Towards Generalized Dominance Structures for Multi-Objective Optimization

Kalyanmoy Deb
University Distinguished Professor and Koenig Endowed Chair Professor
Department of Electrical and Computer Engineering
Director, Computational Optimization and Innovation (COIN) Laboratory
Michigan State University, East Lansing, Michigan USA
kdeb@msu.edu, https://www.coin-lab.org

COIN Report Number 2022014

Abstract

Various dominance structures are proposed in the multi-objective optimization literature. However, a systematic procedure to understand their effect in determining the resulting efficient set (Pareto-optimal front) is lacking. In this paper, we analyze and lay out properties of generalized dominance structures which will result in a non-empty efficient set. We introduce a concept of anti-dominance structure, derived from the chosen dominance structure, to explain how the resulting non-dominated or efficient set can be identified easily compared to using the dominance structure directly. The concept allows a unified explanation of optimal solutions for both single and multi-objective optimization problems. The anti-dominance structure is applied to analyze respective efficient solutions for most popularly-used static and spatially-changing dominance structures. The theoretical and deductive results of this study can be utilized to create more meaningful dominance structures for practical problems, understand and identify resulting efficient solutions, and help develop better test problems and algorithms for multi-objective optimization.

1 Introduction

In a multi-objective optimization, users should have the flexibility in choosing a dominance structure which would consider objective preferences and priorities of users. In most evolutionary multi-objective optimization (EMO) studies, a set of Pareto-optimal solutions are attempted to be found by an evolutionary population-based algorithm and the choice of a single preferred solution from the Pareto-optimal set is deferred as a post-optimality decision-making task [6, 4]. In the recent past, some exceptions to this principle have shown that the original Pareto-dominance structure can be embedded with known preference information for the resulting EMO to converge to a single preferred Pareto-optimal solution or focus to a preferred Pareto-optimal region at the end of the optimization task [8, 14, 2, 18, 33]. The practicalities and merits of both approaches from computational and decision-making points of view can be debated, but if the preference information is easier to obtain, the latter approach can be appealing from a computational view point.

The EMO and classical optimization literature have proposed a number of alternate dominance structures for this purpose [13, 32, 30, 36, 17], not always motivated by the decision-making preferences used in the real world, rather from considering interesting geometric constructions aided by their practical viability. However, the literature lacks a systematic study outlining what properties a generalized domination structure must have so that an EMO will end up finding

an non-empty efficient set with certain desired properties. This study attempts to fill this gap and answer following questions. Does any arbitrarily chosen dominance structure generate a non-empty efficient set? For a given dominance structure, how does one identify the resulting efficient set for a problem? If a dominance structure cannot produce the desired outcome, are there ways to modify it to find a desired non-empty efficient set? Can the existing dominance structures be analyzed using the developed properties to demonstrate their final outcome?

To achieve our goal, we have formalized a concept of *anti-dominance* structure which is intricately dependent on the chosen dominance structure, but argue that it is more useful than the dominance structure in answering the above questions and to provide a better understanding of the outcome of the chosen dominance structure. In addition of conditions for a non-empty efficient set, we discover a number of interesting and useful properties of these structures to determine if the respective efficient set has a single optimum or multiple optima. Although demonstrated using two-objective problems throighout this paper, the concepts of this paper are applicable to higher objective problems as well. Overall, the results of this paper should enable researchers to have a better insight into a direct understanding of generalized dominance structures and the resulting efficient set, which may be useful in various EMO activities, such as multi-objective test problem generation, efficient algorithm development with a knowledge of sources of algorithmic inefficiencies in finding the true efficient set, design of meaningful generalized dominance structures for an effective application.

In the rest of the paper, we discuss the optimality conditions for single-objective optimization problems in a generalized manner through a superiority structure concept in Section 2. Based on uni-modal and multi-modal single-objective optimization, the concept of anti-superiority structure is introduced so the concept of superiority and anti-superiority structures can be carried over to multi-objective optimization in Section 3. Section 4 applies the concept of anti-dominance structure to explain the working of a number of popular generalized dominance principles proposed in the literature. Then, in Section 5, we extend the use of dominance and anti-dominance structures for spatially changing dominance relationships. Finally, in Section 6, we present conclusions of this study.

2 Optimality Principles for Single-Objective Optimization

For a single-objective minimization problem:

$$\begin{array}{ll} \text{Minimize} & f(\mathbf{x}), \\ \text{Subject to} & g_j(\mathbf{x}) \leq 0, \quad j = 1, 2, \ldots, J, \end{array} \tag{1}$$

in which $\mathbf{x} \in \mathbb{R}^n$ is the variable vector, $f(x) : \mathbb{R}^n \to \mathbb{R}$ is the objective function, and $g_j(x) : \mathbb{R}^n \to \mathbb{R}$ is the j-th inequality constraint. A solution is called *feasible*, if all J constraints are satisfied. Let us denote the feasible space $\mathbf{X} = \{\mathbf{x} | g_j(\mathbf{x}) \leq 0, \forall j\}$ is a set of all feasible solutions in the search space. If a solution is not feasible, it is called an *infeasible* solution. First we define a *superior* solution among a pair of feasible solutions, as follows.

Definition 1 (Single-objective superiority condition)**.** For a pair of feasible solutions $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ is *superior* to $\mathbf{y}$ (say, $\mathbf{x} \prec \mathbf{y}$) in a single-objective sense, if $f(\mathbf{x}) < f(\mathbf{y})$.

We also define a superior set $\omega(\mathbf{x})$, to each of which, $\mathbf{x}$ is superior, as follows:

Definition 2 (Single-objective superior set of $\mathbf{x}$)**.** The set of all feasible solutions $\mathbf{y} \in \mathbf{X}$ for which $\mathbf{x} \prec \mathbf{y}$ is defined here as the *superior set* $\omega(\mathbf{x})$ of $\mathbf{x} \in \mathbf{X}$, and the set of respective objective values is defined as the *superior objective set* $\Omega(\mathbf{x}) = \{f(\mathbf{y}) | \mathbf{x} \prec \mathbf{y}, \forall \mathbf{y} \in \mathbf{X}\}$.

For minimization problems, the superior objective set of $\mathbf{x} \in \mathbf{X}$ can also be written as follows:

$$\Omega(\mathbf{x}) = f(\mathbf{x}) + \{\delta | \delta = f(\mathbf{y}) - f(\mathbf{x}) > 0, \forall \mathbf{y} \in \mathbf{X}\}. \tag{2}$$

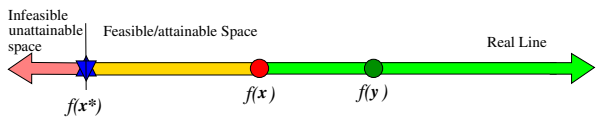
The above superiority condition respects the following properties:

- **Irreflexive Property:** A solution $\mathbf{x}$ is not superior to itself, that is, $\mathbf{x} \not\prec \mathbf{x}$.
- **Asymmetric Property:** If $\mathbf{x} \prec \mathbf{y}$, then $\mathbf{y} \not\prec \mathbf{x}$.
- **Transitive Property:** If $\mathbf{x} \prec \mathbf{y}$ and $\mathbf{y} \prec \mathbf{z}$, then $\mathbf{x} \prec \mathbf{z}$.

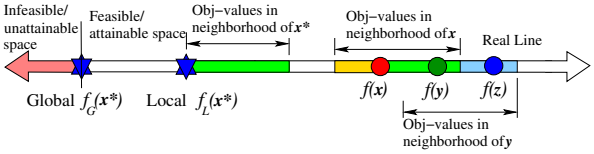
Now, we define the optimality principle for a single-objective optimization problem.

Definition 3 (Single-objective optimality condition)**.** A solution $\mathbf{x}^* \in \mathbf{X}$ is a *minimum* solution in the single-objective sense, if there does not exist any other feasible solution $\mathbf{x} \in \mathbf{X}$ that belongs to the superior set of $\mathbf{x}^*$.

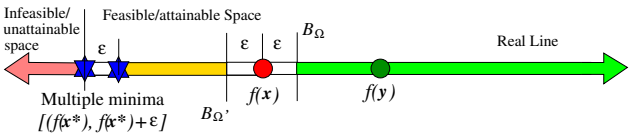
That is, there is no $\mathbf{x} \in \mathbf{X}$, such that $f(\mathbf{x}) < f(\mathbf{x}^*)$. Graphically, the above condition can be demonstrated with a sketch of the objective function in a real line ($\mathbb{R}$), as shown in Figure 1a. The feasible solutions in the search space have objective values larger than and equal to $f(\mathbf{x}^*)$. Since there does not exist any feasible objective value smaller than $f(\mathbf{x}^*)$, $\mathbf{x}^*$ is the minimum solution. The respective objective value is called an minimum point.



(a) Usual minimization.



(b) Global/local minimization.



(c) Epsilon-minimization.

Figure 1: Superiority and optimality definitions are illustrated for a single-objective problem.

Note that the above definition is also valid for each of multiple globally minimum solutions, if exist in a problem. For a locally minimum solution, the condition $\mathbf{x} \in \mathbf{X}$ is restricted to a local neighborhood $\mathcal{B}_\gamma(\mathbf{x}) = \{\mathbf{y} | \ (\|\mathbf{y} - \mathbf{x}\|_2 \leq \gamma) \wedge (\mathbf{y} \in \mathbf{X})\}$ around $\mathbf{x}$. Figure 1b shows the respective objective values in the neighborhood of $\mathbf{x}$. Note that the irreflexive and asymmetric properties are still valid for a locally minimum solution, however, the transitivity solution may not be satisfied among three feasible solutions ($\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{X}$, and $\mathbf{z} \in \mathbf{X}$), but the following semi-transitive property is satisfied:

- **Semi-transitive Property:** If $\mathbf{x} \prec \mathbf{y}$ and $\mathbf{y} \prec \mathbf{z}$, then even if $\mathbf{x} \not\prec \mathbf{z}$, $\mathbf{z}$ must not be superior to $\mathbf{x}$, or $\mathbf{z} \not\prec \mathbf{x}$.

Theorem 1 (Nonexistence of optimum)**.** *If the both transitive and semi-transitive properties are violated by a superiority condition, there can not exist a minimum solution.*

Proof. This can be proven by contradiction. If there exist an minimum solution $\mathbf{x}^*$, then consider two solutions $\mathbf{y}$ and $\mathbf{z}$, in which $\mathbf{y}$ satisfies $\mathbf{x}^* \prec \mathbf{y}$ and $\mathbf{z}$ satisfies $\mathbf{y} \prec \mathbf{z}$. Since, both transitivity and semi-transitivity conditions are not satisfied, $\mathbf{x}^* \not\prec \mathbf{z}$ and $\mathbf{z} \prec \mathbf{x}^*$, thereby violating the optimality condition stated in Definition 3. □

The generality from a unique global minimum solution (Definition 3) to multi-modal minimum solutions allows to smoothly transit to multi-objective case, but before that, we would like to define a new term for single-objective case which will become a non-trivial matter in the multi-objective case.

Definition 4 (Anti-superiority condition)**.** A solution $\mathbf{x} \in \mathbf{X}$ is *anti-superior* to another solution $\mathbf{y} \in \mathbf{X}$, if $\mathbf{y} \prec \mathbf{x}$, or if $f(\mathbf{x}) > f(\mathbf{y})$.

