A Localized High-Fidelity-Dominance based Many-Objective Evolutionary Algorithm

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COIN Report Number 2021020

Abstract—**The inefficacy of Pareto-dominance in many-objective problems can be attributed to its inability to factor the critical Human Decision-making (HDM) elements, including, the number of better objectives; the degree of betterment in objectives; and objectives' relative preference. Other dominance principles have attempted to incorporate *only* the first two HDM elements (one or two at a time), and their utility is marred by the need for new parameters (tuning which is a non-trivial task), and often, disruption in the delicate convergence-diversity balance. This paper proposes *high-fidelity*-dominance principle, that for the *first time* factors all the HDM elements, explicitly and simultaneously, and without requiring tuning of any parameter. This principle has been implemented in a reference vector based framework, leading to a computationally efficient many-objective Evolutionary Algorithm (MaOEA), namely *localized high-fidelity*-dominance based EA (LHFiD). Critically, LHFiD also marks the first MaOEA with an inbuilt mechanism for *intermittent-timing* of nadir point estimation, enabling faster convergence; and *self-termination*, bearing practically utility. This paper is based on 41,912 experiments, and LHFiD compared to other competitive MaOEAs has reported statistically better performance in about 60% instances, making it worthy of further investigation and application.**

Index Terms—**Many-objective, Localized High Fidelity Dominance, Nadir Point Update, Self-termination.**

I. Introduction

Multi-objective optimization problems (MOPs) are characterized by two or more conflicting objectives ($M \geq 2$). The goal of Multi-objective Evolutionary Algorithms (MOEAs) is to iteratively evolve a finite set of solutions, leading to a good Pareto Front (PF) approximation in terms of *convergence* and *diversity* [1]. Critically, research over the past two decades has established that most of the well-known MOEAs face multiple challenges in tackling many-objective problems (MaOPs), which involve more than three objectives. These challenges relate to poor search efficiency; need for an exponentially increasing (with M) population size; and difficulty in visualization and decision-making [2]–[5].

Several Many-objective Evolutionary Algorithms (MaOEAs) based on different principles have been proposed to tackle the above challenges. This paper focuses on the widely used dominance principle that aims to induce convergence to the PF by differentiating the solutions and preferring the better ones. Among the several variants of this principle, *Pareto-dominance* (PD) [1] is one of the most commonly used. In that, a solution X is said to Pareto-dominate another solution Y, if X is not worse than Y in any objective, and is strictly better than Y in at least one objective. Given two solutions, if neither Pareto dominates the other, they are said to be non-dominated. While PD is found to be very effective on two- and three-objective problems, it alarmingly ceases to be effective in tackling MaOPs [2], as highlighted below.

With an increase in M, a higher proportion of solutions in the population becomes non-dominated [3], and PD fails to discriminate the solutions as better or worse. While this leads to diminished selection pressure for convergence (*dominance resistance* [4]), the density-based selection worsens the solution distribution by favoring the remote and boundary solutions (*active diversity promotion* [4]). In a significant contribution, the authors in [6] have argued that - *when dealing with MaOPs, PD is often inefficient in modeling and simulating human decision making (HDM)*. In that, while comparing two solutions, PD fails to account for three critical HDM elements, including:

1) the *number* of improved objectives (n_b): the number of objectives in which a solution is better.
2) the *size* of improvements ($\Delta f_m, \forall m = 1 \ldots M$): the degree by which a solution is better, in the objectives.
3) *objectives' relative preferences* ($p_m, \forall m = 1 \ldots M$).

Plausibly, the recognition of PD's limitations has prompted several variants of the dominance principle. While the definitions of the key variants are presented in Section S2 of the Supplementary Document (S.D.), they have been categorized in Table I based on how they utilize/process the original objective (F) values. These categories relate to the use of (i) Original F values as in PD, (ii) singularized F values leading to a quality-indicator that helps distinguish the solutions,

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(iii) modified F values to enlarge the area dominated by a solution so that the number of non-dominated solutions is reduced, and (iv) HDM elements guided by original F values. Notably, the variants with modified and singularized F values have their strengths. Still, their generality/utility may be marred by: (a) requirement of a new parameter, tuning which is a non-trivial task, and (b) the fact that their inferences on why a particular solution dominates another, may not be intuitive to a practitioner. Interestingly, there do exist dominance principles that utilize the HDM elements. However, (i) those proposed in [6] use either n_b or Δf_m *one-at-a-time*, and (ii) the L-optimality [7] which simultaneously accounts for both n_b and Δf_m, leads to only a subset of the true PF. Notably, both [6], [7] completely ignore incorporation of p_m, and involve multiple tunable parameters (i) settings, each of which is a non-trivial task, and (ii) the impact of whose interplay on the MaOEA performance is not intuitive.

To summarize, no dominance principle (in the context of MaOPs) *till-date* has been able to factor all the HDM elements, explicitly and simultaneously, and without requiring tuning of any parameter. To realize the associated pitfalls, consider a scenario, where a teacher (DM) may need to recommend the students (solutions), for further studies in different academic disciplines, by ranking them based on their performance (F) in six courses ($M = 6$). Clearly, assessing two students as equally good (non-dominated), when one student scores better in one course, while the other in five courses (as PD suggests), may not be fair. Similarly, singularizing or modifying a student's score through a new parameter, may be too subjective. Alternatively, factoring the number of courses in which a student performs better (n_b), and the degree by which a student scores better in each subject (Δf_m) seems a robust and easily interpretable prospect. Furthermore, if the teacher who serves as the decision maker, has well-defined relative preferences (p_ms) for different courses vis-*à*-vis the academic disciplines, then such p_ms can also be factored towards the final recommendation. Else, each subject can be assigned equal preference ($p_m = 1/M$).

In this background, the main contributions of this paper are summarized, below.

- A *high-fidelity*-dominance (*hf*-dominance) principle has ben proposed, which for the first time factors all the HDM elements, explicitly and simultaneously, and without requiring tuning of any parameter.
- A novel MaOEA, namely, *localized high-fidelity*-dominance based EA (LHFiD), has been proposed. It employs a decomposition-based framework, where the F-space is discretized through generation of reference vectors (RVs); clusters of solutions are formed around these RVs; and *hf*-dominance is used in a *localized* manner (cluster-wise).
- To the best of authors' knowledge, LHFiD marks the first MaOEA capable of on-the-fly: *intermittent timing* for nadir point (Z^N) estimation, which promises faster PF approximation; and *self-termination*, avoiding experimental runs with arbitrarily fixed generations. Towards it, the authors' have hired from their earlier established proof-of-concept on stabilization tracking [30].

Table I: Classification of the existing dominance (dm.) principles, factoring their treatment of F values; Human Decision-making (HDM) elements; and need for user-defined parameters (UDP).

