i, \quad \forall i \in \{1, 2, \ldots, m\}. \tag{2.1}$$

and with strict inequality holding for at least one index.

Observe that if $\varepsilon = 0$, the above definition reduces to that of a Pareto optimal. Let us denote the set of ε-Pareto points as $X_{\varepsilon-par}$ and the set of Pareto points as X_{par}.

Definition 2.2 Geoffrion proper Pareto optimal *A point $x_0 \in X$ is called Geoffrion proper Pareto optimal if x_0 is Pareto optimal and if there exists a number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(x_0)$, there exists an index j such that $f_j(x_0) < f_j(x)$ and moreover $(f_i(x_0) - f_i(x))/(f_j(x) - f_j(x_0)) \leq M$.*

Let us denote the set of all Geoffrion properly Pareto optimal solution as X_G.

Lemma 2.1 *A point $x_0 \in X_G$ if and only if there exists $M > 0$ such that the following system is inconsistent (for all $i = 1, 2, \ldots, m$ and for all $x \in X$).*

$$\begin{aligned} -f_i(x_0) + f_i(x) &< 0 \\ -f_i(x_0) + f_i(x) &< M(f_j(x_0) - f_j(x)) \quad \forall j \neq i. \end{aligned}$$

Proof: If $x_0 \in X_G$ then it is clear from the definition that above system is inconsistent.

Suppose the system is inconsistent for some $M > 0$. We claim that $x_0 \in X_{par}$. If not then on the contrary then there exists $x \in X$ such that $f_l(x) < f_l(x_0)$ for some index l, and $f_k(x) \leq f_k(x_0)$, for all $k \neq l$. Thus one easily sees that the above system has a solution for index $i = l$. Hence $x_0 \in X_{par}$. If $x_0 \notin X_G$ then for all $\overline{M} > 0$ there is an index i, and some $x \in X$ satisfying $-f_i(x_0) + f_i(x) < 0$, $-f_i(x_0) + f_i(x) < \overline{M}(f_j(x_0) - f_j(x))$ for all j such that $-f_j(x_0) + f_j(x) > 0$ (such a j exists since $x_0 \in X_{par}$). For j such that $-f_j(x_0) + f_j(x) \leq 0$, $-f_i(x_0) + f_i(x) < \overline{M}(f_j(x_0) - f_j(x))$ is trivially true. Thus the system is consistent for all $\overline{M} > 0$, and hence a contradiction. △

Note that in Geoffrion's definition $x \in X$. However as shown in next lemma, when $Y = f(X)$ is $\mathbb{R}^m_+$-compact (i.e. the sections $(y - \mathbb{R}^m_+) \cap Y$ are compact for all $y \in Y$) then this can be replaced by $x \in X_{par}$.

Lemma 2.2 *Suppose that $Y = f(X)$ is $\mathbb{R}^m_+$-compact, then $x^0 \in X_G$ if x_0 is Pareto optimal and if there exists a number $M > 0$ such that for all i and $x \in X_{par}$ satisfying $f_i(x) < f_i(x^0)$, there exists an index j such that $f_j(x^0) < f_j(x)$ and moreover $(f_i(x^0) - f_i(x))/(f_j(x) - f_j(x^0)) \leq M$.*

Proof: Suppose that x^0 satisfies the conditions of the lemma. Then using lemma 2.1, we obtain that for all $\hat{x} \in X_{par}$ the following system which we mark as (system 1), has no solutions,

$$\begin{aligned} -f_i(x_0) + f_i(\hat{x}) &< 0 \\ -f_i(x_0) + f_i(\hat{x}) &< M(f_j(x_0) - f_j(\hat{x})) \qquad \forall j \neq i. \end{aligned}$$

Take any $x \in X, x \notin X_{par}$. Now since $Y = f(X)$ is $\mathbb{R}^m_+$ compact so there exists $\hat{x} \in X_{par}$ such that

$$\begin{aligned} f_i(\hat{x}) - f_i(x) &\leq 0 \qquad \forall i = 1, 2, \ldots, m \\ f_k(\hat{x}) - f_k(x) &< 0 \qquad \text{for some} \quad k. \end{aligned}$$

Since the system 1 has no solutions, thus we obtain that the following system also has no solutions

$$\begin{aligned} -f_i(x_0) + f_i(\hat{x}) &< f_i(\hat{x}) - f_i(x) \\ -f_i(x_0) + f_i(\hat{x}) &< M(f_j(x_0) - f_j(\hat{x})) + M(f_j(\hat{x}) - f_j(x)) + f_i(\hat{x}) - f_i(x) \qquad \forall j \neq i. \end{aligned}$$

which is equivalent to saying that the following system is inconsistent,

$$\begin{aligned} -f_i(x_0) + f_i(x) &< 0 \\ -f_i(x_0) + f_i(x) &< M(f_j(x_0) - f_j(x)) \qquad \forall j \neq i. \end{aligned}$$

Thus system 1 has no solutions for any $x \in X$. Thus $x \in X_G$ △

Definition 2.3 Liu ε-properly Pareto optimal (Liu [17]) *A point, $x^* \in X$ is called ε-proper Pareto optimal in the sense of Liu [17], if x^* is ε-Pareto optimal and there exists a number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(x^*) - \varepsilon_i$, there exists an index j such that $f_j(x^*) - \varepsilon_j < f_j(x)$ and moreover $(f_i(x^*) - f_i(x) - \varepsilon_i)/(f_j(x) - f_j(x^*) + \varepsilon_j) \leq M$.*

Observe that if $\varepsilon = 0$, the above definition reduces to that of a Geoffrion proper Pareto optimal. Let us denote the set of all Liu properly Pareto optimal solution as $X_L(\varepsilon)$.

Remark 2.1 Let us however observe in the above definition and definition 2.2, M is arbitrary. On the other side M provides a bound on the trade off betwen the components of the objective vector. It is more natural to expect in practice the decision maker will provide a bound on such trade offs. Thus we are motivated to define the following.

Definition 2.4 Geoffrion M properly Pareto optimal *Given a positive number $M > 0$, $x^0 \in X$ is called Geoffrion M proper Pareto optimal if x^0 is Pareto optimal and if for all i and $x \in X$ satisfying $f_i(x) < f_i(x^0)$, there exists an index j such that $f_j(x^0) < f_j(x)$ and moreover $(f_i(x^0) - f_i(x))/(f_j(x) - f_j(x^0)) \leq M$.*

Let us denote the set of all Geoffrion M properly Pareto optimal solution as X_M. It is to be noted that a similar modified definition is also possible for Liu ε-proper Pareto optimal solutions. Let us denote the set of all M ε-proper Pareto optimal as $X_M(\varepsilon)$.