A little thought will lead us to define an *anti-superior objective set* $\Omega'(\mathbf{x}) = f(\mathbf{x}) + \{\delta | \delta = f(\mathbf{y}) - f(\mathbf{x}) < 0, \forall \mathbf{y} \in \mathbf{X}\}$. Note that two sets ($\Omega(\mathbf{x})$ and $\Omega'(\mathbf{x})$) do not have any common element according to asymmetric property; in other words, $\Omega'(\mathbf{x}) \cap \Omega(\mathbf{x}) = \emptyset$. The set $\Omega'(\mathbf{x})$ is important in determining the minimum solution, as optimality means that there does not exist any better solution:

Definition 5 (Alternate single-objective optimality condition)**.** A solution $\mathbf{x}^* \in \mathbf{X}$ is a minimum solution in single-objective sense, if its anti-superior objective set is empty, or $\Omega'(\mathbf{x}^*) = \emptyset$.

This also implies the following:

Theorem 2 (Empty anti-superior set)**.** *If the anti-superior set* $\omega'(\mathbf{x}^*) = \{\mathbf{y} | \mathbf{y} \prec \mathbf{x}^*, \forall \mathbf{y} \in \mathbf{X}\} = \emptyset$*, then* $\mathbf{x}^*$ *is a minimum solution.*

The proof is intuitive.

Theorem 3 (A necessary condition for existence of a minimum)**.** *For a bounded feasible set, a necessary condition for the existence of a minimum solution is* $\Omega \cap \Omega' = \emptyset$*.*

Proof. The condition $\Omega \cap \Omega' = \emptyset$ means that for every solution $\mathbf{x}$, two non-overlapping sets – one better than $\mathbf{x}$ and another worse than $\mathbf{x}$ – exist. Since the feasible set is bounded, all feasible solutions in the search space can be ordered in ascending order using the objective function f. For the top-most solution ($\mathbf{x}^{(1)}$) on the ordered list, by definition, $\Omega'(\mathbf{x}^{(1)}) = \emptyset$. Hence, the proof. □

3 Optimality Principles for Multi-Objective Optimization

We extend the above definitions of a superior and minimum solutions for a multi-objective optimization problem having M conflicting objectives, formulated as follows:

$$\begin{array}{ll} \text{Minimize} & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x})\}, \\ \text{Subject to} & g_j(\mathbf{x}) \leq 0, \quad j = 1, 2, \ldots, J. \end{array} \tag{3}$$

First, we extend and define the superiority of a solution over another, as follows:

Definition 6 (Multi-objective superiority condition)**.** For a pair of solutions $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ is *superior* to $\mathbf{y}$ in a multi-objective sense, if $\mathbf{x}$ *dominates* $\mathbf{y}$ ($\mathbf{x} \prec \mathbf{y}$).

The above calls for a definition of domination. The original dominance condition[1] ($\prec_O$) is defined as follows:

Definition 7 (Original dominance condition)**.** A solution $\mathbf{x} \in \mathbf{X}$ dominates another solution $\mathbf{y} \in \mathbf{X}$, if the following two conditions are true: (i) $f_i(\mathbf{x}) \leq f_i(\mathbf{y})$ for all $i = 1, 2, \ldots, M$, and (ii) $f_i(\mathbf{x}) < f_i(\mathbf{y})$ for at least one $i = 1, 2, \ldots, M$.

[1]The multi-objective optimization literature [25, 11] refers it to as dominance relation, but for our discussions here, we call it the original dominance relation.

Note that for single-objective case, $M = 1$ and the above dominance condition becomes equivalent to the superiority condition defined in Definition 1. Now, we are ready to define an minimum solution for a multi-objective optimization problem:

Definition 8 (Multi-objective optimality condition)**.** A solution $\mathbf{x}^* \in \mathbf{X}$ is *minimum* in multi-objective sense, if there does not exist any other feasible solution $\mathbf{x} \in \mathbf{X}$ that is superior to $\mathbf{x}^*$.

That is, there is no $\mathbf{x} \in \mathbf{X}$, such that $\mathbf{x} \prec_O \mathbf{x}^*$. Note again that there can be multiple minimum solutions according to the above definition. In fact, if the minimum solution $\mathbf{x}^{*,i}$ of the single-objective constrained optimization problem with i-th objective function is different from that ($\mathbf{x}^{*,j}$) of j-th objective function, then $\mathbf{x}^{*,i} \not\prec_O \mathbf{x}^{*,j}$ and $\mathbf{x}^{*,j} \not\prec_O \mathbf{x}^{*,i}$ can both occur. Both solutions then become minimum solutions to the multi-objective optimization problem. Besides these individual minimum solutions, many other *compromise* solutions, trading off the objectives, exist in the search space. All these minimum solutions are called Pareto-optimal solutions and the set is referred to as the Pareto-optimal set ($\mathbf{X}_{\text{eff}}$). The respective set of objective vectors is called the *efficient* set ($\mathbf{Z}_{\text{eff}}$) or is called to constitute a *Pareto-optimal front* (PF), in the parlance of the EMO literature.

Interestingly, a dominance (hence, the superiority) operator for multi-objective optimization problems, must also satisfy irreflexive and asymmetry properties mentioned for single-objective case, and either transitive or semi-transitive property must be satisfied as well. A violation of both transitive and semi-transitive properties will result in an empty Pareto-optimal set, as in the case of single-objective case (Theorem 1). There exist a number of definitions for dominance in the literature and more can be defined to by following the properties of the dominance relations.

3.1 Anti-Superior Objective Set in Multi-objective Optimization

While it was straightforward to define the anti-superior set and anti-superior objective set for the single-objective case, due to the simple switching of '$<$' operator to the '$>$' operator in the superiority condition, it is not so obvious in the multi-objective case. In this section, we develop a relationship between superior and anti-superior objective sets $\Omega(\mathbf{x})$ and $\Omega'(\mathbf{x})$, respectively, for multi-objective case.

3.2 Defining the Anti-dominant Set

Let us consider Equation 2 defining the original superiority condition for single-objective optimization and generalize it with a parameter $\epsilon = 0$:

$$\Omega(\mathbf{x}, \epsilon) = f(\mathbf{x}) + \{\delta | \delta = f(\mathbf{y}) - f(\mathbf{x}) > \epsilon, \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{X}\}. \tag{4}$$

The above allows a generalization of the superior objective set with an epsilon-optimality condition in which a solution $\mathbf{x}$ is considered better than $\mathbf{y}$, if $f(\mathbf{x}) < f(\mathbf{y}) - \epsilon$, allowing a more practice-oriented superiority definition, as shown in Figure 1c. Writing Ω in terms of difference in objective values, we can define a *null* superior set, $\Omega_0(\epsilon) = \{\delta | \delta > \epsilon, \forall \delta \in \mathbb{R}\}$, where $\mathbb{R}$ is a part of the real line allowed by feasible objective values of the problem. Thus, $\Omega(\mathbf{x}, \epsilon) = f(\mathbf{x}) + \Omega_0(\epsilon)$. The defining boundary of the superior set $\Omega_0(\epsilon)$ ($B_\Omega = \epsilon$ in this case) is an important feature for an user to define the desired superiority of a solution in the objective space. The boundary of the superior set can be inclusive (B_Ω^I) to the set or exclusive (B_Ω^E).

It is interesting to note that the respective anti-superior objective set is $\Omega'(\mathbf{x}, \epsilon) = f(\mathbf{x}) + \Omega'_0(\epsilon)$, where $\Omega'_0 = \{\delta | \delta < -\epsilon, \forall \delta \in \bar{\mathbb{R}}\}$ is the null anti-superior set. Writing $\Omega'_0 = \{-\delta | \delta > \epsilon, \forall \delta \in \bar{\mathbb{R}}\}$ we notice that $\Omega'_0(\epsilon) = -\Omega_0(\epsilon)$. The defining boundary of the anti-superior set is denoted as $B_{\Omega'}$ and is equal to $-\epsilon$ for the epsilon-superiority definition. Figure 1c shows that solutions with objective values left of $B_{\Omega'}$ are superior than $\mathbf{x}$. Epsilon-superiority condition

leads to multiple optimal solutions. All solutions $\mathbf{x}$ which are within ϵ from the original optimal solution's objective value $f(\mathbf{x}^*)$ are now epsilon-optimal.

The defining boundary feature in defining superiority is useful to define a generalized dominance condition for multi-objective case. Since $\boldsymbol{\epsilon} \in \mathbb{R}^M$, one can define a boundary $\Omega_0(\boldsymbol{\epsilon})$ at the origin to declare the part of the M-dimensional objective space that are dominated (or worse) than the origin. Figure 2 shows one such generalized dominance definition (Ω_0^{GD}) at the origin. It implies that up to a limit of $-\epsilon_1$ on f_1 from origin, the origin is superior to any point with a trade-off (loss/gain) larger than T_1. A similar trade-off of T_2 exists for a limit of $-\epsilon_2$ in f_2. To be consistent with the EMO literature, superior objective set Ω_0 and anti-superior objective set Ω'_0 can also be referred to as *dominant* objective set and *anti-dominant* objective set, respectively. Clearly, the chosen generalized dominant objective set Ω_0^{GD} is a superset of the original dominant objective set ($\Omega_0^{\text{GD}} \supset \Omega_0^{\text{O}}$). Such a relationship provides a more practical approach to multi-objective optimization.

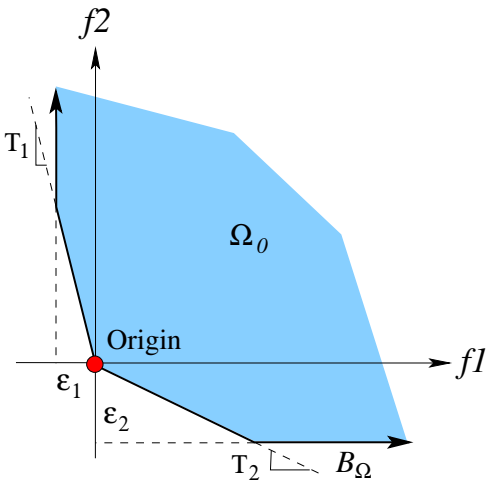


Figure 2: A generalized dominance structure Ω_0^{GD} and its boundary B_Ω^{GD}.

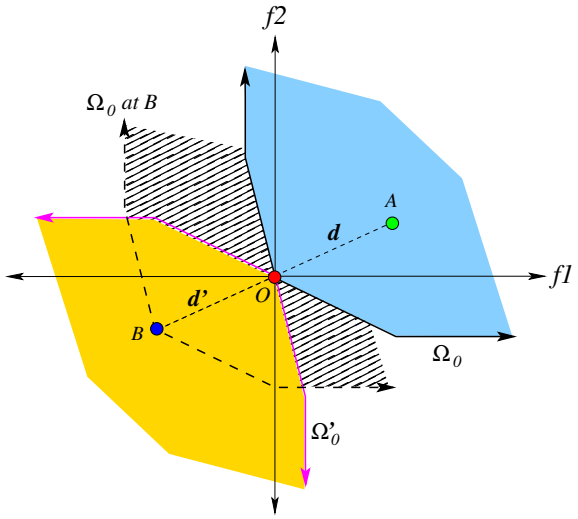


Figure 3: Anti-superior objective set Ω'_0.

3.3 Properties of Generalized Anti-dominant Objective Set

Every member of the dominant objective set Ω_0 is dominated by (or, worse than) the origin and every member of anti-dominant objective set Ω'_0 dominates (or, is better than) the origin. As indicated for the single-objective case, the two sets are related by a simple relation: $\Omega'_0 = -\Omega_0$. The following theorem states that this is a universal property, even for the multi-objective case.