Category	Principle	$\lvert n_b \rvert$	Δf_m	UDP	MaOEA
Original F values	Pareto-dm.	–	–	–	NSGA-III [8] NSGA-II/DM1 [9] SPEA2+SDE [10]
Singularized F values	SDR [11]	–	Yes	$\tilde{\theta}$	NSGA-II/SDR [11]
	θ-dm. [12]	–	Yes	θ	θ-DEA [12]
	D-dm. [13]	–	Yes	β	DrEA [13]
	RP-dm. [14]	–	Yes	–	RPD-NSGA-II [14]
	Volume-dm. [15]	–	–	rSV	–
	Ranking-dm. [16]	–	–	–	–
Modified F values	ϵ-dm. [17]	–	–	ϵ	ϵ-MOEA [17] ϵ-NSGA-II [18]
	α-dm. [19]	–	–	α	NNIA-AD [20]
	Cone-dm. [21]	–	–	a	KPNSGA-II [21]
	Cone ϵ-dm. [22]	–	–	ϵ, κ	coneϵ-MOEA [22]
	CDAS [23]	–	–	S	PPD-MOEA [24] NNIA-CE [20]
	CN-dm. [25]	–	–	H	NNIA-CN [20]
	CNα-dm. [20]	–	–	α, H	NNIA-CNα [20]
	Angle-dm. [26]	–	–	k	NSGA-II+AD [26]
	Grid-dm. [27]	–	–	div	GrEA [27]
	NLAD-dm. [28]	–	–	k, T	BFEA [28]
	GP-dm. [29]	–	–	ϕ	multiGPO [29]
Original F values: HDM elements	(1-k)-dm. [6]	Yes	–	k	–
	(1-k_F)-dm. [6]	–	Yes	ϵ, γ, k_F	–
	Fuzzy-dm. [6]	–	Yes	ϵ, γ, α	–
	L-dm. [7]	Yes	Yes	L, p	MDMOEA [7]

To demonstrate the efficacy of the above propositions:

- The performance of LHFiD has been investigated against other competitive MaOEAs, on commonly used DTLZ [31], WFG [32] and MaF [33] test suites.
- Problems with inverted PFs, such as minus-DTLZ, minus-WFG [34], and pertinent MaF versions have also been considered. Since, the conventional practice of generating the RVs from the ideal point (Z^I) is known to be ineffective in such problems, the RVs for LHFiD and other MaOEAs are generated from Z^N, and the respective variants are denoted through the suffix -i (e.g., LHFiD-i denotes the variant of LHFiD). Notably, the development of the unified versions (generating RVs from both Z^I and Z^N), or adaptive RV generation, capable of tackling both regular and inverted PFs, has been excluded, considering the current scope of the paper.
- 41,912 experimental runs have been conducted, where LHFiD and LHFiD-i are found to outperform their counterparts (with statistical significance) in about 60% instances.

The remaining paper comprises of: a summary of the existing MaOEAs in Section II; the proposition of *hf*-dominance principle and LHFiD (MaOEA) in Section III;

discussion on the use of objectives' relative preferences in Section IV; the experimental settings in Section V; and the experimental results in Sections VI and VII. The paper concludes with Section VIII.

II. Existing Many-objective Evolutionary Algorithms

The existing MaOEAs could broadly be classified as - dominance-based; indicator-based; decomposition-based, and hybrids, as detailed below.

1) Dominance-based MaOEAs: while Table I presents a qualitative characterization of such MaOEAs, they are discussed here with respect to their focus on addressing the challenges of *dominance resistance* and/or *active diversity promotion*. Some MaOEAs have sought to address the former by replacing PD with alternative dominance principles. These include KPNSGA-II [21], NNIA-CNα [20], and NSGA-II+AD [26]. Other MaOEAs (while retaining PD) have focused on addressing the latter (consequently addressing the former indirectly), including, NSGA-II/DM1 [9] by controlled activation of crowding distance, and SPEA2+SDE [10] by assigning high density to poorly converged solutions. Notably, some MaOEAs, such as, GrEA [27], multiGPO [29], and NSGA-II/SDR [11] have directly attempted to address both the challenges simultaneously. In that, GrEA relies on grid-dominance for convergence and an adaptive grid-based density estimator for diversity; multiGPO relies on Generalized Pareto (GP)-dominance for convergence, and a max-min distance mechanism for diversity; and NSGA-II/SDR relies on strengthened dominance relation (SDR) for convergence, and a niching technique for diversity.

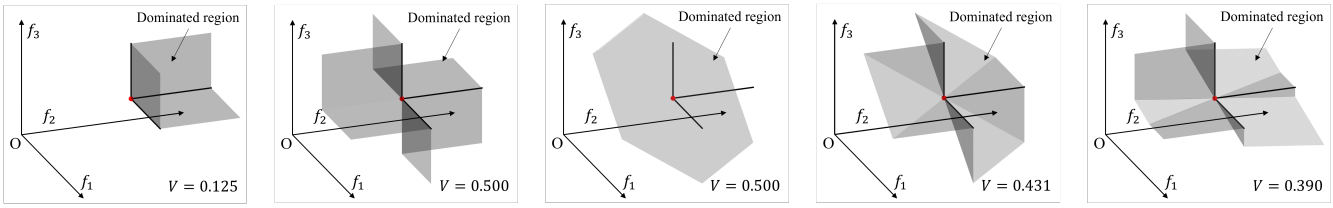
In alignment with Table I, it may be noted that the dominance principles utilizing one or more HDM elements explicitly, are largely restricted. In that, $(1\text{-}k)$-dominance [6] accounts for the number of objectives in which a solution is *better than* (n_b) and *equal to* (n_e) the other, without considering the degree of improvement in any objective (Δf_m). In contrast, two of its extensions based on fuzzy arithmetic, namely, $(1\text{-}k_F)$-dominance and fuzzy-dominance [6] factor only Δf_m in each objective (assuming equal preference to each objective), without considering n_b or n_e explicitly. Despite being the first to recognize the need to account for HDM elements towards countering dominance resistance, the applications of these principles in literature is ironically low. This may be attributed to the fact that these principles have not been included into any MaOEAs. There exists another principle, namely, L-dominance [7] that relies on n_b as well as Δf_m (assuming equal preference to each objective). MDMOEA [7] employs L-dominance to address dominance resistance, and entropy (along with crowding distance) for diversity promotion. However, MDMOEA is only capable of approximating the L-optimal front that constitutes a subset of the true PF [7].

2) Indicator-based MaOEAs: such MaOEAs rely on an *indicator* to guide the search process, instead of a dominance principle. Use of different indicators have led to MaOEAs, such as: SMS-EMOA [35] and HypE [36] using hypervolume (HV); MOMBI [37] using R2-indicator; and MyO-DEMR [38] using inverted generational distance (IGD). These MaOEAs pose a promising alternative to dominance-based MaOEAs since a single indicator can account for both convergence and diversity, however they do have their own limitations. HV is the only known indicator that is monotonic with PD, but imposes a high computational load on the MaOEA [39]. Even if this computational load is neglected, an MaOEA based on HV would return a uniformly distributed population only when the PF is linear [40]. On the other hand, using a computationally efficient but non-monotonic indicator, such as IGD, might lead to performance degradation [39].