Theorem 2.1 *Let $\varepsilon = \varepsilon' e$ where $\varepsilon' \in \mathbb{R}$, $\varepsilon' > 0$ and $e = (1, 1, \ldots, 1)$, then for any fixed M,*

$$X_M = \cap_{\varepsilon' > 0} X_M(\varepsilon) \tag{2.2}$$

Proof: Let $x_0 \in \cap_{\varepsilon'>0} X_M(\varepsilon)$. Hence for any $\varepsilon' > 0$, and for all i, the following system

$$\begin{aligned} -f_i(x_0) + f_i(x) + \varepsilon' &< 0 \\ Mf_j(x) + f_i(x) - Mf_j(x_0) - f_j(x_0) + M\varepsilon' + \varepsilon' &< 0 \end{aligned}$$

has no solutions in $x \in X$

Let $W = \mathbb{R}^m \setminus (-int\mathbb{R}^m_+)$ and consider the vectors $F^i(\varepsilon)$ (for all $i = 1, \ldots, m$) whose 1^{st} component is given by $-f_i(x_0) + f_i(x) + \varepsilon'$ and whose j^{th} component is equal to $Mf_j(x) + f_i(x) - Mf_j(x_0) - f_j(x_0) + M\varepsilon' + \varepsilon'$, for all $j = 2, \ldots, m$ then $F^i(\varepsilon) \in W$ for all $x \in X$. Now since W is a closed cone for each i

$$\lim_{\varepsilon \to 0} F^i(\varepsilon) \in W$$

This shows that the following system

$$\begin{aligned} -f_i(x_0) + f_i(x) &< 0 \\ Mf_j(x) + f_i(x) - Mf_j(x_0) - f_j(x_0) &< 0 \end{aligned}$$

is inconsistent for all $x \in X$. Thus by lemma 2.1 x_0 is M-properly Pareto optimal, or $x_0 \in X_M$. This shows that $\cap_{\varepsilon'>0} X_M(\varepsilon) \subset X_M$

Conversely, let $x_0 \in X_M$, thus for all $i = 1, \ldots, m$ following system

$$\begin{aligned} -f_i(x_0) + f_i(x) &< 0 \\ Mf_j(x) + f_i(x) - Mf_j(x_0) - f_j(x_0) &< 0 \end{aligned}$$

is inconsistent for all $x \in X$, thus the following system

$$\begin{aligned} -f_i(x_0) + f_i(x) &< -\varepsilon' \\ Mf_j(x) + f_i(x) - Mf_j(x_0) - f_j(x_0) &< -M\varepsilon' - \varepsilon' \end{aligned}$$

is also inconsistent for all $x \in X$.Thus x_0 is M ε-properly Pareto for all $\varepsilon' > 0$ Hence $x_0 \in \cap_{\varepsilon'>0} X_M(\varepsilon)$. △

Proposition 2.1 *Consider a (VP) in which X is a finite set. Then there exists an $\varepsilon > 0$, such that $X_M = X_M(\varepsilon)$.*

Proof: Take any $\hat{x} \in X$, $\hat{x} \notin X_M$, using theorem 2.1, there exists an $\varepsilon_0 > 0$, such that $\hat{x} \notin X_M(\varepsilon_0)$. Take ε as the minimum of all of them (since X is finite set). Then we get that for all $\hat{x} \in X$, with $\hat{x} \notin X_M$, $\hat{x} \notin X_M(\varepsilon_0)$. Hence the result. △

We can also consider the space $\mathbb{R}^m$ to be partially ordered by a closed, convex pointed cone C which need not coincide with $\mathbb{R}^m_+$. Under this circumstance we can define a Pareto minimum of (VP) as follows:

Definition 2.5 *A point $x^* \in X$ is said to be a Pareto minimum of (VP) with respect to C if there exists no $x \in X$ such that $f(x) - f(x^*) \in -(C \setminus \{0\})$. Further if $intC \neq \emptyset$ then x^* is called a weak Pareto minimum for (VP) with respect to C if there exists no $x \in X$ such that $f(x) - f(x^*) \in -intC$.*

Definition 2.6 Benson's ε-proper Pareto minimum *A point $x^0 \in X$ is called Benson's ε-proper Pareto minimum, if*

$$cl(cone(f(X) + (C + \varepsilon) - (f(x^0)))) \bigcap (-C) = \{0\}$$

This definition is a modification of Benson's proper efficiency (Benson [2]).

Let us define $C_0^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, x \rangle > 0, \forall x \in C \setminus \{0\}\}$. Since C is pointed and we are in finite dimensional space so $C_0^* \neq \emptyset$.

Lemma 2.3 *If a point x_0 is Benson's ε-proper Pareto optimal then its also ε-Pareto optimal.*

Proof: Since x_0 is ε-properly Pareto optimal (in the sense of Benson) we have, $cl(cone(f(X) + (C + \varepsilon) - (f(x^0)))) \bigcap (-C) = \{0\}$

Suppose on the contrary $\hat{x}$ is not an ε-Pareto optimal w.r.t to C,

$\Rightarrow f(x^0) - \varepsilon - f(x) \in C \setminus \{0\}$ for some $x \in X$

so let $f(x^0) - \varepsilon - f(x) = k$ $(k \neq 0, k \in C)$

let $k^i = (1 - (1)/(i))k$ and $t_i = i$ for $i = 1, 2, \ldots$

Now construct sequences $k^i \in C$ (since C is a cone)

We claim that $-k \in cl(cone(f(X) + (C + \varepsilon) - (f(x^0))))$. Let us observe that

$f(x) + k^i + \varepsilon = f(x^0) - k + (1 - (1)/(i))k = f(x^0) - (k)/(i)$

and $t_i(f(x) + k^i + \varepsilon - f(x^0)) = i((-k)/(i)) = -k \longrightarrow -k$ as $i \longrightarrow \infty$

Hence $\Rightarrow -k \in cl(cone(f(X) + (C + \varepsilon) - (f(x^0))))$

so $-k \in cl(cone(f(X) + (C + \varepsilon) - (f(x^0)))) \cap (-C)$ and moreover $-k \neq 0$.

Thus we get a contradiction of x^0 being properly Pareto optimal in Benson's sense. Hence the result. $\triangle$

Definition 2.7 Henig ε-efficiency *A point $x^* \in X$ is Henig ε-Pareto optimal if*

1. $(f(x^*) - \varepsilon - C \setminus \{0\}) \cap f(X) = \emptyset$, *or equivalently*
2. $(f(X) + \varepsilon - f(x^*)) \cap (-C \setminus \{0\}) = \emptyset$.

where C is the ordering cone, and $\mathbb{R}^m_+ \setminus \{0\} \subseteq intC$

Definition 2.8 Henig ε-weak efficiency *A point $x^* \in X$ is Henig ε-weak efficient point if*

1. $(f(x^*) - \varepsilon - intC) \cap f(X) = \emptyset$, *or equivalently*
2. $\exists$ *no* $x \in X$, *s.t.* $f(x^*) - f(x) - \varepsilon \in intC$

and $\mathbb{R}^m_+ \setminus \{0\} \subseteq intC$

Thus Henig ε-weak efficient points can be seen as weak points obtained when $intC$ is perturbed by an amount ε.