Theorem 4 (Relationship between dominant and anti-dominant sets)**.** *For any dominant objective set Ω_0, its anti-dominant objective set $\Omega'_0 = -\Omega_0$.*

Proof. Let us consider Figure 3 for a proof with a generic dominance structure (Ω_0^{GD}) defined at the origin (point O). Let us consider a generic point B at $\mathbf{d} \in \Omega_0^{\text{GD}}$ from O. Let us construct a point B at $\mathbf{d}' = -\mathbf{d}$ from O, Now, construct the dominance structure Ω_0^{GD} at B (shown in shaded region), as if it is the new origin. Then, the original origin (point O) is now at $\mathbf{d}$ location from the new origin (B). Since $\mathbf{d}$ is within the set Ω_0^{GD}, new origin B dominates the original origin O. This is true for every $\mathbf{d}$ and thus Ω'_0 can be constructed with negative vectors of every member of Ω_0^{GD}. □

Corollary 4.1. *For any dominant objective set Ω_0 with a defining boundary B_Ω, the defining boundary of the anti-dominant objective set $B_{\Omega'} = -B_\Omega$.*

The above corollary is true for both inclusive and exclusive boundary sets. Like for single-objective case, we now discuss the properties that a generalized dominance relationship (Ω_0) must have:

- **Irreflexive Property:** Ω_0 (and its boundary B_Ω) must exclude the **0**-vector (origin) from its set.
- **Asymmetric Property:** $\Omega_0 \cap \Omega'_0 = \emptyset$.
- **Transitive Property:** This requires a chain of Ω_0 consideration and requires further discussions, provided below.

The first two properties indicate that a GD structure can have either an inclusive boundary with common points between B^I_Ω and $B^I_{\Omega'}$, or an exclusive boundary with no common points: $B^E_\Omega \cap B^E_{\Omega'} = \emptyset$. For inclusive boundary GDs, $\Omega_0 \cap \Omega'_0 \neq \emptyset$ is possible.

Like in the case of single-objective optimization, following theorems are true for multi-objective optimization:

Theorem 5 (Multi-objective optimality condition)**.** *A solution $\mathbf{x}^* \in \mathbf{X}$ is a minimum solution in a multi-objective sense with a generalized dominance structure Ω_0, if its anti-superior objective set is empty, or $\Omega'_0(\mathbf{x}^*) = \emptyset$.*

This also implies the following:

Theorem 6 (Empty anti-superior set)**.** *If the anti-superior set $\Omega'_0(\mathbf{x}^*) = \emptyset$ for a given generalized dominance structure Ω_0, then $\mathbf{x}^*$ is a minimum solution for Ω_0 structure.*

Proofs for both the above theorems are intuitive. Extending Theorem 3 to multi-objective optimization, we have

Theorem 7 (A necessary condition for existence of a Pareto-optimal solution)**.** *For a bounded feasible search space, a necessary condition for the existence of a Pareto-optimal solution is $\Omega_0 \cap \Omega'_0 = \emptyset$.*

The proof is similar to that presented for single-objective case, except the ordering can be done using the chosen domination principle. The above also indicates the following corollary is true, as it suggests that there are overlapping solutions between Ω_0 and Ω'_0, arising at least from the dominance boundary points.

Corollary 7.1. *If $\Omega_0 \cup \Omega'_0 = \mathbb{R}^M \backslash \{\mathbf{0}\}$, no Pareto-optimal solution exists.*

Theorem 8 (A necessary condition for existence of multiple Pareto-optimal solutions)**.** *A necessary condition for the existence of multiple Pareto-optimal solutions is to have following properties: $\Omega_0 \cup \Omega'_0 \subset \mathbb{R}^M \backslash \{\mathbf{0}\}$ and $\Omega_0 \cap \Omega'_0 = \emptyset$.*

If boundary points, along with the origin, are excluded from the real space to match the union of Ω_0 and Ω'_0, then one or more Pareto-optimal solutions may exist.

Corollary 8.1. *If $\Omega_0 \cup \Omega'_0 = \mathbb{R}^M \backslash (B_\Omega \cup B_{\Omega'} \cup \{\mathbf{0}\})$, the exclusive boundaries are equal ($B^E_\Omega = B^E_{\Omega'}$) and the efficient set must lie on B_Ω or $B_{\Omega'}$.*

The above theorem is valid for exclusive boundary GDs. The weighted-sum domination structure (Figure 7d) satisfies the above condition and makes all objective vectors on the boundary plane B^E_Ω having a normal vector ($\mathbf{w}$) non-dominated to the origin.

3.3.1 Transitive and semi-transitive properties

Ehrgott [11] has shown that the dominance relationship must be cone to have the transitive property. A cone (C) is a set of points for which if $\mathbf{z} \in C$, then $\lambda\mathbf{z} \in C$ for any $\lambda > 0$. Thus, it is natural that if Ω_0 is a cone satisfying irreflexive and asymmetric properties mentioned above, then the transitive property will be satisfied.

However, we argue here that Ω_0 structure need not be a cone, but can be more generic, but must satisfy irreflexive, asymmetric, and semi-transitive properties to have a non-empty efficient set. Let us first demonstrate this fact graphically with the generalized dominance structure considered in Figure 2. We observe from Figure 4 that when a dominated point $\mathbf{y}$ is chosen from Ω_0^{GD} at $\mathbf{x}$ and another point $\mathbf{z}$ is chosen from Ω_0^{GD} at $\mathbf{y}$, $\mathbf{x}$ may not dominate $\mathbf{z}$, in general. This violates the transitive property, but we observe from the figure that this specific dominance structure satisfies the semi-transitive property in that $\mathbf{z}$ does not dominate $\mathbf{x}$. Thus, an important task is to determine the true nature of the *effective* domination structure when intermediate points such as $\mathbf{y}$ are allowed in the optimization process.

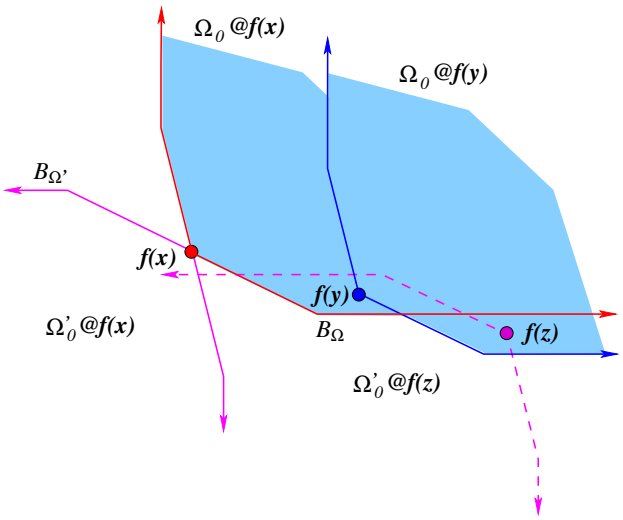


Figure 4: Ω_0^{GD} is not transitive, but follows the semi-transitive property.

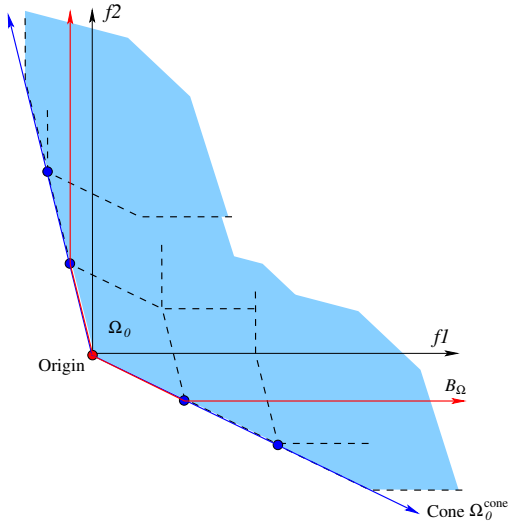


Figure 5: Repeated application of semi-transitive Ω_0 creates a transitive dominance condition Ω_0^{cone}.

For a population-based optimization algorithm, such as evolutionary multi-objective optimization (EMO) [6, 4], the set of non-dominated population members (which are not dominated by any other population member) is determined by a comparing every population member with every other for domination. Thus, in an EMO population, if all three solutions ($\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$) exist, $\mathbf{z}$ will not be in the same non-dominated set with $\mathbf{x}$ due to the presence of $\mathbf{y}$ as a catalyst in the population. This may not be possible for a point-based multi-objective optimization approach, which it would usually check two competing solutions at every iteration. If we continue the chain of domination structure on the specific problem in Figure 2, we observe that with the presence of catalyst solutions, the effective dominant structure of $\mathbf{x}$ is a cone, shown in Figure 5, which has the transitive property.

This example illustrates how a semi-transitive dominance structure can exhibit an effective transitive dominance (ETD) behavior in presence of a population of catalyst solutions, such as solution $\mathbf{y}$. This is an important distinction between population-based and point-based multi-objective optimization algorithm, allowing EMO researchers and practitioners to consider a more relaxed dominance structures to suit their practical needs. This concept leads to the following theorem.

Theorem 9 (Semi-transitive and effective transitive dominance structure relation)**.** *For a semi-transitive dominance structure having* $\Omega_0 \cap \Omega'_0 = \emptyset$*, its effective transitive dominance structure also follows the same principle:* $\Omega_0^{\text{ETD}} \cap \Omega'^{\text{ETD}}_0 = \emptyset$*.*

Proof. Consider three points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in Figure 4. The semi-transitive property indicates that $\mathbf{x} \prec_{\text{GD}} \mathbf{y}$, and $\mathbf{y} \prec_{\text{GD}} \mathbf{z}$, but $\mathbf{z} \not\prec_{\text{GD}} \mathbf{x}$. The final property indicates that $\mathbf{z}$ cannot lie in Ω'^{GD}_0 of $\mathbf{x}$. Clearly, $\mathbf{y} \in \Omega_0^{\text{ETD}}$ and $\mathbf{z} \in \Omega_0^{\text{ETD}}$ of $\mathbf{x}$, as the ETD structure is the collection of all such dominated points, thereby yielding $\mathbf{x} \prec_{\text{ETD}} \mathbf{z}$. If $\mathbf{z}$ must lie in Ω'^{ETD}_0 too, it means that there exist a chain of points starting with $\mathbf{z}$, making $\mathbf{z} \prec_{\text{GD}} \mathbf{y}'$ and $\mathbf{y}' \prec_{\text{GD}} \mathbf{x}$. Since each and every point $\mathbf{y}'$ which is dominated by $\mathbf{z}$ is also a member of Ω_0^{ETD} (by semi-transitive property of GD structure), $\mathbf{y}' \not\prec_{\text{GD}} \mathbf{x}$. Hence, the chain breaks, and $\mathbf{z}$ cannot be a member of both Ω_0^{ETD} and Ω'^{ETD}_0. □

Let us consider a GD structure Ω_0 at origin indicating the region inside the blue circle dominated by the origin, as shown in Figure 6. The origin is at $5\pi/4$ radian from the positive f_1-axis at the center of the circle. Its anti-dominant structure Ω'_0 is shown in golden color. Clearly, $\Omega_0 \cap \Omega'_0 = \emptyset$. Thus, this circle dominance structure is expected to produce a non-empty efficient set. However, to understand the exact nature of the efficient set, we notice that the above circle dominance structure is semi-transitive. An analysis reveals that the effective transitive dominance (ETD) structure is a region above the -45^{deg} line: $\Omega_0^{\text{ETD}} = \{\mathbf{f}(\mathbf{x}) | f_1(\mathbf{x}) + f_2(\mathbf{x}) > 0\}$, shown in the figure. The respective anti-dominant structure is $\Omega'^{\text{ETD}}_0 = \{\mathbf{f}(\mathbf{x}) | f_1(\mathbf{x}) + f_2(\mathbf{x}) < 0\}$. A careful thought reveals that the ETD structure is identical to the weighted-sum dominance structure with equal weight to each objective and according to Theorem 9, the respective ETD and anti-ETD structures are also non-overlapping. Thus, although we wanted to establish a circle-dominance concept to find respective efficient points, but effectively an EMO will establish a weighted-sum dominance structure with equal weight to find the efficient points.