3) Decomposition-based MaOEAs: such MaOEAs rely on decomposing an M-objective problem into a set of subproblems, guided by a set of RVs. Each of these subproblems (one per RV) is optimized using an aggregation function, in collaboration with neighbouring solutions. These MaOEAs, including MOEA/D [41], MOEA/D-DE [42], and MOEA/D-AS [43], provide the scope for approximating an M-dimensional PF with a reasonably sized solution set, while the convergence is guided by the choice of aggregation function, such as weighted-sum, PBI, Tchebycheff [41], or LWS [44]. Notably, the performance of such MaOEAs strongly depends on the choice of neighbourhood size [45]. To alleviate the associated challenge, I-DBEA [46] considers the entire population as neighbourhood, while MOEA/D-LWS [44] defines the neighbourhood using a hypercone, and avoids an adhoc fixation of neighbourhood size.

4) Hybrid MaOEAs: such MaOEAs rely on the use of different dominance principles (pursuing convergence), within an RV based *decomposition* architecture (enabling diversity). For convergence, while NSGA-III [8], [47] relies on PD despite its known limitations, BFEA [28] uses NLAD-dominance, and RPD-NSGA-II [14] uses RP-dominance. Other MaOEAs rely on use of two different dominance principles, for instance, θ-DEA [12] uses PD and θ-dominance, while DrEA [13] uses PD and D-dominance. Notably, each of the above MaOEAs: (i) performs *non-dominated sorting* of the entire population, once in each generation, based on underlying dominance principle, (ii) associates solutions to different RVs based on their perpendicular distance, and (iii) selects a solution for each RV, by factoring its perpendicular distance and/or the solution's projection along the RV.

It is important to note that the performance of *RV-based MaOEAs (decomposition based or hybrids)* strongly depends on the PF shape [34]. Here, the performance of several MaOEAs, including, NSGA-III and θ-DEA, is shown to deteriorate on problems with inverted PFs. The reason



(a) Pareto-dominance **(b)** $|n_b|$ in *hf*-dominance **(c)** Δf in *hf*-dominance, with $p_m = 1/M$ **(d)** *hf*-dominance, with $p_m = 1/M$ **(e)** *hf*-dominance, with p_ms $\equiv \{0.2, 0.1, 0.7\}$

Figure 1: Dominance envelopes: *hf*-dominance versus Pareto-dominance.

is, that such MaOEAs rely on the uniform generation of RVs, through Z^I, with an aim to obtain a diverse set of solutions across the true *PF*. Hence, the implicit *assumption* is that a major proportion of these RVs would pass through the true *PF*. However, this assumption stands violated on problems with inverted *PF*s. The remedial approaches broadly relate to: (i) *adapting the distribution* of RVs, while retaining their source in Z^I [48], and (ii) *modifying the source* of RVs (Z^I to Z^N), while retaining their uniform distribution [49], [50].

III. Proposed Dominance Principle and MaOEA

The core contributions of this paper, are presented in this section. They relate to the proposition of *hf*-dominance, leading up to LHFiD.

A. hf-dominance: Definition

For solutions X and Y, let n_b^X and let n_b^Y denote the number of objectives in which X is better than Y, and vice-versa. Then, X is said to *hf-dominate* Y, denoted as $X \succ_{hf} Y$, if *both* the following conditions are met:

- X is better in more or equal number of objectives than Y, implying,
$$n_b^X \geq n_b^Y. \tag{1}$$
- For X, the weighted gain in objectives exceeds the weighted loss in objectives, when compared to Y. If p_m is the relative preference for the m^{th} objective, and $\Delta f_m = f_m^X - f_m^Y$, then for a minimization problem, the above can be described by Equation 2.
$$\Delta F_{X,Y} = \sum_{m=1}^{M} p_m \cdot \Delta f_m < 0, \tag{2}$$

In this background, X^* is said to be *hf*-optimal if there exists no feasible solution X that *hf*-dominated X^*. The salient features of *hf*-dominance are highlighted below.

1) Dispelling the Challenge of Parameter-tuning: as highlighted earlier, all existing dominance principles that account for the HDM elements even partially, involve additional parameter(s), tuning which is a non-trivial task. It is significant to note, that the proposed *hf*-dominance, not only accounts for all the three HDM elements *explicitly and simultaneously*, but does it *without requiring tuning of any parameter*, as highlighted below.

It is evident that, for any two solutions X and Y in the EA population, the evaluation of whether or not: (i) $n_b^X \leq n_b^Y$, and (ii) $\Delta f_m = f_m^X$-$f_m^Y \leq 0$, is purely dependant on their objective values, and *no additional parameter is involved, hence, the question of its tuning does not even arise*.

It ought to be noted that the third HDM element gains significance primarily in real-world situations, where a DM based on one's domain expertise and experience has well-defined relative preferences over the objectives, and he/she expects those preferences to be inbuilt into an MOEA's/MaOEA's *search*, so that the final solutions are an outcome of the desired preference order. In such situations, the inability of existing dominance principles to account for such relative preferences presents a critical bottleneck. Hence, the overcoming of this limitation by *hf*-dominance, marks a major advancement. More so, because, *given the desired and well-defined p_ms by the DM, the challenge of parameter tuning does not arise*. It is imperative to ponder, what if the DM's preferences are not available. In such situations, guided by the goal of avoiding the *challenge of parameter tuning*:

- equal preferences ($p_m = 1/M$, in Equation 2) may be set, implying no relative preferences over the objectives.
- alternatively, the rationale-based and efficient choice available for p_ms, owing to the implementation of *hf-dominance* in a RV-based framework, may be used. A detailed discussion on this is deferred to Section IV.

2) Enhancement of Dominance Area: towards revelation of how the area dominated by a solution as per PD, is enhanced by *hf*-dominance, the solution's dominance envelopes alongside the corresponding dominance volume (V), are highlighted in the Figure 1. For ease of visualization, a three-objective space is chosen. In that, the dominance envelope, corresponding to:

- PD is highlighted in Figure 1a, offering $V = 0.125$.
- the $|n_b|$ criteria in *hf*-dominance, *alone*, is highlighted in Figure 1b, offering $V = 0.50$.
- the Δf criteria in *hf*-dominance, *alone*, is highlighted in Figure 1c, offering $V = 0.50$.
- *hf*-dominance, with no relative preferences in the objectives ($p_m = 1/M$), is highlighted in Figure 1d, offering $V = 0.431$.