Theorem 2.2 *Let us consider the problem (VP) where $f : \mathbb{R}^n \to \mathbb{R}^m$ is C-convex and X be a closed convex subset of $\mathbb{R}^n$. Let $\varepsilon = \varepsilon' e$, where $\varepsilon' > 0$. Let $x_0 \in X$ be Henig ε-weak minimum, then there exists $\mu \in C^* \setminus \{0\}$, with $\langle \mu, e \rangle = 1$ such that x_0 is a ε-minimum for the following scalar minimum problem (MP)*

$$\min_{x \in X} \langle \mu, f(x) \rangle$$

Proof: The proof follows along the lines of Deng [8] and is thus ommitted here.

Definition 2.9 Henig ε-proper efficiency. *A point $x^* \in X$ is said to be Henig ε-proper Pareto minimum (with respect to the cone C) if $f(x^*) - \varepsilon - (\Theta \setminus \{0\}) \cap f(X) = \emptyset$ where $C \setminus \{0\} \subseteq int\Theta$, and Θ is a closed, convex and pointed cone in $\mathbb{R}^m$.*

This definition is a modification of Henig's *global proper efficiency* (Henig [11])

Lemma 2.4 *Let $\varepsilon = \varepsilon' e$ and H denote the set of all Henig weak minimum of the program (VP) and for any given $\varepsilon > 0$, let H_ε denote the set of all Henig ε-weak minimum of (VP). Then,*

$$H = \cap_{\varepsilon' > 0} H_\varepsilon \tag{2.3}$$

Proof: Let us consider $x_0 \in \cap_{\varepsilon'>0} H_\varepsilon$ then from Kazmi [13] we have $\cap_{\varepsilon'>0} H_\varepsilon \subset H$. Consider, $x_0 \in H$, hence there exists no $x \in X$ $s.t.$, $f(x) - f(x_0) \in -intC$. Hence for any $\varepsilon' > 0$, there does not exists $x \in X$ $s.t.$, $f(x) - f(x_0) \in -intC - \varepsilon$. Hence x_0 is Henig ε-weak Pareto of (VP). Since ε' was arbitrary we have $x_0 \in \cap_{\varepsilon'>0} H_\varepsilon$. △

When the ordering cone is $\mathbb{R}^m_+$, the above theorem reduces to

Corollary 2.1 Lemaire [15]

Let $\varepsilon = \varepsilon' e$ and E denote the set of all weak vector minimum of the program (VP) and for any given $\varepsilon > 0$, let E_ε denote the set of all ε-weak minimum of (VP). Then,

$$E = \cap_{\varepsilon > 0'} E_\varepsilon \tag{2.4}$$

We get a proposition (which is a generalization of Lemaire [15]) similar to Proposition 2.1.

Proposition 2.2 *Consider a (VP) in which X is a finite set. Then there exists an $\varepsilon > 0$, such that $H = H_\varepsilon$.*

Let $(f_i)'_\varepsilon(x; d)$ denote the ε-directional derivative of a convex function f_i at x in the direction d. The next theorem is a generalization of Theorem 2.3 (Deng [8]) for any ordering cone C.

Lemma 2.5 *Consider the problem (VP) where each f_i is convex. Let $\varepsilon = \varepsilon' e$. If*

$$((f_1)'_\varepsilon(y; x-y), \dots, (f_m)'_\varepsilon(y; x-y)) \in W = \mathbb{R}^m \setminus (-intC) \quad \forall x \in X \tag{2.5}$$

then $y \in H_\varepsilon$. When $\varepsilon = 0$, the converse is also true.

Proof: The proof follows along the same lines of theorem 2.3 (Deng [8]) and is thus omitted here.

3 Modified Benson method for approximate solutions

In this section we introduce a scalarization scheme to compute and check approximate solutions. The scheme may be thought of as a modification of Benson's scheme for usual Pareto minimum solutions. We define the problem $P_\varepsilon(\varepsilon^*, x_0)$ as follows,

$\max \sum_{i=1}^m \varepsilon_i$

sub to $f_i(x_0) - \varepsilon_i - f_i(x_i) - \varepsilon_i{}^* = 0$

$\varepsilon_i \geq 0 \ \forall i = 1, 2 \dots, m$

$x \in X$

where $\varepsilon^* \geq 0$ is a user defined modulus of approximation and $x_0 \in X$ is some initial solution. Analogous to Benson's method, we can prove the following theorems for our modified Benson's method.

Theorem 3.1 *The point $x_0 \in X \neq \emptyset$ is ε^*-Pareto optimal if $P_\varepsilon(\varepsilon^*, x_0)$ is infeasible.*

Proof: Let $P_\varepsilon(\varepsilon^*, x_0)$ be infeasible, hence there does not exists any $x \in X$ and $\varepsilon \geq 0$ for which $f_i(x_0) - \varepsilon_i - f_i(x_i) - {\varepsilon_i}^* = 0\ \forall i$

$\Rightarrow \nexists$ any $x \in X$, for which $\varepsilon_i = f_i(x_0) - f_i(x_i) - {\varepsilon_i}^* \geq 0\ \forall i$

$\Rightarrow \nexists$ any $x \in X$, for which $f_i(x_0) - {\varepsilon_i}^* \geq f_i(x_i)\ \forall i$

Three cases arise,

Case 1. $f_i(x) \geq f_i(x_0) - {\varepsilon_i}^*\ \forall i \Rightarrow f(x)$ is not ε^*-dominating $f(x^0)$.

Case 2. $f_k(x) > f_k(x_0) - {\varepsilon_k}^*$ for some indices k, and $f_l(x) \leq f_l(x_0) - {\varepsilon_l}^*\ \forall l$, and atleast one index r such that $f_r(x) < f_r(x_0) - {\varepsilon_l}^* \Rightarrow f(x^0)$ and $f(x)$ are not ε^*-comparable.

Case 3. $f_k(x) > f_k(x_0) - {\varepsilon_k}^*$ for some indices k, and $f_l(x) = f_l(x_0) - {\varepsilon_l}^*$ for all indices $l \neq k$, $\Rightarrow x^0$ ε^*-dominates $x \Rightarrow x^0 \in \varepsilon^* - X_{par}$ △

Theorem 3.2 *Suppose the problem $P_\varepsilon(\varepsilon^*, x_0)$ is feasible, then the point $x_0 \in X$ is ε^*-Pareto optimal if and only if the optimal objective value of $P_\varepsilon(\varepsilon^*, x_0)$ is 0.*

Proof: Let us assume x_0 be ε^*-Pareto optimal of (VP). Hence there does not exists $x \in X$ for which, $f_i(x) \leq f_i(x^0) - \varepsilon_i^*, \quad \forall i \in \{1, 2, \ldots, m\}$. with at least one strict inequality.