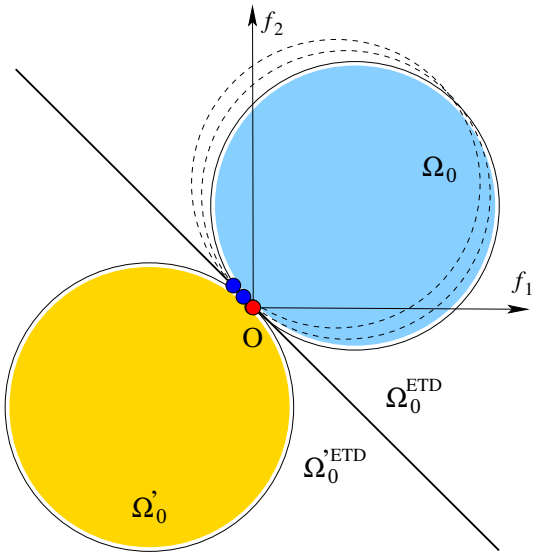


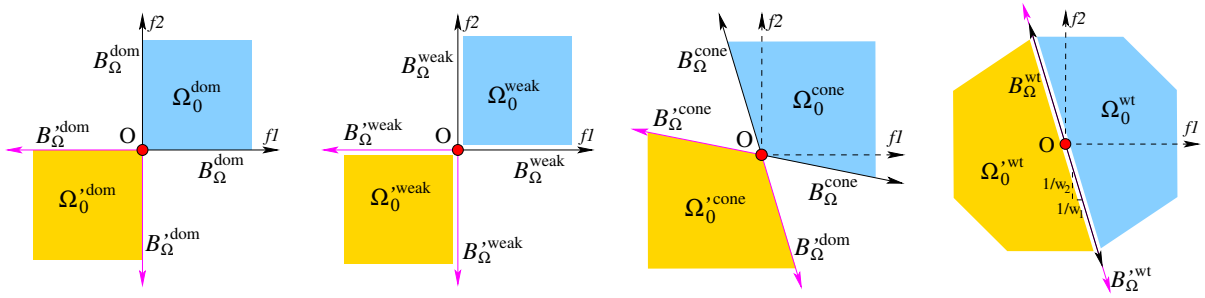
Figure 6: A user-specified circle dominance Ω_0^{circle} results in weighted-sum dominance as an effective transitive dominance structure.

For semi-transitive dominance structures, the results (both theoretical and experimental) of this paper extend to their effective transitive dominance structures.

3.4 Commonly-used Dominance Structures

Figure 7 shows the Ω_0 for a number of commonly used domination structures in the literature and their respective Ω'_0 set for two-objective problems. They are extendable to higher dimensions as well.

First, the original domination structure ($\Omega_0^{\text{dom}} = \{\mathbf{d} |\ d_i \geq 0, \forall i \wedge\ \mathbf{d} \neq \mathbf{0}\}$), in which a point dominates all points in its first quadrant (for two objectives) without itself, is shown in Figure 7a. It's respective $\Omega'^{\text{dom}}_0 = -\Omega_0^{\text{dom}} = \{\mathbf{d} |\ d_i \leq 0, \forall i \wedge\ \mathbf{d} \neq \mathbf{0}\}$ is, by definition, the third quadrant without itself. Since $\Omega_0 \cup \Omega'_0 \subset \mathbb{R}^M \backslash \{\mathbf{0}\}$ (second and third quadrants are not in the set $\Omega_0 \cup \Omega'_0$, hence the subset symbol $\subset$), the efficient set is likely to have multiple points, according to Theorem 8.



(a) Original domination. (b) Weak domination. (c) Cone domination. (d) Weighted-sum domination.

Figure 7: Commonly-used domination structures. In each case, $\Omega_0 \cap \Omega'_0 = \emptyset$. For Figs. 7b and 7d, the boundary is exclusive, while in the other two the boundary is inclusive.

Second, the weakly domination structure ($\Omega_0^{\text{weak}} = \{\mathbf{d} |\ d_i > 0, \forall i\}$), in which a point dominates all points in the interior of its first quadrant (for two objectives), is shown in Figure 7b. It's respective $\Omega'^{\text{weak}}_0 = -\Omega_0 = \{\mathbf{d} |\ d_i < 0, \forall i\}$ is the interior of the third quadrant.

Next, Ω_0^{cone} and its respective Ω'^{cone}_0 of the commonly-used cone domination structure [11, 25] are shown in Figure 7c. Note that for a wider cone structure (cone angle more than 180 degrees for two objectives), the asymmetric property is violated, meaning $\Omega_0^{\text{cone}} \cap \Omega'^{\text{cone}}_0 \neq \emptyset$ Such a structure will result in an empty efficient set, resulted from Theorem 7 .

For the weighted-sum approach, $\Omega_0^{\text{wt}} = \{\mathbf{d} |\ \sum_{i=1}^{M} w_i d_i > 0\}$, the resulting $\Omega'^{\text{wt}}_0 = \{\mathbf{d} |\ \sum_{i=1}^{M} w_i d_i < 0\}$ and is shown Figure 7d. Here, $B^E_{\Omega'} = B^E_\Omega$. According to Corollary 8.1, efficient point(s) must lie on the boundary B^E_Ω.

We discuss a few more existing generalized domination structures in Section 4, but next we discuss an important matter of identifying the generalized efficient set for a given generalized dominance structure.

3.5 Identifying Generalized Non-dominated Set

For a given Ω_0 structure, a member of the generalized non-dominated (GND) set in a finite population of objective vectors P is defined as follows:

Definition 9 (Generalized non-dominated set)**.** A feasible objective vector $\mathbf{f}$ in a finite set P is a member of the generalized non-dominated set $P^{\text{GND}} \subset P$, if no other member of P lies in the anti-dominant set at $\mathbf{f}$, or $\mathbf{z} \notin (\mathbf{f} + \Omega'_0)\,,\ \forall \mathbf{z} \in P$.

Let us reconsider Figure 8 with a GD structure with four members in set P. The above definition can be used to identify if the point O is non-dominated. There can be two approaches for checking it. First, the Ω_0 set can be translated to every other feasible point in P, such as A, B

and C, and check if O lies on the respective Ω_0 set. With three other points in P, three translations of Ω_0 are needed. As clear from the figure, point O does not lie in the Ω_0 set of all three population members. Only at the end of three checks, we know that Point O is a non-dominated point in the population. In the second approach with anti-dominance structure, the set Ω'_0 can be put only on O and check if there is any other population member which lies on the respective Ω'_0 set. It is seen that $\Omega'_0 = \emptyset$, meaning that point O is a non-dominated point in the population. Since the latter involves a single translation of the Ω'_0 set, rather than translating Ω_0 multiple times, the second approach is a computationally faster non-domination check approach [7]. In this regard, the creation of Ω'_0 from a given Ω_0 becomes an important task for non-domination check or also for non-dominated sorting purposes. The above discussion also results in the following theorem.

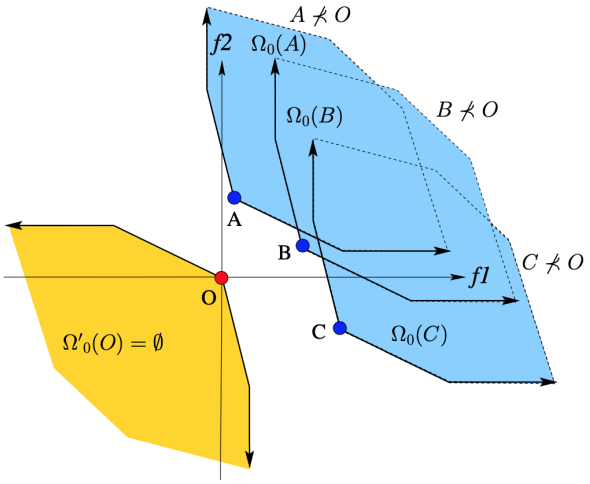


Figure 8: The set Ω_0 needs to be applied many times to identify a GND point, but $\Omega_0{}'$ needs to be applied once at Point O to identify if it is on the GND set.

Theorem 10 (Identical efficient sets)**.** *For two identical Ω_0 (or, two identical Ω'_0 sets), the respective efficient sets are identical.*

Proof. Since for a given Ω_0, Ω'_0 set is unique and since the non-domination check is performed with Ω'_0, if the resulting Ω'_0 for two domination principles are identical, the resulting efficient sets will also be identical. □

3.6 Identifying Generalized Efficient Set

If Ω_0 satisfies all three properties (irreflexive, asymmetry and transitive or semi-transitive), then there will exist a non-empty efficient set. Ω'_0 can be used directly with the following theorem to identify an efficient point:

Theorem 11 (Generalized efficient set)**.** *A feasible point* $\mathbf{x}$ *is efficient with respect to a generalized dominance structure Ω_0, if $(\mathbf{f}(\mathbf{x}) + \Omega'_0) \cap \mathbf{Z} = \emptyset$.*

Proof. Since there does not exist any feasible objective vector in the anti-superior objective set of $\mathbf{x}$ in the entire feasible search space $\mathbf{Z}$, there is no solution to dominate (in the sense of Ω_0) it. Hence, $\mathbf{x}$ is an efficient point. □

The above theorem can used to achieve following tasks computationally or theoretically.

1. First, it can be used to test if an objective vector $\mathbf{f}$ is a potential efficient point, as discussed above, but instead of restricting the check in a finite sampled set P, every feasible point from

the search space must be considered. Although it is a computationally challenging task, the concept can be used theoretically or in a geometric sense.

2. Second, Ω'_0 can be used to identify the entire efficient set for a given feasible objective set $\mathbf{Z}$. This task will be useful for studies involving test problems and requires an identification of the exact efficient set ($\mathbf{Z}^{\text{eff}}$) from $\mathbf{Z}$ for a given dominance structure. The theoretical procedure is to apply Ω'_0 at every point of $\mathbf{Z}$ systematically by repeating the test only to Ω'_0 solutions in a nested manner. This will allow a faster computational procedure to identify the efficient set.

3. Third, knowing one or more efficient points, Ω_0 can help identify further efficient points quickly by eliminating the dominated solutions from its Ω_0 set and narrowing down the search for further efficient points. However, in such a task, often the relevant boundary points of $\mathbf{Z}$ are tested for their Pareto-optimality. Starting with extreme boundary points of $\mathbf{Z}$, Ω'_0 can immediately test if the point is a member of the efficient set. If yes, the test can continue to the neighboring extreme boundary point, and so on. If no, the test will identify the points in the Ω'_0 set that dominated the extreme point and a new test can be executed to members of the Ω'_0 set.