Algorithm 1: Overall LHFiD procedure

Input: Objectives M, population size N, RV set $\mathcal{R}$, stabilization tracking parameters ψ^N, and ψ^{TM}.

```
1  begin
2      Randomly create P_1 (P_t at t = 1)
3      Z^I ← min_{x∈P_1} f_k(x) ∀ k ∈ [1, M]
4      Z^N ← ∅
5      foreach t ∈ [1, 2, ...] do
6          P_mating ← Random Selection (P_t)
7          Q_t ← Reproduction (P_mating)
8          U_t ← P_t ⋃ Q_t
9          Z^I ← min(Z^I_k, min_{x∈U_t} f_k(x)) ∀ k ∈ [1, M]
10         F ← All objective vectors in U_t
11         if Z^N ≠ ∅ then
12             F̃ ← Normalized F using Z^I & Z^N
13         else
14             F̃ ← F − Z^I
15         end
16         P_{t+1} ← Environment Selection(U_t, F̃, N, R)
17         C^N ← Stabilization Tracking(P_t, P_{t+1}, R, ψ^N)
18         if C^N is True then
19             Z^N ← Update Nadir(P_{t+1}, Z^I)
20         C^TM ← Stabilization Tracking(P_t, P_{t+1}, R, ψ^TM)
21         if C^TM is True then
22             Terminate LHFiD (t_TM = t)
23     end
24     return P_t at t = t_TM
25 end
```

- *hf*-dominance, with objectives' relative preferences arbitrarily chosen as $p_m\text{s} \equiv \{0.2, 0.1, 0.7\}$, is highlighted in Figure 1e, offering $V = 0.390$.

The above leads to the following key revelations:

- Each HDM element partially contributes to the final dominance envelope. The contributions of n_b and Δf are evident in Figures 1b and 1c. Furthermore, the difference in Figures 1d and 1e reveals that the dominance envelope also gets impacted by the changes in the objectives' relative preferences.
- *hf*-dominance's envelope if larger than that of PD.

3) Properties: notably, *hf*-dominance manifests the following properties (derived in Section S3, S.D.).

(a) The *hf*-dominance relation is *not reflexive* since any solution X does not *hf*-dominate itself ($X \not\succ_{hf} X$).
(b) The *hf*-dominance relation is *not symmetric* because $X \succ_{hf} Y$ does not imply $Y \succ_{hf} X$.
(c) The *hf*-dominance relation is *not transitive* since $X \succ_{hf} Y$ and $Y \succ_{hf} Z$ does not imply $X \succ_{hf} Z$, but implies $Z \not\succ_{hf} X$. Thus, it is *semi-transitive*.
(d) X Pareto-dominates Y ($X \succ Y$), implies $X \succ_{hf} Y$.
(e) If X^* is *hf*-optimal, then X^* is Pareto-optimal too.

B. Proposed MaOEA: LHFiD

The overall procedure of LHFiD is summarized in Algorithm 1 (AL.1). Its inputs include: the MOP/MaOP definition with objectives M; population size N; the RV set $\mathcal{R}$, sized N; reproduction (crossover and mutation) parameters; and two other parameter sets, i.e., ψ^N and ψ^{TM}, influencing the timing of Z^N estimation/update and LHFiD's termination, respectively. Initially (lines 2–4, AL.1): the first population (P_1) is created randomly; its Z^I is estimated as the minimum value of each objective; and the nadir point is set as $\emptyset$. Following this, the generation-wise procedure (lines 5–23, AL.1) starts. Any generation t of LHFiD can broadly be split into 4 phases, as discussed below:

1) Creation and inclusion of offspring (lines 6–8, AL.1): first, a mating pool of solutions is created using random selection on the population P_t (line 6, AL.1). Then, reproduction operators are applied on the mating pool to yield the offspring population Q_t (sized N), followed by its merger with P_t to yield the combined population U_t (lines 7–8, AL.1).

2) Normalization/Translation (lines 9–15, AL.1): first, the ideal point is updated using U_t and the last estimated ideal point (line 9, AL.1). Then, if the nadir point is available, the population is normalized, else, it is simply translated such that the ideal point coincides with the origin (lines 11–15, AL.1).

Intently, the nadir point is not updated in every generation. Rather, it is estimated intermittently, corresponding to *only* those generations where, the inbuilt stabilization tracking algorithm [30] detects mild population stabilization. The rationale is that during the earlier generations, the population evolves rapidly and the extreme points fluctuate significantly. This may render the nadir point estimates unstable, adversely impacting the diversity [30], [51]. Hence, until the first detection of mild population stabilization, the nadir point remains unavailable, implying just the translation of population. However, in generations subsequent to the first stability detection, the population is normalized using the last estimated nadir point.

3) Environmental selection (line 16, AL.1): with an aim to select the N best solutions (P_{t+1}) from the combined population U_t, the environmental selection procedure invoked in line 16, AL.1, is summarized in Algorithm 2 (AL.2). This procedure relies on a set of RVs for ensuring diversity, and on dominance principles for ensuring convergence *locally*. First, the normalized/translated solutions are associated to their respective nearest RVs based on perpendicular distance $d^{\perp}$ (lines 3–4, AL.2). The remaining procedure can be read in two steps.

The first step addresses the selection of solutions on non-empty RVs - that have at least solution associated with them (lines 5–18, AL.2). In that, for any RV, the associated solutions are clustered together ($\mathcal{C}$) and the Pareto-dominated solutions within $\mathcal{C}$ are removed (lines 5–7, AL.2). From the solutions left in $\mathcal{C}$, the solution nearest to the underlying RV (by perpendicular distance) is referred to as the α-solution. If the underlying RV is *not an axis vector*, the *hf*-dominance based selection is used (lines 9–16, AL.2). In that, all solutions in $\mathcal{C}$ that *hf*-dominate the α-solution are identified and referred

Algorithm 2: Environment Selection$(U_t, \tilde{F}, N, \mathcal{R})$

```
1  begin
2      S ← ∅, S^L ← ∅, R^L ← R
3      D^⊥ : d^⊥_{i,j} ← d^⊥(F̃_i, R_j) ∀ i ∈ [1,2N] & j ∈ [1,N]
4      Associate each solution U_t with its nearest RV
5      foreach R_j ∈ R do
6          C ← All solutions in neighbourhood of R_j
7          Remove pareto-dominated solutions in C
8          α ← Solution in C with minimum d^⊥
9          if R_j is not an axis vector then
10             β ← ∅
11             for s ∈ C do
12                 if s hf-dominates α then
13                     β ← β ⋃ s
14             end
15             if count(β) ≥ 1 then
16                 α ← Solution in β with minimum d^⊥
17         S ← S ⋃ α;   U_t ← U_t \ α;   R^L ← R^L \ R_j
18     end
19     F̃ ← Normalized/Translated Objective vectors from U_t
20     foreach r ∈ R^L do
21         D^⊥ : d^⊥_i ← d^⊥(F̃_i, r) ∀ F̃_i ∈ F̃
22         α ← Solution in U_t with minimum d^⊥
23         S^L ← S^L ⋃ α
24     end
25     return P_{t+1} ← S ⋃ S^L
26 end
```

to as β-solutions (lines 10–13, AL.2). Further, the cluster $\mathcal{C}$'s representative solution is selected as per one of the following scenarios (lines 15–17, AL.2):

- No β-solution exists: the α-solution is selected.
- One β-solution exists: that β-solution is selected.
- More than one β-solutions exist: the β-solution nearest to the underlying RV is selected.