$\Rightarrow \nexists x \in X$ for which, $\varepsilon_i = f_i(x_0) - f_i(x) - \varepsilon_i^* \geq 0, \quad \forall i \in \{1, 2, \ldots, m\}$. and $\varepsilon_j > 0$ for some indices j.

$\Rightarrow \nexists x \in X$ for which, $\sum_{k=0}^{m} \varepsilon_k > 0$ holds.

(but $P_\varepsilon(\varepsilon^*, x_0)$ is feasible so $\sum_{k=0}^{m} \varepsilon_k \geq 0$ holds for feasible point.

$\Rightarrow$ optimal objective value of $P_\varepsilon(\varepsilon^*, x_0) = 0$ Conversely, suppose that optimal objective value of $P_\varepsilon(\varepsilon^*, x_0)$ is zero. Hence $\sum_{k=0}^{m} \varepsilon_k = 0$. Thus $\varepsilon_i = 0$ where ε_i, $i = 1, 2, \ldots m$ are the optimal variables.

Now we have to show that x_0 is ε^*-Pareto optimal, or equivalently we have to show that

$\nexists x' \in X$ for which the following is true,

$$f_i(x') \leq f_i(x^0) - \varepsilon_i^*, \quad \forall i \in \{1, 2, \ldots, m\}.$$

$$f_j(x') < f_j(x^0) - \varepsilon_j^* \text{ for some } j \in \{1, 2, \ldots, m\}.$$

so assume on the contrary that such an $x^i \in X$ exists.

Define $\varepsilon_i^0 = f_i(x^0) - \varepsilon_i^* - f_i(x')$, Now $\varepsilon_i^0 \geq 0 \forall i$, and $\varepsilon_j^0 > 0$ thus we would have that,

(x', ε^0) is feasible to $P_\varepsilon(\varepsilon^*, x_0)$ and for which $\sum_{k=0}^m \varepsilon_k^0 > 0$, which is a contradiction to optimality of ε. Hence our assumption of existence of such an $x \in X$, was wrong thus $x^0 \in \varepsilon^*\text{-}X_{par}$ △

Theorem 3.3 *If problem $P_\varepsilon(\varepsilon^*, x_0)$ has an optimal solution with finite objective value and if this value is attained at $(\bar{x}, \bar{\varepsilon})$ then $\bar{x} \in X_{par}$.*

Proof: Suppose $\overline{x} \notin X_{par}$. Then for some $\hat{x} \in X$, $f_i(\hat{x}) \leq f_i(\overline{x})$ $\forall i = 1, 2 \ldots, m$ and $f_k(\hat{x}) < f_k(\overline{x})$ for some index k. Define $\hat{\varepsilon}_i = f_i(x^0) - f_i(\hat{x}) - \varepsilon_i^*$

$(\hat{x}, \hat{\varepsilon})$ is feasible for $P_\varepsilon(\varepsilon^*, x_0)$ because

$\hat{\varepsilon_i} = f_i(x^0) - f_i(\hat{x}) - \varepsilon_i^* \geq f_i(x^0) - f_i(\overline{x}) - \varepsilon_i^* = \overline{\varepsilon_i} \geq 0$

Furthermore

$\sum_{i=1}^m \hat{\varepsilon}_i > \sum_{i=1}^m \overline{\varepsilon}_i$, a contradiction of optimality of $\overline{\varepsilon}$. △

Theorem 3.4 *If $P_\varepsilon(\overline{\varepsilon}, x_0)$ has no finite optimal objective value then $\overline{\varepsilon}\text{-}X_{p-par} = \emptyset$.*

Proof: The proof follows along the lines of Benson [1] and is thus ommited here. △

Remark 3.1 *If $P_\varepsilon(\overline{\varepsilon}, x_0)$ has no finite optimal objective value then $\overline{\varepsilon}\text{-}X_{p-par} = \emptyset$, and thus $\varepsilon^*\text{-}X_{p-par} = \emptyset$ $\forall \varepsilon^* \leq \overline{\varepsilon}$. In particular $X_{p-par} = \emptyset$. Thus we see that this theorem can be used to find bad situations in which the gain to loss ratio is unbounded.*

We will now illustrate the above results through an example.

Example 3.1 *Consider the following Vector Optimization problem (VP)*

$\min(x^2 - 4, (x-1)^4)$

$s.t$ $-x - 100 \leq 0$

Let $\varepsilon^* = (.1, .1)$ and we want to find the set $\varepsilon^*\text{-}X_{par}$

Sol : The Benson's modified problem $P_\varepsilon(\varepsilon^*, x_0)$ is

max $\varepsilon_1 + \varepsilon_2$

s.t. $x_0^2 - 4 - \varepsilon_1 - x^2 + 4 - 0.1 = 0$

$(x_0 - 1)^4 - \varepsilon_2 - (x-1)^4 - 0.1 = 0$

$-x - 100 \leq 0$

$\varepsilon_1, \varepsilon_2 \geq 0$

First let us obtain the values of x_0 for which the problem above is infeasible.

Now, $\varepsilon_1, \varepsilon_2 \geq 0 \Rightarrow 2$ cases

case 1 $x^2 \leq x_0^2 - 0.1 \Rightarrow x_0^2 - 0.1 \geq 0$

$\Rightarrow x_0 \in [-100, -0.316] \cup [0.316, \infty)$

case 2 $(x-1)^4 \leq (x_0 - 1)^4 - 0.1 \Rightarrow (x_0 - 1)^4 - 0.1 \geq 0$

$\Rightarrow x_0 \in [-100, 0.437] \cup [1.562, \infty)$

Thus we obtain that, $P_\varepsilon(\varepsilon^*, x_0)$ is infeasible for $x_0 \in (-0.316, 0.316) \cup (0.437, 1.562)$. Thus using Theorem 3.1 we obtain that $(-0.316, 0.316) \cup (0.437, 1.562) \subset \varepsilon^*\text{-}X_{par}$

Now to find $\varepsilon^*\text{-}X_{par}$ completely consider the 4 cases

case A: $x_0 \in [-100, -0.316]$, let $x_0 = -0.4$ (say)

The Benson's modified problem $P_\varepsilon(\varepsilon^*, -0.4)$ is

$max\ \varepsilon_1 + \varepsilon_2$

s.t. $(-.4)^2 - 4 - \varepsilon_1 - x^2 + 4 - 0.1 = 0$

$(-0.4-1)^4 - \varepsilon_2 - (x-1)^4 - 0.1 = 0$

$-x - 100 \leq 0$

$\varepsilon_1, \varepsilon_2 \geq 0$

Now from the constraints we obtain that $0 \leq \varepsilon_1 \leq 0.06$, and $0 \leq \varepsilon_2 \leq 3.7416$. Thus the problem $P_\varepsilon(\varepsilon^*, -0.4)$ is bounded. Also $x = 0, \varepsilon_1 = 0.06, \varepsilon_2 = 2.7416$ is feasible so optimal objective value is non-zero and thus $x_0 = -0.4 \notin \varepsilon^*\text{-}X_{par}$ Similarly if we take any $x_0 \in [-100, -0.316]$ we would obtain that optimal objective value is nonzero implying that $[-100, -0.316] \notin \varepsilon^*\text{-}X_{par}$.