Clearly, Ω'_0 enables a faster way to identify the efficient set than Ω'_0, simply because of its ability to identify points that dominate the current point under consideration, thereby not only allowing to determine if the current point is an efficient point and also narrowing down the search for an efficient point. For the generalized dominance structure (Ω_0^{cone}) shown in the inset of Figure 9, the resulting $\mathbf{Z}_{\text{eff}}^{\text{cone}}$ is determined for a hypothetical $\mathbf{Z}$ in the figure. The $\mathbf{Z}_{\text{eff}}^{O}$ according to the original dominance principle is the entire line 1-10, whereas the $\mathbf{Z}_{\text{eff}}^{\text{cone}}$ consists of line segments 2-4, 6-7, and 9-10. For the extreme boundary point 1, Ω'_0 is not empty, thus, point 1 cannot be an efficient point. It also helped to identify the boundary region (points between 1 and a) which must be tested next. Since the relevant boundary in this problem comes from piece-wise linear segments and a cone-domination structure is used, we restrict our testing only to extreme points of the line segments. By testing Point 2 with Ω'_0, it is clear that $\Omega'_0 = \emptyset$. Thus, Point 2 is a member of the efficient set. This can continue systematically to identify the entire efficient set (line segments 2-4, 6-7, and 9-10).

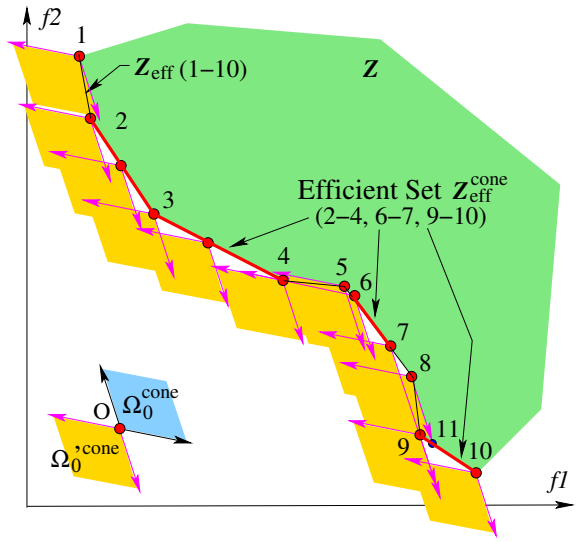


Figure 9: Identification process of $\mathbf{Z}_{\text{eff}}^{\text{cone}}$ using a generalized dominance structure Ω_0^{cone}.

The set Ω_0 at any feasible point $\mathbf{f}$ can also be used to identify a special scenario:

Theorem 12 (Singleton efficient point)**.** *If* $\mathbf{Z} - (\mathbf{f}(\mathbf{x}) + \Omega_0) = \{\mathbf{f}(\mathbf{x})\}$*, then the dominance structure* Ω_0 *produces a single efficient point (*$\mathbf{f}(\mathbf{x})$*) for the problem.*

Proof. The condition signifies that $\mathbf{x}$ dominates every point in $\mathbf{Z}$ and thus, no member of $\mathbf{Z}$ dominates $\mathbf{x}$. Hence, it is the only efficient point. □

If the weighted-sum dominance structure, demonstrated in Figure 7d, satisfies the above condition for a specific solution $\mathbf{x}$, it is the singleton efficient point for the problem [25]. The following generic properties of dominance structures are also useful for identifying efficient sets [11].

Theorem 13. *If* $\Omega_0^{(1)}$ *is a subset of* $\Omega_0^{(2)}$*, then the resulting efficient set* $\mathbf{Z}_{\text{eff}}^{(2)}$ *is a subset of* $\mathbf{Z}_{\text{eff}}^{(1)}$*.*

Proof. Since $\Omega_0^{(1)} \subset \Omega_0^{(2)}$, ${\Omega'}_0^{(1)} \subset {\Omega'}_0^{(2)}$. Thus, every member of ${\Omega'}_0^{(1)}$ that dominates $\mathbf{x}$, also exists in ${\Omega'}_0^{(2)}$ and dominates $\mathbf{x}$. Moreover, there can be additional members of ${\Omega'}_0^{(2)}$ that dominate $\mathbf{x}$. Hence, the efficient set $\mathbf{Z}_{\text{eff}}^{(2)}$ is a subset of $\mathbf{Z}_{\text{eff}}^{(1)}$. □

Two corollaries follow from this theorem.

Corollary 13.1. *For any* $\Omega_0 \subset \Omega_0^O$, $\mathbf{Z}_{\text{eff}}^O \subset \mathbf{Z}_{\text{eff}}$.

The above corollary means that any dominance structure that is weaker than the original dominance condition may declare originally dominated points as efficient points, thereby having a larger efficient set. An example is when $\Omega_0 = \Omega_0^{\text{weak}}$. The original efficient set is a subset of the *weakly* efficient set.

Corollary 13.2. *For any* $\Omega_0^O \subset \Omega_0$, $\mathbf{Z}_{\text{eff}} \subset \mathbf{Z}_{\text{eff}}^O$.

The above indicates that any dominance structure that is stronger than the original dominance structure may not find some originally efficient points as efficient, thereby having a reduced efficient set. An example is when $\Omega_0 = \Omega_0^{\text{cone}}$. The cone-efficient set ($\mathbf{Z}_{\text{eff}}^{\text{cone}}$) is a subset of original efficient set ($\mathbf{Z}_{\text{eff}}^O$).

3.7 Theoretical and Practical Efficient Sets

Besides identifying the theoretical efficient set by the above procedure, there is a practical aspect which we discuss next. For practical reasons, one can define a generalized dominance structure that has an overlapping dominant and anti-dominant sets having $\Omega_0 \cap {\Omega'}_0 \neq \emptyset$. As discussed in Theorem 7, for such a structure no theoretical efficient point exists. However, an use of such a dominance structure can still produce artificial efficient points by an multi-objective optimization algorithm due to certain algorithmic inaccuracies and an often-used implementational adjustment. We illustrate these aspects with the epsilon-dominance structure [20].

The epsilon-dominance was proposed to obtain original efficient solutions with a certain pre-specified (ϵ_i) difference in the i-th objective values, even in continuous search space problems. The epsilon-dominance definition is as follows:

Definition 10 (Epsilon Dominance)**.** A feasible solution $\mathbf{x} \in \mathbf{X}$ epsilon-dominates[2] another feasible solution $\mathbf{y} \in \mathbf{X}$, if $f_i(\mathbf{x}) \leq f_i(\mathbf{y}) + \epsilon_i$ for all $i = 1, \ldots, M$.

The respective Ω_0 and ${\Omega'}_0$ are shown in Figure 10a in blue and golden shaded region, respectively. It is clear that $\Omega_0 \cap {\Omega'}_0 \neq \emptyset$, having a small overlapping rectangular region around the point O. Thus, there does not exist a theoretical efficient set for this dominance structure.

[2]The original study [20] defined with a product term.

However, the original domination structure may produce an efficient set, from which the user may be interested in detecting a few with certain regularity using the epsilon-dominance structure. For this purpose, we propose three different algorithmic and implementational approaches using a different generalized dominance structure (epsilon-dominance, for our illustration) which can lead to a set of efficient points from the original efficient set – ideas which can be used with other GD structures, as well.

3.7.1 Algorithmic Inefficiency

For a point (say, $\mathbf{x}$) close to the efficient front based on original domination structure, Ω'_0 dictates the objective space that GD-dominates the point $\mathbf{x}$. However, if this objective space is relatively small or difficult to discover by an algorithm's search operators, the point $\mathbf{x}$ may be declared as an GD-based efficient point. Consider the point 8 in Figure 9. It is not an efficient point based on Ω_0^{cone}, due to the existence of a tiny objective space (region 8-9-11). If a multi-objective optimization algorithm fails to find any point in this tiny region which would dominate Point 8, Point 8 will be wrongly declared as an efficient point.

3.7.2 Compatibility of Dominance Structure with Discreteness in Search Space

Artificial efficient points may also emerge if the search space is discrete, causing certain critical points to stay non-dominated due to unavailability of other points in the feasible search space to dominate them. Figure 10b shows two scenarios with epsilon-dominance structure with $\epsilon_i = 0.1$ for $i = 1, 2$ to find efficient points for a discrete search space problem having a linear efficient set in $f_i \in [0, 1]$ for $i = 1, 2$. In the first scenario (the main part of the figure), each objective value comes at an interval of $1/9$. The epsilon-dominance structure finds 10 efficient points, each of whose Ω'_0 is empty, as shown in the figure. Each discrete point clears the Ω'_0 region for other discrete points to constitute 10 final efficient points. The second scenario (shown in the inlet figure) has a finer discreteness (f_i values are available at an interval of $1/19$). The inlet figure shows that now no efficient point is discovered, as for every discrete point on the boundary of $\mathbf{Z}$, there are a few points in its Ω'_0 set, making it dominated. This exercise illustrates that the discreteness of the search space and the epsilon-dominance structure must have to be coordinated in a certain way to discover a non-empty efficient set.

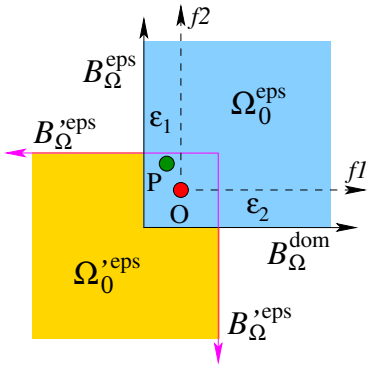
3.7.3 Implementation of Adjusted Domination Principle

The overlapping dominant structure can be modified so that a non-overlapping Ω_0 and Ω'_0 sets are achieved to produce a non-empty efficient set in a problem. For the epsilon-dominance structure, the overlapping region between Ω_0 and Ω'_0 is removed from both sets in the following adjustment of the sets, as shown in Figure 11a:

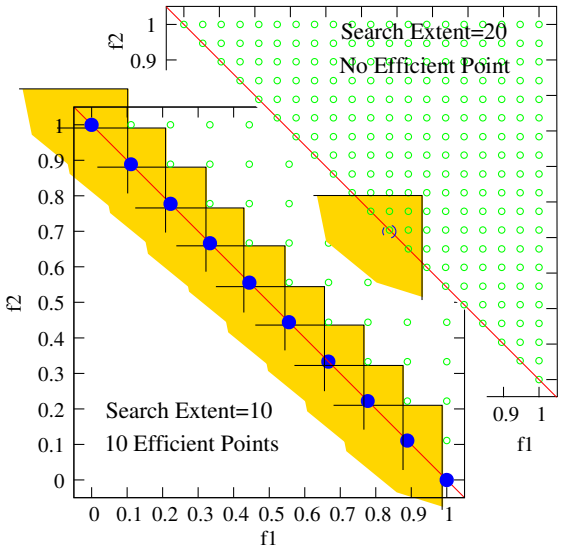
$$\Omega_0 \leftarrow \Omega_0 \backslash (\Omega_0 \cap \Omega'_0), \tag{5}$$

$$\Omega'_0 \leftarrow \Omega'_0 \backslash (\Omega_0 \cap \Omega'_0). \tag{6}$$

The above guarantees that adjusted sets are non-overlapping: $\Omega_0 \cap \Omega'_0 = \emptyset$. Figure 11b identifies the same 10 efficient solutions as in Figure 10b with the above adjusted dominance structure for the coarse search space (interval of $1/9$). However, when a finer search space (interval of $1/19$) is used, 20 efficient points are found with the same epsilon vector. No discrete boundary point lies on the Ω'_0 region of another. Note that algorithmic inefficiency is not a matter here, but change of dominance structure, choice of epsilon-vector, and location of boundary points make the efficient set non-empty.