However, if the underlying RV is *an axis vector*, the α-solution is directly selected as the cluster $\mathcal{C}$'s representative, allowing a better approximation of PF extremes; better nadir point update; and eventually better diversity.

The second step addresses the selection of solutions on empty RVs - that do not have any solution associated (lines 19–24, AL.2). Towards it, first, the cluster representatives selected above (for non-empty RVs) are removed from the population U_t. From the leftover solutions in U_t, the solutions nearest to the empty RVs are selected as their respective representatives (lines 20–23, AL.2).

On the whole, the above steps ensure selection of one representative for each RV. Since the number of RVs is same as the population size N, it implies selection of N best solutions that constitute P_{t+1}.

4) Two pronged stabilization tracking (lines 17-22, AL.1): as evident in Algorithm 1, the *stabilization tracking* algorithm is invoked with dual goal (lines 17 and 20, AL.1) of determining the timing for: (i) intermittent nadir point update, and (ii) LHFiD's termination. While complete details of the stabilization tracking algorithm can be found in [30], its key formulations are provided in Section S4 of S.D.. The gist is that, at any generation t:

- the current population P_t when subjected to mating selection, reproduction, and environmental selection leads to the new population P_{t+1}.
- the relative stability of these successive populations (P_t and P_{t+1}) is assessed through a distance measure $\mathcal{D}_t$, whose mean ($\mu_{\mathcal{D}_t}$) and standard deviation ($\sigma_{\mathcal{D}_t}$) *from the first till the* t^{th} *generation* are computed.
- as $\mu_{\mathcal{D}} = [\mu_{\mathcal{D}_1}, \ldots, \mu_{\mathcal{D}_t}]$ and $\sigma_{\mathcal{D}} = [\sigma_{\mathcal{D}_1}, \ldots, \sigma_{\mathcal{D}_t}]$, are available, the population is considered to have *stabilized*, if each of $\mu_{\mathcal{D}_t}$ and $\sigma_{\mathcal{D}_t}$ conform with their respective values over n_s successive generations, up to n_p decimal places.

It may be noted that the parameter set $\psi \equiv \{n_p, n_s\}$ is used to track different degrees of population stability, through appropriate settings of n_p and n_s. In that, ψ^N aiming at detecting mild stability is used for timing the nadir point update (line 17, AL.1). If detected, the nadir point is updated as per the method in [8] (lines 18–19, AL.1). Similarly, ψ^{TM} aiming at detecting stricter stability is used for final termination of LHFiD (lines 20–22, AL.1).

IV. Use of Objectives' Relative Preferences in LHFiD

This section first discusses, how the implementation of *hf-dominance* in RV-based LHFiD, makes possible, an alternative choice for p_ms, which when compared to the use of $p_m = 1/M$ (no relative preferences), is arguably more rational and efficient, and still does away with the challenge of *subjectivity* and *parameter tuning*.

This choice relates to *the use of the components of the RVs towards objectives' relative preferences*. The rationale is that the components of any RV represents a unique preference-order in the F-space, and hence could synergistically be utilized in the corresponding solution cluster as p_ms. In other words, while weighing the degree of betterment in the objectives (associating p_m with Δf_m, in Equation 2) for solutions in a cluster, it is only natural that the objectives' preference-order embedded in the underlying RV be utilized. Hence, this paper, in the absence of real DM's preferences, recommends the use of of $p_m = 1/w_m$, where w_m is the m^{th} component of an RV. Its implication is that objectives' relative preferences will vary from one RV to another depending on the unique preference order that the underlying RV represents. Needless to assert, this recommendation is neither subjective, nor suffers from the challenge of parameter tuning.

The remaining section aims to reveal, with respect to a simple example:

- how the inbuilt scope in LHFiD (enabled by *hf*-dominance) to incorporate real DM's preferences, when ever available, can be meaningfully realized.
- how in the absence of DM's preferences, the recommended use of $p_m = 1/w_m$ may help select better solutions compared to the use of $p_m = 1/M$.

Towards it, LHFiD's environmental selection is executed for the example in Figure 2, and presented in Table II.

Table II: Application of LHFiD's environmental selection with different preferences, on solution clusters $\mathcal{C}_A$ and $\mathcal{C}_B$ in Figure 2.

LHFiD's environmental selection steps			Cluster $\mathcal{C}_A$	Cluster $\mathcal{C}_B$	$\{\mathcal{C}_A, \mathcal{C}_B\}$ Representatives
Step A.1: Pareto non-dominated solutions			S_1, S_2, S_4 and S_5 are Pareto non-dominated	S_6, S_7, S_9 and S_{10} are Pareto non-dominated	
Step A.2: α-solution (based on $d^{\perp}$)			S_2 is the α-solution	S_6 is the α-solution	
Step A.3	(a): Solutions better in n_b, than α		S_1 & S_5 ($n_b^{S_1}$ & $n_b^{S_5} > n_b^{S_2}$)	S_7 & S_{10} ($n_b^{S_7}$ & $n_b^{S_{10}} > n_b^{S_6}$)	
	(b): ΔF with	$p_m = 1/M$	$\Delta F_{S_5,S_2} > 0$; $\Delta F_{S_1,S_2} < 0$	$\Delta F_{S_7,S_6} > 0$; $\Delta F_{S_{10},S_6} < 0$	$\{S_1, S_{10}\}$
		$p_m = 1/w_m$	$\Delta F_{S_5,S_2} < 0$; $\Delta F_{S_1,S_2} > 0$	$\Delta F_{S_7,S_6} < 0$; $\Delta F_{S_{10},S_6} < 0$	$\{S_5, S_{10}{}^*\}$
		$p \equiv \text{DM} \equiv \{0.1, 0.6, 0.3\}$	$\Delta F_{S_5,S_2} > 0$; $\Delta F_{S_1,S_2} < 0$	$\Delta F_{S_7,S_6} < 0$; $\Delta F_{S_{10},S_6} > 0$	$\{S_1, S_7\}$

* Step A.4 was necessary only in the case of $\mathcal{C}_B$ corresponding to $p_m = 1/w_m$, since both $F_{S_7,S_6} < 0$ and $\Delta F_{S_{10},S_6} < 0$, and the choice of $\mathcal{C}_B$'s representative (S_{10}) had to be based on the proximity to R_B. Step B, is not required since R_A and R_B do not coincide with any objective-axis.

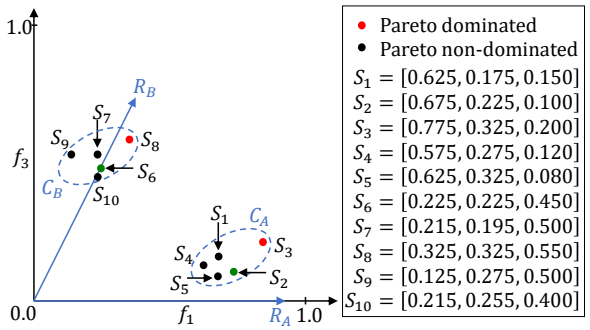


Figure 2: An example to illustrate LHFiD's environmental selection with different objectives' relative preferences.