The minimum value of $P_\varepsilon(\varepsilon^*, x_0)$ occurs at $x = 0.2445$ and thus from theorem 3.3, $x = 0.2445 \in X_{par}$.

case B: $x_0 \in (-0.316, 0.316)$, as stated earlier in this case we obtain $(-0.316, 0.316) \in \varepsilon^*\text{-}X_{par}$.

case C: $x_0 \in [0.316, 0.437]$, let $x_0 = 0.4$ (say)

The Benson's modified problem $P_\varepsilon(\varepsilon^*, 0.4)$ is

$max\ \varepsilon_1 + \varepsilon_2$

s.t. $(0.4)^2 - 4 - \varepsilon_1 - x^2 + 4 - 0.1 = 0$

$(0.4-1)^4 - \varepsilon_2 - (x-1)^4 - 0.1 = 0$

$-x - 100 \leq 0$

$\varepsilon_1, \varepsilon_2 \geq 0$

Now from the constraints $\varepsilon_1, \varepsilon_2 \geq 0$ we obtain that $0.06 - x^2 \geq 0$ and $0.0296 - (x-1)^4 \geq 0$ implying that $x \leq 0.2444$ and $x \in [.5852, 1.414]$, which is not possible simultaneously. Thus using Theorem 3.2 we obtain that $[0.316, 0.437] \in \varepsilon^*\text{-}X_{par}$.

case D: $x_0 \in (0.437, 1.562)$, as stated earlier in that case we obtain that $(0.437, 1.562) \in \varepsilon^*\text{-}X_{par}$.

case E: $x_0 \in [1.562, \infty)$, let $x_0 = 2.0$ (say)

The Benson's modified problem $P_\varepsilon(\varepsilon^*, 2)$ is

$max\ \varepsilon_1 + \varepsilon_2$

s.t. $(2)^2 - 4 - \varepsilon_1 - x^2 + 4 - 0.1 = 0$

$(2-1)^4 - \varepsilon_2 - (x-1)^4 - 0.1 = 0$

$-x - 100 \leq 0$

$\varepsilon_1, \varepsilon_2 \geq 0$

Now from the constraints we obtain that $0 \leq \varepsilon_1 \leq 3.9$, and $0 \leq \varepsilon_2 \leq 0.9$. Thus the problem $P_\varepsilon(\varepsilon^*, 2)$ is bounded. Also $x = 1, \varepsilon_1 = 2.9, \varepsilon_2 = 0.9$ is feasible so optimal objective value is non-zero and thus $x_0 = 2.0 \notin \varepsilon^*\text{-}X_{par}$ Similarly if we take any $x_0 \in [1.562, \infty)$ we would obtain that optimal objective value is nonzero implying that $[1.562, \infty) \notin \varepsilon^*\text{-}X_{par}$.

The minimum value of $P_\varepsilon(\varepsilon^*, 2)$ occurs at $x = 0.4102$ and thus from Theorem 3.3, $x = 0.4102 \in X_{par}$.

Thus we obtain $\varepsilon^*\text{-}X_{par} = (-0.316, 1.562)$. Note that $X_{par} = (0, 1)$.

4 Benson ε-proper Pareto optimality and scalarization.

In this section we will be studying some of the properties of Benson's ε-proper efficient solutions. Liu's ε-Pareto optimality can be related with our new definition.

Theorem 4.1 *When the ordering cone $C = \mathbb{R}^m_+$ then a point x^0 is Liu ε-proper Pareto optimal if and only if it is Benson ε-proper Pareto.*

Proof: The proof follows along the lines of Ehrgott [10] and is thus ommitted here. △

Before we derive some theorems some cone separation results are given below (Borwein, [3])

Lemma 4.1 *Suppose N, S are closed convex cones in $\mathbb{R}^n$ and that $N \cap S = 0$. Suppose that the dual cone S^* has nonempty interior. Then there is some $s^* \in S_0^*$ with $-s^* \in N^*$, thus*

$$\langle s^*, s\rangle > 0 \forall s \in S \setminus \{0\} \tag{4.1}$$

Theorem 4.2 *Suppose that x^0 is $\langle s^*, \varepsilon\rangle$ minimum for $(P(s^*))$*

$\min_{x\in X}\langle s^*, f(x)\rangle$

for some $s^* \in C_0^*$. *Then* x^0 *is Benson* ε*-proper Pareto.*

Proof: Suppose some $h \in cl(cone(f(X) + C + \varepsilon - f(x^0)))$. Take a sequence

$h_n = t_n(f(x_n) + s_n + \varepsilon - f(x^0)) \to h$

with $t_n \geq 0$, with $x_n \in X$, $s_n \in C$.

Now since x^0 is $\langle s^*, \varepsilon\rangle$-minimum for $(P(s^*))$, thus for each n,

$\langle s^*, f(x^0)\rangle \leq \langle s^*, f(x_n)\rangle + \langle s^*, \varepsilon\rangle$

$\Rightarrow \lim_{n\to\infty} t_n(\langle s^*, (f(x_n) + s_n - f(x_0) + \varepsilon)\rangle) \geq 0$

$\Rightarrow \langle s^*, h\rangle \geq 0\ \forall h \in cl(cone(f(X)+C+\varepsilon-f(x^0)))$. Also note that if $h \in -C\setminus\{0\}$ then one would not have $\langle s^*, h\rangle \geq 0$ (since $s^* \in C_0^* \Rightarrow \forall(-h) \in C$ we have that $\langle(-h), s^*\rangle > 0$),

thus $cl(cone(f(X) + C + \varepsilon - f(x^0))) \bigcap -C = \{0\}$, showing that x^0 is Benson ε-proper Pareto optimal. △

Theorem 4.3 *Suppose that* f *is a* C*-convex function and that* X *is a convex set. Then,* x^0 *is* ε*-proper Pareto minimum for (VP) in Benson sense if and only if* x^0 *is* $\langle s^*, \varepsilon\rangle$*-optimal for* $(P(s^*))$ *for some* $s^* \in C_0^*$.