(a) Epsilon domination. $\Omega_0'^{eps}$ also includes blue rectangle around O.



(b) Epsilon efficient set due to discreteness in search space.

Figure 10: Ω_0^{eps} does not produce any theoretical efficient point due to the overlap between Ω_0 and Ω'_0, while in practice it may produce an artificial efficient set.

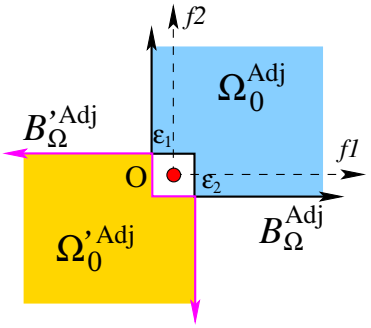
3.7.4 Implementation of Grid Domination Principle

Another approach adopted by EMO researchers is the use of a fixed grid structure in the objective space with size ϵ_i along i-th objective axis [21, 9, 15]. Every objective vector $\mathbf{f}$ is now replaced with it's grid vector (the lower left corner point of the grid in which $\mathbf{f}$ lies). In the grid domination structure, original domination is applied with grid vector of points, and not with the points themselves. For two points in the same grid results in the same grid vector. In this case, the one closer to the grid vector dominates the other, as shown in Figure 12a. Note that $\Omega_0^{grid} \cap \Omega_0'^{grid} = \emptyset$. This change in the domination structure produces a non-empty efficient set with well-distributed points. Figure 12b shows that irrespective the discreteness in the search space, the same number of efficient points are obtained for both levels of discreteness in the search space (interval of $1/9$ and $1/19$). Since the same epsilon-vector is used in the dominance structure, the final outcomes with the respective grid vectors are the same for both scenarios.

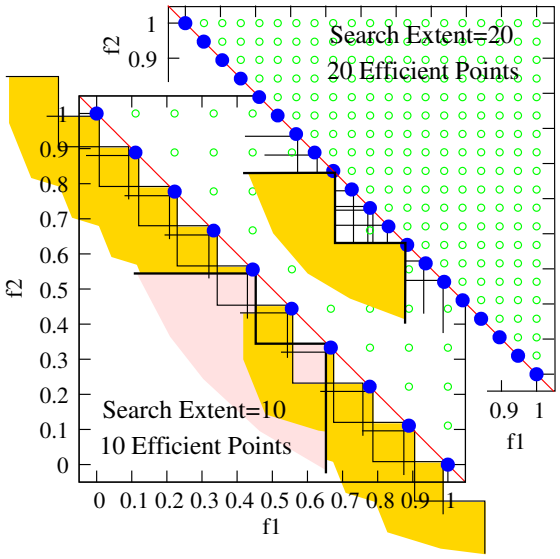
These algorithmic and implementational adjustments of a generalized dominance structure may produce a different outcome that their theoretical solutions. A proper analysis of the derived Ω'_0 for the chosen Ω_0 structure may reveal the expected outcome from an EMO run. Moreover, such a thought process may allow new and innovative adjustments in the dominance structure or algorithmic implementation to be discovered for different problems.

4 Other Dominance Structures

Next, we consider a few other existing dominance structures from the EMO literature and attempt to reveal their properties based on above fundamental principles of generalized dominance structures. Visual descriptions of the boundary for the dominant objective space (Ω_0) of some of these domination structures were presented in [32].



(a) Adjusted epsilon-domination.



(b) Adjusted epsilon-efficient set.

Figure 11: Adjusted epsilon-domination structure can produce an efficient set.

4.1 α-Domination

The α-domination was proposed in [17] in 2001 and is identical to the cone-domination structure, described in Figure 7c.

4.2 Cone-epsilon Domination

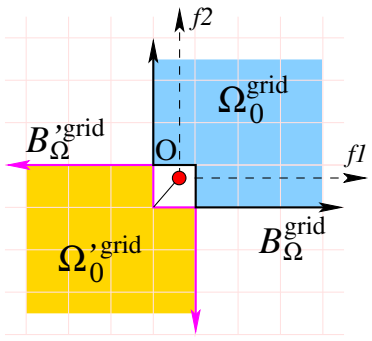
The cone-epsilon domination structure [1] uses the grid-based epsilon-domination concept, discussed in Section 3.7.4, but instead of distance-based domination for points within the occupying grid, it uses an acute cone domination principle, as shown in Figure 13(a). Clearly, $\Omega_0^{\text{cone-e}} \cap \Omega'^{\text{cone-e}}_0 \neq \emptyset$, thereby producing no theoretical efficient solution. However, adjusted domination principle (shown in Figure 13(b)), discussed in Section 3.7.3, can be implemented to find a set of efficient solutions.

4.3 CN and CN-alpha Domination

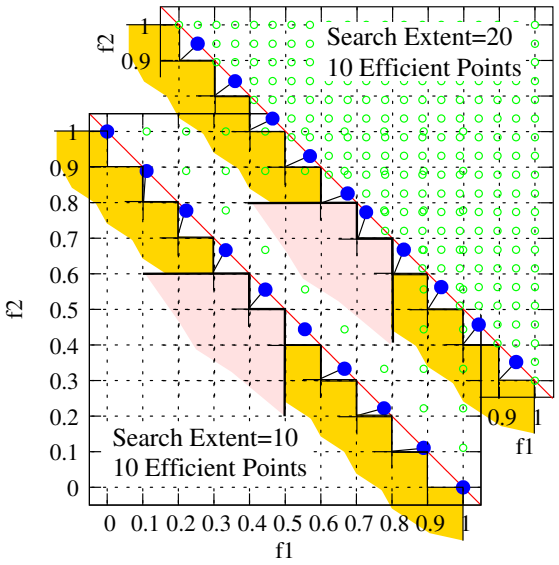
The CN-domination structure is shown in Figure 14(a). Clearly, $\Omega_0^{\text{CN}} \cap \Omega'^{\text{CN}}_0 \neq \emptyset$, making no theoretical efficient solution. The adjusted $\Omega'^{\text{CN}}_0 \leftarrow \Omega'^{\text{CN}}_0 \backslash (\Omega_0^{\text{CN}} \cap \Omega'^{\text{CN}}_0)$ structure will produce a set of CN-efficient solutions and is equivalent to adjusted epsilon-domination structure (shown in Figure 14(b)), discussed in Section 3.7.3. The grid-based CN-domination structure is also possible to construct.

4.4 CN-α Domination

The CN-α domination structure uses the cone domination with a predefined cone in the vicinity (within ϵ-vector), as shown in Figure 15(a). The overlap between $\Omega_0^{\text{CN-}\alpha}$ and $\Omega'^{\text{CN-}\alpha}_0$ exists. To find an efficient set of solutions, the adjusted anti-domination structure (Figure 15(b)) can be used.

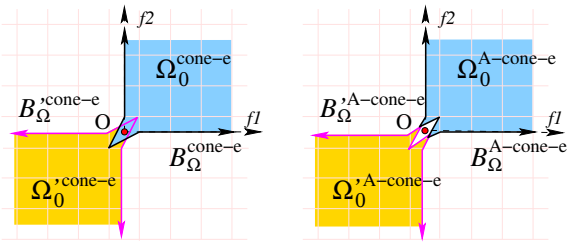


(a) Grid domination.



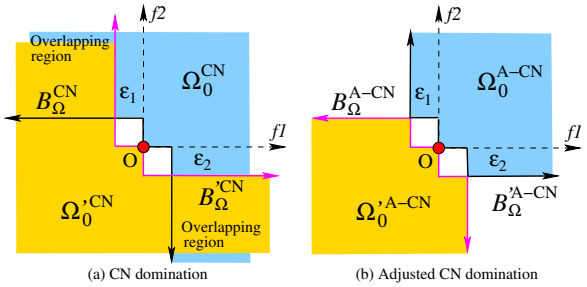
(b) Grid efficient set.

Figure 12: Grids are fixed in objective space. Ω_0^{grid} can produce a well-distributed set of efficient points.



(a) Cone–epsilon domination (b) Adj. cone–epsilon domination

Figure 13: Cone-epsilon domination.



(a) CN domination (b) Adjusted CN domination

Figure 14: CN domination.

4.5 Nonlinear Domination NLAD

A nonlinear domination structure was proposed in [23]. It uses a cubic domination boundary to define the dominated region. For certain cubic parameters, a non-overlapping $\Omega_0'^{NLAD}$ can be formulated and used to find the respective efficient set.

Other nonlinear domination definitions exist that combines objectives in a nonlinear manner $\phi(\mathbf{f})$ and impose dominance based on smaller value of ϕ [22, 5] and others that simply map every objective into a nonlinear function $\phi_i(f_i)$ and check original Pareto-dominance based on ϕ_i, such as CDAS-dominance structure [29].

4.6 D-Domination

The D-domination structure was defined in [3] and was based on the PBI metric, shown in Figure 17:

Definition 11 (D-domination)**.** A solution $\mathbf{x} \in \mathbf{X}$ D-dominates another solution $\mathbf{y} \in \mathbf{X}$, if OA $<$ OB, or $d_1(\mathbf{x}) + d_2(\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}))\cot\beta < d_1(\mathbf{y})$.

The distances d_1 and d_2 are along and orthogonal directions of a weight vector $\mathbf{w}$ from

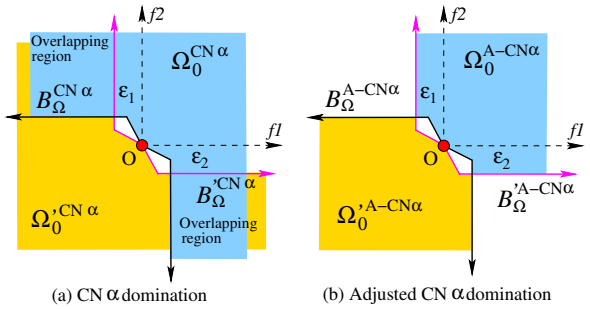


Figure 15: CN-alpha domination.

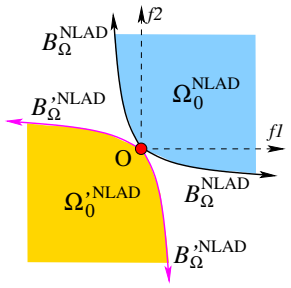


Figure 16: NLAD domination.

a reference point (usually, the ideal point), as shown in Figure 17. It is similar to the cone-domination structure, except in more than two-objective problems, the dominated region is conical, rather than prismatic. For a pair of solutions $\mathbf{x}$ and $\mathbf{y}$, authors defined a condition for which $\mathbf{x}$ is better and also a condition for which $\mathbf{y}$ is better and if both conditions are not satisfied, then both solutions are non-dominated according to D-dominance structure. In some sense, the dominant and anti-dominant sets can be defined from the definition, making this study as one of the precursors of the anti-dominance concept.

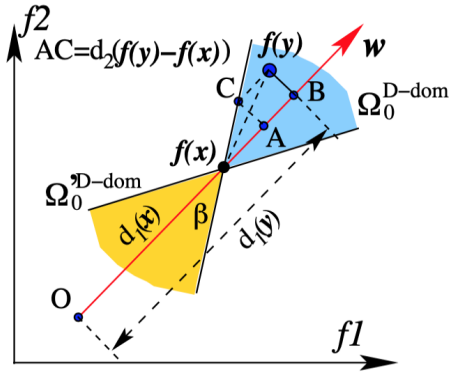


Figure 17: D-domination.