For interpretability of the steps in Table II, the steps in LHFiD's selection ((Section 1)) are re-iterated below, with conforming notation (Steps A.1-A.4; and B).

Step A: *For RVs that do not coincide with any objective-axis, the convergence-diversity trade-off is managed through the following*:

A.1 Only the Pareto non-dominated solutions are considered (emphasizing convergence).

A.2 The α-solution becomes the reference for selection in the cluster (emphasizing diversity).

A.3 The solutions which *hf*-dominate α are identified (emphasizing convergence), by factoring:

(a) the number of better objectives (n_b)

(b) the size of betterment (ΔF), guided by p_ms

A.4 Selection of the *hf*-dominant solution, nearest to the RV (emphasizing diversity).

Step B: *For RVs that coincide with an objective-axis*: the α-solution is selected (emphasizing diversity).

In the context set above, Figure 2 may be referred to. Notably, it portrays a two dimensional projection (f_1-f_3 subspace) of a three-objective F space, and details the 10 solutions (S_1-S_{10}) to be considered with respect to two RVs (R_A and R_B). In that: (i) $R_A \equiv [0.75, 0.25, 0.00]$ is a boundary vector, where $\{S_1, \ldots, S_5\}$ constitute its cluster $\mathcal{C}_A$, and (ii) $R_B \equiv [0.25, 0.25, 0.50]$ is a non-boundary vector, where $\{S_6, \ldots, S_{10}\}$ constitute its cluster $\mathcal{C}_B$. Since R_A and R_B do not coincide with any objective-axis, only the Step A (above) is applicable, and its self-explanatory implementation for both clusters, is presented in Table II. Notably, the role of objectives' relative preferences comes into play in Step A. 3(b) towards which three options are considered for further insights, including:

- $p_m = 1/M$: no relative preferences.
- $p_m = 1/w_m$: relative preferences aligned with the components of the RVs, recommended in this paper, if real DM's preferences are not available.
- $\{p_1, p_2, p_3\} \equiv \{0.1, 0.6, 0.3\}$: emulating DM's preferences, as if they were available.

1) Advantage of incorporating (HDM's third element) objectives' relative preferences in alignment with the components of RV: notably, use of $p_m = 1/M$, implying, equal preferences for the objectives, practically means that the HDM's third element is not accounted for. In this case the $\{\mathcal{C}_A, \mathcal{C}_B\}$ representatives are $\{S_1, S_{10}\}$. In contrast, when the components of the RVs are used towards the relative preferences, the $\{\mathcal{C}_A, \mathcal{C}_B\}$ representatives are $\{S_5, S_{10}\}$. Clearly, use of $p_m = 1/w_m$ promises broader coverage (diversity) than the use of $p_m = 1/M$. More importantly, this is not a coincidence, but the result of a judicious choice, discussed below. Notably, $R_A \in f_1$-f_2 subspace with $w_3 = 0$, implying that while computing ΔF, the contribution of f_3/w_3 will overpower the contributions of f_1/w_1 and f_2/w_2. This, in conjunction with the fact that $f_3(S_5) < f_3(S_1)$, ensures that $\Delta F_{5,2} < \Delta F_{1,2}$. This potential for better diversity, via higher selection pressure on solutions closer to the boundary, justifies the recommendation for use of $p_m = 1/w_m$ in scenarios, where DM's preferences may not be available.

2) Utility of incorporating (HDM's third element) DM's preferences, over the entire PF approximation: notably, the $\{\mathcal{C}_A, \mathcal{C}_B\}$ representatives, corresponding to the DM's preferences, arbitrarily chosen as $\{0.1, 0.6, 0.3\}$ turn out to be $\{S_5, S_7\}$. Clearly, such representatives will vary with a change in the specified p_ms. However, what is more significant is that, *hf*-dominance provides the scope to incorporate p_ms, which no other dominance principle does, till date. Interestingly, by incorporating *hf*-dominance in a RV-based framework, LHFiD presents one of the rare opportunities, where DM's preferences can be applied, without compromising on the approximation of the *entire* PF. Hence, from a practical perspective,

LHFiD offers the DM the confidence, that whenever a solution other than the α-solution is chosen as a cluster representative, his/her preferences have been accounted for, over the entire PF.

V. Experimental Settings

This section prepares the ground for the computational experiments and analysis of the results.

A. Test Problems

The test problems considered here comprise of three to 15-objective versions of non-redundant problems, with both regular and inverted PFs, from the DTLZ [31], WFG [32], DTLZ^{-1} [34], WFG^{-1} [34], and MaF [33] test suites[1]. For the DTLZ, DTLZ^{-1}, and MaF problems, the number of variables are kept as $n_{\text{var}} = (M-1)+k$, where $k = 20$, except for MaF14 where $n_{\text{var}} = 20 \times M$. Similarly, for WFG and WFG^{-1} problems, the number of variables are $n_{\text{var}} = 2 \times (M-1)+k$, where $k = 20$. Notably, $k = 20$ makes these problems quite difficult compared to the generally used $k = 5$ or 10.

B. MaOEA suite for comparison with LHFiD

The performance of LHFiD on regular problems is compared with other MOEAs, including NSGA-III, θ-DEA, RPD-NSGA-II, MOEA/D-LWS, multiGPO, the nale for which is evident in Table III. In that, the chosen MaOEAs cover different categories of dominance principles, and also different MaOEA architectures. Notably, the

Table III: MaOEAs to be compared with LHFiD

Category of Dominance Principle	MaOEA	MaOEA Architecture
Original F values	NSGA-III	Hybrid (Dominance + Decomposition)
Singularized F values	θ-DEA RPD-NSGA-II	
Modified F values	multiGPO	Pure Dominance
– Not Applicable –	MOEA/D-LWS MOEA/AD	Pure Decomposition

HDM elements based MaOEAs could not be chosen for comparison, since: (i) $1-k$, $1-k_F$, and Fuzzy dominance principles have not been incorporated into any MaOEA, and (ii) L-dominance based MDMOEA offers only a subset of the true PF. Moreover, the choice of these MaOEAs is also guided by: (i) availability of their source codes, which helps avoid inadvertent bias/errors, and (ii) the fact that these have already been exhaustively tested against several other[2] MaOEAs, implicitly allowing for their indicative comparison with LHFiD too. Additionally, the variants of LHFiD and other MaOEAs have also been considered in different contexts, highlighted below.

Notably, LHFiD's performance may be attributed to its two core constituents, namely: (i) *hf*-dominance based environmental selection, and (ii) intermittent Z^N update based on stabilization tracking algorithm. To enable an isolated assessment of the efficacy of *hf*-dominance based environmental selection *alone*, a variant of LHFiD, namely LHFiD-n where Z^N is updated in every generation (not guided by the stabilization tracking algorithm) has been introduced, and compared with the other MaOEAs.