Proof: ($\Leftarrow$) is proved in above Theorem 4.2, so it suffices to prove only ($\Rightarrow$)

Since X is convex and f is C-convex, so

$$f(X) + C + \varepsilon - f(x^0) \subset cl(cone(f(X) + C + \varepsilon - f(x^0))) = K \tag{4.2}$$

the cone K is convex. Since x^0 is Benson ε-properly Pareto optimal so $K \cap (-C) = 0$, equivalently $-K \cap C = 0$. Using Lemma 4.1 (with $N = -K$) there exists $s^* \in (K^*)$ satisfying $\langle s^*, s\rangle > 0\ \forall s \in C\setminus\{0\}$. Thus $\langle s^*, (f(x)+s+\varepsilon-f(x^0))\rangle \geq 0$ for all $x \in X, s \in C$ Putting $s = 0$ we obtain, x^0 is $\langle s^*, \varepsilon\rangle$-minimum for $(P(s^*))$. △

Let $S_\varepsilon(Y) = \{y \mid y \in \langle s^*, \varepsilon\rangle\text{-}argmin_x\{\lambda_0^t f(x) \mid x \in X\}\}$. Then, we can prove that

Theorem 4.4 *Let* $Y_{\varepsilon-p-eff}$ *be the set of all Benson* ε*-proper efficient points (in the objective space). If* $Y = f(X)$ *is* $\mathbb{R}_+^m$ *− convex then* $Y_{\varepsilon-p-eff} = S_\varepsilon(Y)$

Proof: ($\Leftarrow$) This has been proved in Theorem 3, Liu [17] and is thus omitted here.

($\Rightarrow$) Since the ordering cone $S = \mathbb{R}_+^n$ then a point x^0 is Liu ε-properly Pareto optimal if and only if it is Benson ε-Pareto optimal. Thus for proving the theorem we may equivalently use our modified Benson's sense of proper Pareto optimality.

Let $y^* \in Y_{\varepsilon-p-eff}$, i.e. $cl(cone(f(X) + (\mathbb{R}^m + \varepsilon) - (y^*))) \cap -\mathbb{R}^m = \{0\}$ (vector minimization problem considered).

The rest of the proof is on the similar lines of proof in Ehrgott [10], using separation theorem and is thus omitted here. △

Finally we observe that for nonconvex cases, Benson ε-proper efficiency can also be characterized along the same lines as in Chen *et al* [5].

5 Kuhn Tucker type optimality conditions

In this section we intend to develop optimality conditions for ε-proper efficiency in vector optimization. Using theorem 4.3 we can derive the necessary and sufficient Kuhn Tucker type optimality conditions for Benson ε-proper Pareto optimal solutions.

Theorem 5.1 *Consider the problem (VP) with the set X described as with inequality constraints. Let $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ and $g(x) = (g_1(x), g_2(x), \ldots, g_l(x))$. Suppose that f is a convex function with respect to C and that $g_1, g_2, \ldots, g_m$ are convex functions. Assume that the Slater Constraint Qualification holds. Then $x_0 \in X$ is an ε-properly Pareto optimal in Benson's sense if and only if there exists scalars $\mu_j \in C_0^*, j \in T = \{1, 2, \ldots, m\}$, $\lambda_i \geq 0$, $i \in L = \{1, 2, \ldots, l\}$, $\delta_{j^*} \geq 0, j \in T = \{1, 2, .., m\}$ and $\varepsilon_{i^*} \geq 0, i \in L = \{1, 2, \ldots, l\}$ such that*

1. $0 \in \sum_{j=1}^{m} \partial_{\delta_{j^*}}(\mu_j f_j)(x_0) + \sum_{i=1}^{l} \partial_{\varepsilon_{i^*}}(\lambda_i g_i)(x_0)$, *and*
2. $\sum_{j=1}^{m} \delta_{j^*} + \sum_{i=1}^{l} \varepsilon_{i^*} - \langle \mu, \varepsilon \rangle \leq \sum_{i=1}^{l} \lambda_i g_i(x_0) \leq 0$

Proof: To prove the necessary part we take x_0 to be a Benson ε-proper Pareto optimal thus by theorem 4.3, there exists $\mu = (\mu_1, \ldots, \mu_m) \in C_0^*$ such that x_0 is an $\langle \mu, \varepsilon \rangle$ minimum of scalar problem $P(\mu)$. Applying Theorems 5.1 and 5.2 (Hiriart-Urruty [12]) since the Slaters Constraint qualification holds, we get that there exists scalars $\varepsilon_0 \geq 0, \varepsilon_p \geq 0$ and $\varepsilon_{i^*} \geq 0$, $i \in L = \{1, 2, \ldots, l\}$ such that $\varepsilon_0 + \varepsilon_p + \sum_{j=1}^{m} \varepsilon_{i^*} = \langle \mu, \varepsilon \rangle$ and $\lambda_i \geq 0, i \in L$, such that

1. $0 \in \partial_{\varepsilon_0}(\sum_{j=1}^{m} \mu_j f_j)(x_0) + \sum_{i=1}^{l} \partial_{\varepsilon_{i^*}}(\lambda_i g_i)(x_0)$
2. $\varepsilon_p + \sum_{i=1}^{l} \lambda_i g_i(x_0) \geq 0$

Using Theorem 2.1 in [12] we obtain scalars $\delta_{j^*} \geq 0, J \in T$ and $\varepsilon_0 = \sum_{j=1}^{l} \delta_{j^*}$ with

1. $0 \in \sum_{j=1}^{m} \partial_{\delta_{j^*}}(\mu_j f_j)(x_0) + \sum_{i=1}^{l} \partial_{\varepsilon_{i^*}}(\lambda_i g_i)(x_0)$
2. $\varepsilon_p + \sum_{i=1}^{l} \lambda_i g_i(x_0) \geq 0$

using $\varepsilon_0 + \varepsilon_p + \sum_{j=1}^{m} \varepsilon_{i^*} = \langle \mu, \varepsilon \rangle$, we obtain that $\varepsilon_p + \sum_{j=1}^{m} \delta_{j^*} + \sum_{j=1}^{m} \epsilon_{i^*} = \varepsilon$

thus we obtain condition 2 as

$$\sum_{j=1}^{m} \delta_{j^*} + \sum_{j=1}^{m} \varepsilon_{i^*} - \langle \mu, \varepsilon \rangle \leq \sum_{i=1}^{l} \lambda_i g_i(x_0) \leq 0 \tag{5.1}$$

To we prove the sufficiency part, let us assume that $x_0 \in X$ and there exists scalars $\delta_{j^*} \geq 0, j \in T, \varepsilon_{i^*}, \mu_j \geq 0, j \in T$ with $\sum_{j=1}^{l} \mu_j = 1$ and $\lambda_i \geq 0, i \in L$ and conditions 1 and 2 of theorem holds. From 1 we get that there exists $\phi_j \in \partial_{\delta_{j^*}}(\mu_j f_j)(x_0)\ \forall j \in T = \{1, 2, \ldots, m\}$ and $\eta_i \in \partial_{\epsilon_{i^*}}(\lambda_i g_i)(x_0)\ \forall i \in L = \{1, 2, \ldots, l\}$ such that

$$0 = \sum_{j=1}^{m} \phi_j + \sum_{i=1}^{l} \eta_i$$

Since each $f_1, \ldots, f_m$ are C convex functions and each $g_1, \ldots, g_l$ are convex functions we have for all $x \in X$

$$\sum_{j=1}^{m} \mu_j f_j(x) - \sum_{j=1}^{m} \mu_j f_j(x_0) \geq \langle \sum_{j=1}^{m} \phi_j, x - x_0 \rangle - \sum_{j=1}^{m} \delta_{j^*} \tag{5.2}$$

and

$$\sum_{i=1}^{l} \mu_i g_i(x) - \sum_{i=1}^{m} \mu_i g_i(x_0) \geq \langle \sum_{i=1}^{l} \eta_i, x - x_0 \rangle - \sum_{i=1}^{l} \varepsilon_{i^*} \tag{5.3}$$