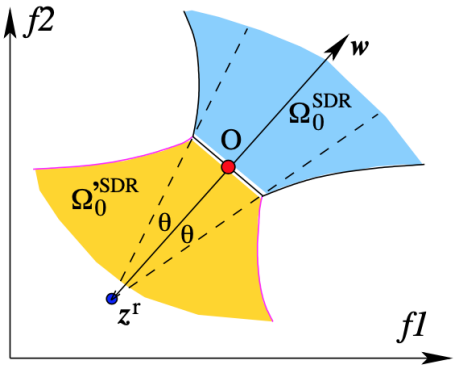


Figure 18: SDR domination.

4.7 Strength Dominance Relationship (SDR)

For the reference vector based optimization, [32] proposed a new dominance structure in which points associated with the reference vector are compared with the sum of objective functions:

Definition 12 (SDR-domination)**.** A solution $\mathbf{x} \in \mathbf{X}$ SDR-dominates another solution $\mathbf{y} \in \mathbf{X}$, if $\sum_{i=1}^{M} f_i(\mathbf{x}) < \sum_{i=1}^{M} f_i(\mathbf{y})$ for $\angle(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \theta$; else if $\sum_{i=1}^{M} f_i(\mathbf{x}) \frac{\angle(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y}))}{\theta} < \sum_{i=1}^{M} f_i(\mathbf{y})$, otherwise.

However, when an associated point is compared with an un-associated point, angles between the points are considered, resulting in a dominance structure Ω_0^{SDR} shown in Figure 18. If the entire objective space is checked for SDR-domination, the anti-dominant objective set Ω'_0 can be constructed from Ω_0, as shown in the figure. Clearly, they are non-overlapping sets with no common points on boundaries of these sets and are expected to produce a set of efficient solutions.

4.8 $(1-k)$-Domination

For two solutions $\mathbf{x}$ and $\mathbf{y}$, three quantities are first computed [13]: (i) $n_b(\mathbf{x},\mathbf{y})$, count of number of objectives in which $\mathbf{x}$ is better than $\mathbf{y}$, (ii) $n_e(\mathbf{x},\mathbf{y})$, count of number of objectives in which $\mathbf{x}$ is identical to $\mathbf{y}$, and (iii) $n_w(\mathbf{x},\mathbf{y})$, count of number of objectives in which $\mathbf{x}$ is worse than $\mathbf{y}$.

Definition 13. A solution $\mathbf{x} \in \mathbf{X}$ $(1-k)$-dominates another solution $\mathbf{y} \in \mathbf{X}$, if (i) $n_e(\mathbf{x},\mathbf{y}) < M$, and (ii) $n_b(\mathbf{x},\mathbf{y}) \geq \frac{M-n_e(\mathbf{x},\mathbf{y})}{k+1}$.

For $k = 0$, $(1-k)$-domination becomes the original Pareto-dominance, but for $k > 1$, it results in a reduced efficient set. Figures 19 and 20 show Ω_0 and Ω'_0 for three-objective case having $k = 1$ with $n_e = 1$ and $n_1 = 0$, respectively. Both sets are non-overlapping in both cases.

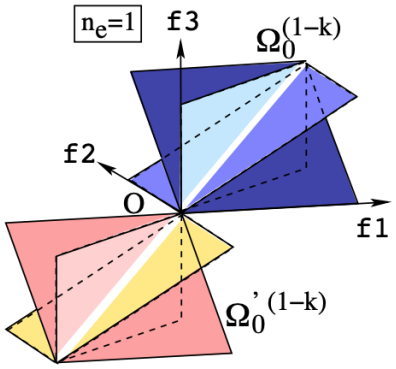


Figure 19: $(1-k)$-domination with $M = 3$, $k = 1$, and $n_e = 1$.

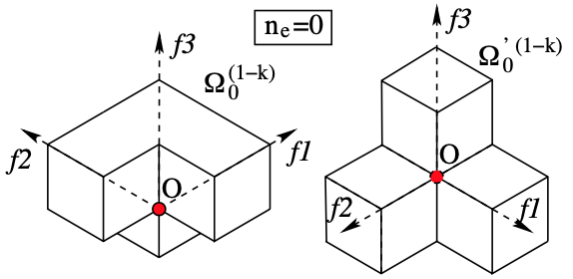


Figure 20: $(1-k)$-domination with $M = 1$, $k = 1$, and $n_e = 0$.

This dominance is also known as the fuzzy-dominance structure. Its effect becomes prominent for more than two objectives.

4.9 L-Dominance

Borrowing the idea from the above $(1-k)$-domination structure, [39] proposed the L-domination:

Definition 14 (L-Dominance)**.** A solution $\mathbf{x}$ L-dominates another solution $\mathbf{y}$, if (i) $n_b(\mathbf{x},\mathbf{y}) > n_w(\mathbf{x},\mathbf{y})$ and (ii) $\|\mathbf{f}(\mathbf{x})\| < \|\mathbf{f}(\mathbf{y})\|$.

Normalization of objectives is performed before computing the above checks. L-dominance causes more solutions to be dominated than the original Pareto-dominance mainly due to the first condition, hence L-dominance is likely to cause a reduced L-efficient set compared to the original efficient set. The contribution of the second condition in defining the dominance structure is not obvious.

4.10 $(M-1)$-Generalized Pareto Dominance

Extending the concept of cone-domination, [38] proposed a cone which does not extend the cone along one of the objective axis. Thereafter, they suggested to use M GPD cones to determine the overall non-dominated set. The concept of $(M-1)$-GPD is similar to the cone-domination described before.

5 Spatially-dependent Ω_0 Structure

All the above discussions are applicable to static dominance structures in which the same dominance structure is applicable at every point in the objective space. However, multi-objective optimization researchers have developed other dominance structures, which change with the location of **x** in the objective space. A few such dominance structures are reference vector (RV) based domination, such as achievement scalarization function (ASF) based domination [34], PBI metric based domination [37], LHiFD [30], w-domination [31], and location based domination, such as angle-domination [24]. Some of these structures are applied paralelly to a number of pre-defined RVs in the objective space and are expected to produce a single efficient solution for every RV. We attempt to explain here how our proposed anti-dominant structure concept can be used to identify respective efficient solution(s).

5.1 ASF-based Domination

Consider the ASF approach [34] first, in which the minimization of the following scalarizing function produces a weakly efficient point:

$$\begin{array}{ll} \text{Minimize} & \text{ASF}(\mathbf{x}) = \max_{i=1}^{M} \frac{f_i(\mathbf{x}) - z_i^r}{w_i}, \\ \text{subject to} & \mathbf{x} \in \mathbf{X}. \end{array} \tag{7}$$

The reference vector **w** and reference point $\mathbf{z}^r$ are fixed in the objective space, as shown in Figure 21. Since points above or below the reference vector **w** has a different domination principles, theoretically Ω_0 and Ω'_0 sets vary from point to point and also depend on the chosen RV. However, considering the contour of ASF function, we perform a mapping of **f**-vector on to the RV **w**, as represented in the RV line in the figure. Thereafter, we can define fixed Ω_0 and Ω'_0 on **w**, as follows. For a point A, its mapped point *a* on **w**-line is first found. All objective vectors that are mapped above *a* (away from $\mathbf{z}^r$) on **w**-line belong to point A's Ω_0 set. Similarly, all points that are mapped below (near $\mathbf{z}^r$) belong to Ω'_0 at A, thereby applying the proposed dominant and anti-dominant objective set definitions along the RV line. A little thought will reveal that the respective objective vectors of Ω'_0 set will come from the region marked in golden color in the objective space, meaning the any point in the golden region ASF-dominates point A. These adjusted and mapped dominant and anti-dominant sets are non-overlapping to each other and when applied to the entire feasible **Z**, will result in a single efficient point, marked with a red circle. Other reference-point based domination structures, such as g-dominance [26], r-dominance

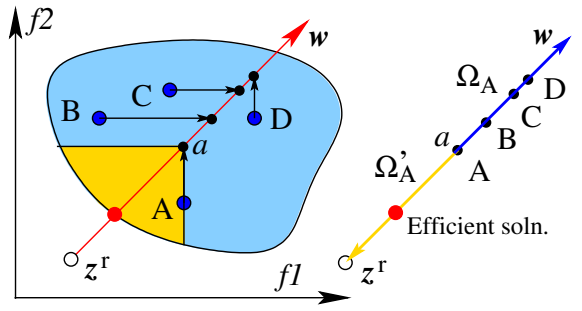


Figure 21: ASF domination.

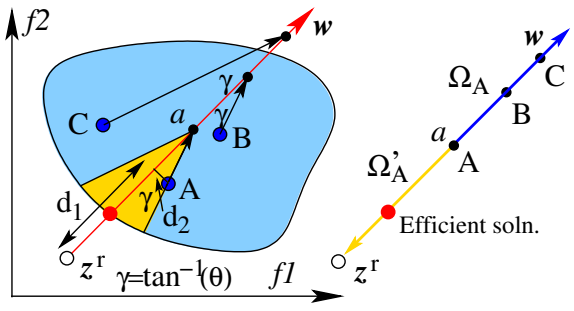


Figure 22: PBI domination.

[28], p-dominance [16], and ar-dominance [35] can be also be analyzed using above concept.

5.2 θ-Domination or PBI-Domination

A similar adjusted and mapped dominance structure can also be applied to the PBI metric based domination structure [36], as shown in Figure 22. The PBI metric uses a user-specified parameter θ, which is equal to $\tan(\gamma)$ (γ is half of the cone angle) to create a penalty function to combine perpendicular and parallel distances along **w**-line. It is clear that point A PBI-dominates B, which again PBI-dominates C. The region that dominates point A is marked in golden color. The Ω_A created at mapped Point A contains mapped Points B and C and the respective Ω'_A represents all points from the golden region, thereby following the principle of PBI-based dominance structure.

5.3 Angle-domination

Instead of the objective values, angles ($\alpha_i, i = 1, \ldots, M$) from M anchor points on objective axes can be computed for any **f**-vector by extending the nadir point by k times (> 1) and locating its coordinates on the objective axes. The angle-dominance [24] is then defined as follows:

Definition 15 (Angle-domination)**.** A solution $\mathbf{x} \in \mathbf{X}$ angle-dominates another solution $\mathbf{y} \in \mathbf{X}$, if $\alpha_i(\mathbf{x}) \leq \alpha_i(\mathbf{y})$ for all $i = 1, 2, \ldots, M$ and $\alpha_i(\mathbf{x}) < \alpha_i(\mathbf{y})$ for at least one $i = 1, 2, \ldots, M$.

As can be seen from Figure 23, it is similar to the cone-domination principle, except that the cone depends on the point **x**. To determine the efficient set, **f**-vectors can be converted to $\boldsymbol{\alpha}$-vectors, as shown in Figure 23 and original Pareto-dominance based check can be performed on the angle-space.

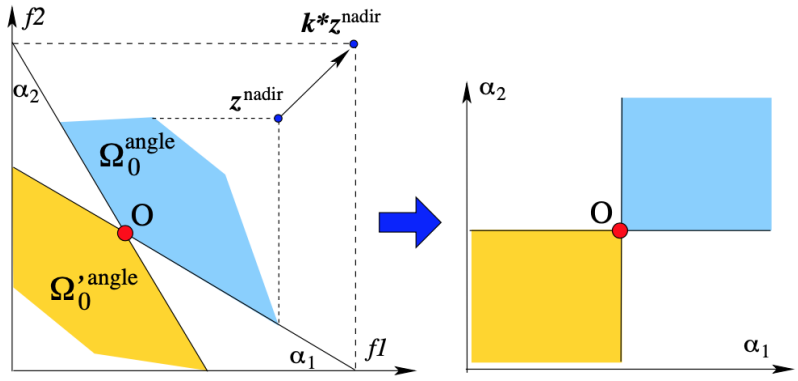


Figure 23: Angle-domination structure can be converted to angle-space and standard Pareto-dominance can be applied.