It may also be noted that unlike other MaOEAs, MOEA/D-LWS uses an offline unbounded external archive (UEA) to store all the non-dominated solutions found from the initial till the termination generation, and the corresponding HV is used as the performance measure. The reported computational complexity of MOEA/D-LWS (in [44]) is $\mathcal{O}(N^2)$. However, updating of UEA towards performance evaluation, entails a complexity of $\mathcal{O}(M\tilde{N}^2)$, where, $\tilde{N} = N \times t_{TM}$. This plausibly explains why the experiments in [44] were restricted to WFG test suite, $M = 7$ and $t_{TM} = 250$. In this background, the experiments for two different settings, as below:

- without UEA: here, the DTLZ, WFG, and MaF problems are considered (with $M = 3,\ 5,\ 8$, and 10); and the MaOEAs without UEA, are compared at t_{TM} generations, where t_{TM} corresponds to on-the-fly termination of LHFiD.
- with UEA: here, the DTLZ and WFG problems are considered (with $M = 3,\ 5$ and 8), and the MaOEAs with UEA, are compared at $t_{TM} = 500$.

Finally, to cater to the *inverted* problems, another variant of LHFiD, namely, LHFiD-i has been proposed. Considering that LHFiD is a hybrid MaOEA, it is also vulnerable to the challenges cited in Section II, calling for either the adaptation of distribution of RVs, or modifying the source of RVs. However, as the primary focus in this paper is to evaluate the efficacy of *hf*-dominance based selection in an RV-based architecture, and not to propose a method for adapting the distribution of RVs, minimal changes are incorporated in LHFiD, leading up to LHFiD-i. These include:

- generation of RVs from Z^N, instead of Z^I.
- estimation of Z^N using the naïve normalization [54], involving only the current non-dominated solutions.
- update of Z^N in each generation initially, until the first mild stabilization is detected, following which, the Z^N update is intermittently timed.

[1]Notably, the redundant problems such as DTLZ5, DTLZ6, WFG3, and MaF6 have been excluded, since: (i) they yield the claimed degenerate fronts only after incorporation of constraints [52], [53], (ii) the performance of different MaOEAs will also depend on their underlying constraint-handling approaches, given which comparison on the efficacy of environmental selection procedures alone, becomes difficult. While MaF1 and MaF4 are covered in Inverted PF category, MaF10–12 have been omitted since they overlap with WFG problems. For paucity of space, only one (MaF14) of the two large scale problems (MaF14-15) has been considered.

[2]These include: (i) GrEA, POGA, SPEA2+SDE, MOEA/D, dMOPSO, HypE, MOMBI [12]; (ii) MOEA/DD, MOEA/D-DE, θ-NSGA-III [14]; (iii) MOEA/D-AS, MOEA/D-SS, PICEA-g [44]; and (iv) 1by1EA and NSGA-II/SDR [29].

For a fair performance comparison with LHFiD-i:

- the earlier used RV-based MaOEAs have been modified, such that their RVs also emanate from Z^N, similar to LHFiD-i. The modified MaOEAs are renamed as NSGA-III-i, θ-DEA-i, RPD-NSGA-II-i and MOEA/D-LWS-i, respectively.
- multiGPO is retained, as such, since it is purely dominance-based and does not rely on RVs at all.
- MOEA/AD [50], which was developed dedicatedly to cater to inverted problems, has been added.
- all MaOEAs have been compared: (i) at t_{TM} determined on-the-fly for LHFiD-i, and (ii) without UEA.

C. Parameter Settings

Notably, all considered MaOEAs (except multiGPO) require a set of reference points, for which the Das and Dennis (DD) method [55] is used. The number of gaps (p) for DD method and the corresponding population sizes N are highlighted in Table IV. Where two values of p are shown, the first value is used to generate boundary points while the second value is used to generate interior points [8]. RVs are generated by joining the origin with each of these reference points. The size of RVs is kept identical to N. Even though multiGPO is not aided by RVs, the same population size is used, as given in Table IV.

Table IV: Parameter settings for the Das and Dennis method.

M	3	5	8	10	15
N	105	210	156	275	135
p (gaps)	13	6	3, 2	3, 2	2, 1

For all the experiments across the MaOEAs, the SBX crossover ($p_c = 0.9$ and $\eta_c = 20$) and polynomial mutation ($p_m = 1/n_{\text{var}}$ and $\eta_m = 20$) are used. Some problems, namely, DTLZ1, DTLZ1^{-1}, DTLZ3, and DTLZ3^{-1}, are exceptions where $\eta_m = 5$ is used. This is done, considering that these problems are highly multimodal, and ensuring diversity during the early generations is critical, towards which a lower distribution index for mutation, helps. Apart from these, the MaOEA specific parameters (as applicable) are highlighted below.

- LHFiD, LHFiD-n and LHFiD-i: $p_m = 1/w_m$; and $\psi^N = \{2, 20\}$ and $\psi^{TM} = \{3, 50\}$ [30], are used.
- θ-DEA: the penalty parameter (θ) is set to $\theta = 10^6$ for axis RVs, and $\theta = 5$ for other RVs [12].
- MOEA/D-LWS: neighbourhood size (T= 10% of N); mating restriction probability $\delta = 0.8$; and unlimited offspring replacement size nr [44].
- multiGPO: expanding angle (ϕ) is set to $\phi = \eta M$, where $\eta = 3$ for $M \leq 10$ and $\eta = 2.5$ for $M > 10$ [29].
- MOEA/AD: neighbourhood size $T = 20$; mating restriction probability $\delta = 0.9$; and offspring replacement sizes, $nr_1 = 1$ and $nr_2 = 2$ [50].

For each test instance, the performance of each MaOEA is assessed over its runs with 31 random seeds, under two different termination settings, including: (i) t_{TM} inferred by the stabilization tracking of LHFiD (and its variants, as applicable) for comparisons without UEA, and (ii) a priori specified $t_{TM} = 500$ for comparisons with UEA.

D. Performance Indicators

In this paper, different sets of performance indicators are used, as summarized below.

- *For regular problems:* HV is used as the performance indicator with reference point, $r = 1 + 1/p$, where p is given in Table IV [56]. Further, $d^{\parallel}$ and $d^{\perp}$ are used for interpretation of HV measures, where $d^{\parallel}$ and $d^{\perp}$ are population averages of d_1 and d_2, respectively, as used in PBI [41].
- *For inverted problems:* HV with $r = 2.0$, is used as the performance indicator, since $r = 2.0$ (compared to lower values of r) enables the contribution of boundary and interior points towards HV, to be more equable [34], [56].

The indicator values are subjected to a statistical analysis: first using Kruskal-Wallis test [57] with a threshold p-value of 0.05 to infer if the overall differences are insignificant or not. If not, Wilcoxon rank-sum test [58] is used for pairwise comparisons, with LHFiD as reference. Furthermore, the threshold p-value is adjusted using standard Bonferroni correction [59], to retain the same overall confidence. In addition, the rank biserial effect size (r_b) [60] has also been reported.