By adding 5.2 and 5.3, using condition 2 we conclude that $\forall x \in X$

$$\sum_{j=1}^{m} \mu_j f_j(x) - \sum_{j=1}^{m} \mu_j f_j(x_0) \geq -\langle \mu, \varepsilon \rangle$$

It shows that x_0 is an $\langle \mu, \varepsilon \rangle$ minimum of $(P(\mu))$, thus by theorem 4.3 we obtain that its Benson ε-proper Pareto optimal. △

Theorem 5.2 *Consider the problem (VP) where $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ be such that each f_i is C convex and X is a closed convex subset of $\mathbb{R}^n$. Let $x_0 \in X$ be a Benson's ε-proper minimum of (VP) with $\varepsilon > 0$. Then there exists an x^* in a closed ball centered at x_0 and radius $\sqrt{\langle \mu, \varepsilon \rangle}$ such that x^* is also an ε-proper minimum of (VP) with $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in C_0^*$ such that*

$$0 \in \sum_{j=1}^{m} \mu_j \partial f_j(x^*) + N_X(x^*) + \sqrt{\langle \mu, \varepsilon \rangle} B^*.$$

where $\partial f_j(x^*)$ denote the sub-differential of the convex function f_j at x^*, $N_X(x^*)$ denote the normal cone to the convex set X at $x^* \in X$ and B^* denote the unit ball in $\mathbb{R}^n$ and C is the ordering cone.

Proof: Let $\mu^T\varepsilon = \hat{\varepsilon}$. Observing that X is a closed convex set and $x_0 \in X$ is an Benson ε-proper Pareto optimal, thus by Theorem 4.3, there exists $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in C_0^*$ such that x_0 is an $\langle\mu, \varepsilon\rangle$-minimum of $(P(\mu))$: $\min_{x\in X}\langle\mu, f(x)\rangle$

Thus x_0 is an $\langle\mu, \varepsilon\rangle$-minimum of the following unconstrained problem

$$inf_{x\in\mathbb{R}^n}\langle\mu, f(x)\rangle + \delta_X(x) \tag{5.4}$$

where δ_X is the indicator function of the convex set X, which is a proper lower-semi-continuous function. Since x_0 is an $\langle\mu, \varepsilon\rangle$-minimum of 5.4 so

$$\langle\mu, f(x_0)\rangle + \delta_X(x_0) \leq min_{x\in\mathbb{R}^n}\langle\mu, f(x)\rangle + \delta_X(x) + \hat{\varepsilon}$$

Using Ekeland Variational Principle, corresponding to $\hat{\varepsilon} > 0$, there exists a point x^*, such that $\|x - x^*\| \leq \sqrt{\hat{\varepsilon}}$ and

$$\langle\mu, f(x^*)\rangle + \delta_X(x^*) \leq \langle\mu, f(x_0)\rangle + \delta_X(x_0) \tag{5.5}$$

and also for all $x \neq x^*$

$$\langle\mu, f(x^*)\rangle + \delta_X(x^*) \leq \langle\mu, f(x)\rangle + \delta_X(x) + \sqrt{\hat{\varepsilon}}\|x - x^*\| \tag{5.6}$$

From 5.5 we get that $x^* \in X$, x^* is $\hat{\varepsilon}$-minimum of 5.4 and hence x^* is an $\hat{\varepsilon}$-minimum of (MP). Thus using theorem 4.3 x^* is also an Benson's proper $\hat{\varepsilon}$-minimum of (VP). From 5.6 we get that

$$0 \in \partial(\langle\mu, f(x^*)\rangle + \delta_X(x^*) + \sqrt{\hat{\varepsilon}}\|x - x^*\||_{x=x^*})$$

By using Moreau-Rockafeller Theorem and noting that $\partial\delta_X(x^*) = N_X(x^*)$ and $\partial(\|x - x^*\|)|_{x=x^*} = B^*$ we have

$$0 \in \sum_{j=1}^{m}\mu_j\partial f_j(x^*) + N_X(x^*) + \sqrt{\hat{\varepsilon}}B^* \tag{5.7}$$

△

Remark 5.1 If we assume that Slaters Constraint Qualification holds and all the functions are differentiable then (5.7) reduces to

$$0 \in \sum_{j=1}^{m}\mu_j\nabla f_j(x^*) + \sum_{i\in I}\lambda_i\nabla g_i(x^*) + \sqrt{\hat{\varepsilon}}b^*$$

where I is the set of active indices at x^* and $b^* \in B^*$. This implies that

$$\|\sum_{j=1}^{m}\mu_j\nabla f_j(x^*) + \sum_{i\in I}\lambda_i\nabla g_i(x^*)\| \leq \sqrt{\hat{\varepsilon}} \tag{5.8}$$

Thus we obtain that in order to calculate a Benson's ε-proper minima for (VP) numerically under the assumptions of convexity and differentiability, one has to develop an algorithm that after successive iterations generates a point x^* and $\mu \in C_0^*$ such that condition (5.8) holds.

It is also to be noted that when the ordering cone is $\mathbb{R}_+^m$ then Benson's ε-proper points which are equivalent to Liu's so they are good in the sense that trade-offs are bounded from above, thus such an algorithm will find points which are better than ε-Pareto one's.

Lemma 5.1 *(Tanino [21])Let ε-$E(Y|C)$ denote the set of all ε-efficient points of Y when ordering cone is C. If $0 \in A \subseteq C \cup \{0\}$. then*

$$\varepsilon - E(Y|C) = \varepsilon - E(Y + A|C)$$

Theorem 5.3 *It C is closed and acute, then Benson ε-proper Pareto optimality is equivalent to Henig ε-proper efficiency.*

Proof: ($\Rightarrow$) Let x_0 be a Benson ε-properly Pareto optimal point. Thus

$$cl(cone(f(X) + (C + \varepsilon) - (f(x_0))) \bigcap (-C) = \{0\}$$

thus by lemma 3.4 (Henig [11]), there exists a convex cone Θ such that $C \subseteq \text{int } \Theta \cup \{0\}$ and $cl(cone(f(X) + (C + \varepsilon) - (f(x^0))) \bigcap (-\Theta) = \{0\}$. Hence

$$cone(f(X) + (C + \varepsilon) - (f(x_0)) \bigcap (-\Theta) = \{0\}$$

using lemma 5.1 $f(x_0) \in \varepsilon$-$E(f(X)|\Theta)$, thus we obtain $f(x_0)$ is Henig ε-proper efficient point.