Besides the above positionally-defined dominance structures in the objective space, domination structures that depend on a given sample of points, such as, ranking domination [19, 27], and dominance structures that do not use objective vector and are restricted to variable space relationships, are difficult to analyze to have a comprehensive insight on resulting efficient sets, however, certain innovative mapping techniques, as proposed above, may be necessary in such cases.

6 Conclusions

This study has introduced a concept of anti-dominant structure for a chosen static dominant structure, by extending the optimality conditions in a single-objective problem to multi-modal problems and then to multi-objective problems. It has been shown that the anti-dominant structure can be useful for identifying the respective non-dominated set in a finite population or perceiving

the true efficient set in two-objective problems. Importantly, the study has brought out certain fundamental principles which every dominant structure should have for it to generate a non-empty efficient set. However, adjustments to chosen dominance structures have been proposed to make them suitable and more practicable. Also, sources for algorithmic inaccuracies in finding non-efficient solutions have been explained through the concept of anti-dominant structures.

To demonstrate their use, the dominant and anti-dominant objective sets have been identified for most popularly-used dominance structures in the EMO and classical multi-objective optimization literature. In some occasions, an adjustment with mapped objective vectors has been proposed to analyze the resulting efficient solutions.

It has been shown that with a population-based multi-objective optimization algorithm, transitivity of the dominance structure need not be a strict requirement for EMO studies. Semi-transitivity property of dominance structures allows a transitive relationship to be implicitly formed from the existence of intermediate population members. In this sense, population-based evolutionary multi-objective optimization algorithms have an edge to using more flexible dominant structures with semi-transitive properties.

Although not used in this paper, the anti-superiority concept can be extended to find special and practically relevant (non-optimal) solutions by developing generalized superiority conditions for single-objective optimization. An extension of anti-dominance structure for constrained dominance principles [12, 10] will motivate further development of efficient constrained multi-objective optimization algorithms. Developing a GUI-based system for providing a dominance structure, an automated procedure for developing the respective anti-dominance structure, and linking with an EMO algorithm should a flexible and practically useful multi-objective optimization software. Finally, providing preference elicitation through a number of pair-wise comparison of objective vectors and creating a meaningful generalized dominance structure from them, following the principles observed in this paper, would be another interesting practically significant study.

References

[1] Lucas S Batista, Felipe Campelo, Frederico G Guimaraes, and Jaime A Ramírez. Pareto cone ε-dominance: improving convergence and diversity in multiobjective evolutionary algorithms. In *International conference on evolutionary multi-criterion optimization*, pages 76–90. Springer, 2011.

[2] J. Branke, S. Greco, R. Slowinski, and P. Zielniewicz. Interactive evolutionary multiobjective optimization using robust ordinal regression. In *Proceedings of the Fifth International Conference on Evolutionary Multi-Criterion Optimization (EMO-09)*, pages 554–568. Berlin: Springer-Verlag, 2009.

[3] Lei Chen, Hai-Lin Liu, and Kay Chen Tan. Decomposition based dominance relationship for evolutionary many-objective algorithm. In *2017 IEEE Symposium Series on Computational Intelligence (SSCI)*, pages 1–6. IEEE, 2017.

[4] C. A. C. Coello, D. A. VanVeldhuizen, and G. Lamont. *Evolutionary Algorithms for Solving Multi-Objective Problems*. Boston, MA: Kluwer, 2002.

[5] Cai Dai, Yuping Wang, and Miao Ye. A new evolutionary algorithm based on contraction method for many-objective optimization problems. *Applied Mathematics and Computation*, 245:191–205, 2014.

[6] K. Deb. *Multi-objective optimization using evolutionary algorithms*. Wiley, Chichester, UK, 2001.

[7] K. Deb, S. Agrawal, A. Pratap, and T. Meyarivan. A fast and elitist multi-objective genetic algorithm: NSGA-II. *IEEE Transactions on Evolutionary Computation*, 6(2):182–197, 2002.

[8] K. Deb, A. Sinha, P. Korhonen, and J. Wallenius. An interactive evolutionary multi-objective optimization method based on progressively approximated value functions. *IEEE Transactions on Evolutionary Computation*, 14(5):723–739, 2010.

[9] Kalyanmoy Deb, Manikanth Mohan, and Shikhar Mishra. Evaluating the -domination based multi-objective evolutionary algorithm for a quick computation of pareto-optimal solutions. *Evolutionary computation*, 13(4):501–525, 2005.

[10] Kalyanmoy Deb, Amrit Pratap, and T Meyarivan. Constrained test problems for multi-objective evolutionary optimization. In *International conference on evolutionary multi-criterion optimization*, pages 284–298. Springer, 2001.

[11] M. Ehrgott. *Multicriteria Optimization*. Berlin: Springer, 2005.

[12] Z. Fan, Y. Fang, W. Li, X. Cai, C. Wei, and E. D. Goodman. MOEA/D with angle-based constrained dominance principle for constrained multi-objective optimization problems. *Appl. Soft Comput.*, 74:621–633, 2019.

[13] Marco Farina and Paolo Amato. A fuzzy definition of "optimality" for many-criteria optimization problems. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 34(3):315–326, 2004.

[14] J. W. Fowler, E. S. Gel, M. Koksalan, P. Korhonen, J. L. Marquis, and J. Wallenius. Interactive evolutionary multi-objective optimization for quasi-concave preference functions. Submitted to European Journal of Operational Research, 2009.

[15] A. G. Hernández-Díaz, L. V. Santana-Quintero, C. A. C. Coello, and J. Molina. Pareto adaptive ϵ-dominance. *Evolutionary Computational Journal*, 15(4):493–517, 2007.

[16] Jianjie Hu, Guo Yu, Jinhua Zheng, and Juan Zou. A preference-based multi-objective evolutionary algorithm using preference selection radius. *Soft Computing*, 21(17):5025–5051, 2017.

[17] K. Ikeda, H. Kita, and S. Kobayashi. Failure of Pareto-based MOEAs: Does non-dominated really mean near to optimal? In *Proceedings of IEEE Congress on Evolutionary Computation*, pages 957–962, 2001.

[18] Miłosz Kadziński, Michał K Tomczyk, and Roman Słowiński. Preference-based cone contraction algorithms for interactive evolutionary multiple objective optimization. *Swarm and Evolutionary Computation*, 52:100602, 2020.

[19] Saku Kukkonen and Jouni Lampinen. Ranking-dominance and many-objective optimization. In *2007 IEEE Congress on Evolutionary Computation*, pages 3983–3990. IEEE, 2007.

[20] M. Laumanns, G. Rudolph, and H. P. Schwefel. A spatial predator-prey approach to multi-objective optimization: A preliminary study. In *Proceedings of the Parallel Problem Solving from Nature, V*, pages 241–249, 1998.

[21] M. Laumanns, L. Thiele, K. Deb, and E. Zitzler. Combining convergence and diversity in evolutionary multi-objective optimization. *Evolutionary Computation*, 10(3):263–282, 2002.

[22] Khoi Le and Dario Landa-Silva. Obtaining better non-dominated sets using volume dominance. In *2007 IEEE congress on evolutionary computation*, pages 3119–3126. IEEE, 2007.

[23] Junhua Liu, Yuping Wang, Shiwei Wei, Xiangjuan Wu, and Wuning Tong. A parameterless penalty rule-based fitness estimation for decomposition-based many-objective optimization evolutionary algorithm. *IEEE Access*, 7:81701–81716, 2019.

[24] Yuan Liu, Ningbo Zhu, Kenli Li, Miqing Li, Jinhua Zheng, and Keqin Li. An angle dominance criterion for evolutionary many-objective optimization. *Information Sciences*, 509:376–399, 2020.

[25] K. Miettinen. *Nonlinear Multiobjective Optimization*. Kluwer, Boston, 1999.

[26] Julián Molina, Luis V Santana, Alfredo G Hernández-Díaz, Carlos A Coello Coello, and Rafael Caballero. g-dominance: Reference point based dominance for multiobjective metaheuristics. *European Journal of Operational Research*, 197(2):685–692, 2009.

[27] Syed Zaffar Qasim and Muhammad Ali Ismail. Rode: Ranking-dominance-based algorithm for many-objective optimization with opposition-based differential evolution. *Arabian Journal for Science and Engineering*, 45(12):10079–10096, 2020.

[28] Lamjed Ben Said, Slim Bechikh, and Khaled Ghédira. The r-dominance: A new dominance relation for interactive evolutionary multicriteria decision making. *IEEE transactions on Evolutionary Computation*, 14(5):801–818, 2010.

[29] Hiroyuki Sato, Hernán E Aguirre, and Kiyoshi Tanaka. Controlling dominance area of solutions and its impact on the performance of moeas. In *International conference on evolutionary multi-criterion optimization*, pages 5–20. Springer, 2007.

[30] D. K. Saxena, S. Mittal, S. Kapoor, and K. Deb. A localized high-fidelity-dominance based many-objective evolutionary algorithm. *IEEE Transactions on Evolutionary Computation*, 2022.

[31] Rafal Szlapczynski and Joanna Szlapczynska. W-dominance: Tradeoff-inspired dominance relation for preference-based evolutionary multi-objective optimization. *Swarm and Evolutionary Computation*, 63:100866, 2021.

[32] Ye Tian, Ran Cheng, Xingyi Zhang, Yansen Su, and Yaochu Jin. A strengthened dominance relation considering convergence and diversity for evolutionary many-objective optimization. *IEEE Transactions on Evolutionary Computation*, 23(2):331–345, 2018.

[33] Michał K Tomczyk and Miłosz Kadziński. Robust indicator-based algorithm for interactive evolutionary multiple objective optimization. In *Proceedings of the Genetic and Evolutionary Computation Conference*, pages 629–637, 2019.

[34] A. P. Wierzbicki. The use of reference objectives in multiobjective optimization. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Applications*, pages 468–486. Berlin: Springer-Verlag, 1980.

[35] Jun Yi, Junren Bai, Haibo He, Jun Peng, and Dedong Tang. ar-MOEA: A novel preference-based dominance relation for evolutionary multiobjective optimization. *IEEE Transactions on Evolutionary Computation*, 23(5):788–802, 2018.

[36] Yuan Yuan, Hua Xu, Bo Wang, and Xin Yao. A new dominance relation-based evolutionary algorithm for many-objective optimization. *IEEE Transactions on Evolutionary Computation*, 20(1):16–37, 2015.

[37] Q. Zhang and H. Li. MOEA/D: A multiobjective evolutionary algorithm based on decomposition. *Evolutionary Computation, IEEE Transactions on*, 11(6):712–731, 2007.

[38] Shuwei Zhu, Lihong Xu, Erik D Goodman, and Zhichao Lu. A new many-objective evolutionary algorithm based on generalized pareto dominance. *IEEE Transactions on Cybernetics*, 2021.

[39] Xiufen Zou, Yu Chen, Minzhong Liu, and Lishan Kang. A new evolutionary algorithm for solving many-objective optimization problems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 38(5):1402–1412, 2008.