($\Leftarrow$) Let x_0 be a Henig ε-proper Pareto optimal point. Thus using lemma 5.1 we obtain that there exists a convex cone Θ such that $C \subseteq \text{int } \Theta \cup \{0\}$, and further $x_0 \in \varepsilon - E(Y + A|\Theta)$ implying $cone(f(X) + (C + \varepsilon) - (f(x_0)) \bigcap (-\Theta) = \{0\}$ Now if we assume on the contrary that x_0 is not Benson ε-proper Pareto optimal then there exists some $\alpha \in cone(f(X) + (C + \varepsilon) - (f(x_0)) \bigcap (-C\{0\})$. Now since $C\{0\} \subseteq \text{int } \Theta$, one would get a sequence $\{\alpha_i\} \neq \{0\}$ such that $\alpha_i \in cone(f(X) + (C + \varepsilon) - (f(x_0)) \bigcap (-\Theta)$ which is a contradiction. △

Finally we remark that a ε-saddle point theorem (similar to theorem 3.1 Dutta [9]) will hold for Henig ε-weak efficient points with multipliers $\tau_0 \in C^* \setminus \{0\}$.

6 Multiobjective Linear Programming (MOLP)

Consider the following multiobjective linear program,

$(MOLP)$ minBx

sub to $Ax = b$ $\qquad B : m \times n$ matrix, $A : k \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$

$x \geq 0$

Theorem 6.1 *The set of ε^*-efficient solution for MCLP is the same as the set of ε^*-properly efficient solutions. That is ε^*-$Y_{eff} = \varepsilon^*$-Y_{p-eff}*

Proof: The proof follows along the lines of Theorem in Jahn [14]. By definition of ε^* proper Pareto optimality, if $y \in \varepsilon^*$-$Y_{p-eff} \Rightarrow y \in \varepsilon^*$-$Y_{eff}$ To prove the other way, let $x^* \in \varepsilon^* X_{par}$. Let $Z = (-C) \cap (BX + \varepsilon^* - Bx^*)$, where $BX = \{Bx \in \mathbb{R}^m \mid x \in S\}$ and $S = \{x \in \mathbb{R}^n \mid Ax = b\}$. Then there does not exists $\overline{z} \in Z$ with $\overline{z} \neq 0$ (since existence of such a $\overline{z}$ contradicts the ε^*-Pareto optimality of x^*.) Thus $(-C) \cap (BX + \varepsilon^* - Bx^*) = \{0\}$ (if there exists $x \in X$ such that $Bx = Bx^* - \varepsilon^*$), otherwise, $(-C) \cap (BX + \varepsilon^* - Bx^*) = \emptyset$.

Thus in both the cases $(-C) \cap cone(BX + \varepsilon^* - Bx^*) = \{0\}$ (since $cone(BX + \varepsilon^* - Bx^*)$ is a polyhedral cone).

By using separation theorem for convex cones Theorem 3.22, Jahn [14], we obtain that there exists real numbers $t_1, t_2, \dots, t_m$ with $t = (t_1, t_2, \dots, t_m) \neq 0$, such that

$$\sum_{i=1}^{m} t_i y_i \leq 0 \leq \sum_{i=1}^{m} t_i z_i \qquad \forall y \in -C, \forall z \in cone(BX + \varepsilon^* - Bx^*) \tag{6.1}$$

with

$$\sum_{i=1}^{m} t_i y_i < 0 \qquad \forall y \in -C, y \neq 0 \tag{6.2}$$

If we take the vector $y_i = (0, 0, \dots, -1, \dots 0)$ with -1 at i^{th} place then we obtain, $t_i > 0$. Since i was arbitrary so we obtain that $t > 0$. Thus $t \in C_0^*$

The right inequality in 6.1 gives that,

$\sum_{i=1}^{m} t_i(Bx_i + \varepsilon_i^* - Bx_i^*) \geq 0, \forall x \in S$

which implies that

$\sum_{i=1}^{m} t_i(Bx_i + \varepsilon_i^*) \geq \sum_{i=1}^{m} t_i(Bx_i^*), \forall x \in S$

thus Bx^* is an $\langle t, \varepsilon^* \rangle$ minimum of scalar problem, $min_{x \in S} t^t Bx$, thus by Theorem 4.2, we obtain that $x^* \in \varepsilon^*$-X_{p-eff} $\triangle$

Corollary 6.1 *When the ordering cone is $\mathbb{R}^m_+$, then every ε-efficient solution of linear multiobjective problem is Liu ε-properly efficient.*

Corollary 6.2 *When the ordering cone is $\mathbb{R}^m_+$, and $\varepsilon = 0$, then every efficient solution of linear multiobjective problem is Geoffrion properly efficient.*

Corollary 6.3 Theorem 3.1.7 Sawaragi [20] *When the ordering cone is C, and $\varepsilon = 0$, then every Henig efficient solution of linear multiobjective problem is Benson properly efficient.*

7 Conclusions

The results of this paper can be used effectively in Multiobjective Evolutionary Algorithms (MOEA's). It can be easily seen that Lemma 2.2 is also valid for ε-properly Pareto optimal (Liu [17]). This lemma (when used with ε-dominance concept) will be useful in developing $\varepsilon-$*MOEA's* which incorporate Liu's (or Geoffrion's) concept of proper Pareto optimality with an upper bound on M. In such a case we can only check the boundedness condition with only the best non-dominated set of solutions. This can be easily be done since MOEA's are population based approaches. Thus the trade off information can easily be incorporated in an MOEA. The concept of M ε-proper Pareto optimality is useful among other concepts like ε-Pareto optimality, weak ε-Pareto optimality and proper ε-Pareto optimality. Theorem 2.1 shows that if we take the limit of any M ε-proper Pareto solution then it will give only the set of M proper solutions. This cannot be said of any other concepts like ε-Pareto optimality, weak ε-Pareto optimality and proper ε-Pareto optimality, in the limit they get to weak Pareto optimal solutions. In MOEA's the concept of Henig ε-efficiency can be thought of as combining an ε-EMOA with Branke's guidance approach ([4]). Finally the optimality conditions developed in section 5 can be used effectively in MOEA's as a convergent metric for differentiable multiobjective problems, condition 5.8 can be used as a stopping criteria for MOEA's when the ε's are less than certain value. It is felt that such a condition will be useful since checking them a decision maker will become more confident in his solutions. Another advantage of the conditions are that they are valid for any ordering cone and not just the usual $\mathbb{R}^m_+$, thus these conditions can effectively be used as stopping criteria in distributed computing system of Pareto optimal solutions [7].

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