E. Computational complexity

For the proposed LHFiD, the worst-case time complexity is approximately $\mathcal{O}(MN^2)$ corresponding to the $2N \times N$ computations of $d^{\perp}$ (line 3, AL.2), with a complexity of $\mathcal{O}(M)$). The complexities of other MaOEAs are highlighted below, and it can be seen that LHFiD fares equivalent or better than others.

- NSGA-III: $\max\{\mathcal{O}(N \log^{M-2} N), \mathcal{O}(MN^2)\}$
- θ-DEA and RPD-NSGA-II: $\mathcal{O}(MN^2)$
- MOEA/D-LWS: $\mathcal{O}(MN^2 t_{TM}^2)$ with UEA, and $\mathcal{O}(MN^2)$ without UEA
- multiGPO: $\mathcal{O}(M^2N^2)$
- MOEA/AD: $\max\{\mathcal{O}(N^2 \log N), \mathcal{O}(MN^2)\}$

VI. Experimental Results

This section reports the experimental results, first for the regular problems, followed by inverted problems.

A. Results on Regular Problems

1) Setting I: t_{TM} determined on-the-fly; without UEA: The performance of LHFiD vis-*à*-vis other MAOEAs is presented in Table V. In that, for any given problem, the reported t_{TM} is the average (rounded, to the next integer) of the t_{TM} values determined on-the-fly, for the 31 LHFiD

Table V: HV based comparison of LHFiD and other MaOEAs (all without UEA), on DTLZ, WFG and MaF problems, with t_{TM} determined *on-th-fly* for LHFiD: the entries are formatted as $med(r_b)$, followed by "–" or "=" or "+". Here, med and r_b represent the median and rank biserial effect size for HV over 31 runs, while −/=/+ inform that the underlying performance is statistically worse than, comparable to, or better than LHFiD, respectively.

Problem	M	t_{TM}	NSGA-III	θ-DEA	RPD-NSGA-II	MOEA/D-LWS	multiGPO	LHFiD
DTLZ1	3	624	1.247307 (-0.55)=	1.247823 (-0.55)=	1.248936 (-0.91)+	1.248775 (-0.91)+	1.248941 (-0.89)+	1.244145
	5	772	2.161328 (-0.24)=	2.161389 (-0.56)=	2.161394 (-1.00)+	2.161394 (-0.95)+	2.161394 (-0.99)+	2.161377
	10	751	17.757294 (0.64)–	17.757701 (-0.23)=	17.757727 (-0.99)+	17.757727 (-1.00)+	17.757713 (0.15)=	17.757679
	15	881	437.875666 (1.00)–	437.421046 (1.00)–	437.893517 (0.34)=	437.893890 (-1.00)+	405.517655 (1.00)–	437.893739
DTLZ2	3	500	0.666954 (-1.00)+	0.484388 (1.00)–	0.654990 (1.00)–	0.634640 (1.00)–	0.663860 (-1.00)+	0.660475
	5	570	1.855361 (-1.00)+	1.857332 (-1.00)+	1.847444 (-0.80)+	1.770959 (1.00)–	1.856145 (-1.00)+	1.844668
	10	669	17.663406 (1.00)–	17.674449 (1.00)–	17.672550 (1.00)–	17.454792 (1.00)–	17.656295 (1.00)–	17.687847
	15	634	437.828041 (1.00)–	437.836784 (1.00)–	437.768033 (1.00)–	436.767130 (1.00)–	437.749547 (1.00)–	437.846235
DTLZ3	3	591	1.185201 (-0.60)=	0.000000 (1.00)–	1.247395 (-0.93)+	1.242750 (-0.93)+	1.247756 (-0.96)+	1.050165
	5	697	2.133900 (-0.59)=	2.161364 (-0.95)+	2.161390 (-0.99)+	2.161365 (-0.92)+	2.161391 (-1.00)+	1.819034
	10	679	17.318663 (0.44)–	17.757155 (-0.67)+	17.757727 (-0.99)+	17.757726 (-0.84)+	17.757727 (-0.89)+	17.751664
	15	803	402.777146 (0.98)–	437.686825 (0.77)–	430.463631 (0.93)–	437.893890 (-1.00)+	20.213373 (1.00)–	437.892573
DTLZ4	3	904	0.667274 (-0.41)+	0.089205 (1.00)–	0.655611 (0.87)–	0.638791 (0.87)–	0.665079 (-0.24)+	0.661308
	5	852	1.859055 (-1.00)+	1.859478 (-1.00)+	1.847690 (0.11)=	1.770793 (1.00)–	1.858606 (-1.00)+	1.847690
	10	817	17.677868 (1.00)–	17.678846 (1.00)–	17.679297 (1.00)–	17.503898 (1.00)–	17.662487 (1.00)–	17.684803
	15	706	437.854045 (0.26)–	437.854448 (0.14)–	437.725849 (1.00)–	436.893889 (1.00)–	437.780444 (1.00)–	437.854993
WFG1	3	969	0.520021 (1.00)–	0.852932 (-1.00)+	1.128620 (-1.00)+	0.592511 (-0.44)=	1.130172 (-1.00)+	0.580169
	5	1123	0.871345 (1.00)–	1.474983 (-1.00)+	2.040545 (-1.00)+	1.168945 (-0.99)+	2.147911 (-1.00)+	0.999174
	10	1429	7.109197 (0.93)–	15.814711 (-1.00)+	17.729133 (-1.00)+	8.653199 (0.28)=	17.740183 (-1.00)+	9.166877
	15	1533	223.574018 (-1.00)+	385.132369 (-1.00)+	437.775264 (-1.00)+	186.039760 (0.38)–	437.338113 (-1.00)+	194.808723
WFG2	3	801	1.150280 (0.53)–	0.749987 (1.00)–	1.151542 (0.45)–	1.159375 (0.03)=	1.153670 (0.53)–	1.158393
	5	937	2.133862 (0.06)–	2.134063 (0.06)–	2.138288 (0.21)–	2.152752 (-0.06)=	2.102151 (0.35)–	2.154220
	10	1014	17.543773 (0.06)=	17.365137 (0.42)–	17.536336 (-0.19)=	15.066749 (0.58)=	17.581732 (-0.02)=	17.618647
	15	1096	429.357511 (-0.80)+	371.027931 (0.29)=	433.087393 (-0.99)+	259.737917 (0.98)–	433.993435 (-0.81)+	366.091523
WFG4	3	689	0.645502 (0.96)–	0.440520 (1.00)–	0.635518 (0.99)–	0.641464 (1.00)–	0.634373 (1.00)–	0.650313
	5	790	1.761016 (1.00)–	1.765717 (1.00)–	1.800656 (1.00)–	1.806596 (1.00)–	1.814901 (1.00)–	1.841264
	10	807	16.281904 (1.00)–	16.518346 (1.00)–	16.928771 (1.00)–	17.577037 (-0.52)+	17.416539 (-0.99)+	17.275279
	15	763	406.133546 (1.00)–	410.993476 (0.95)–	419.402362 (-0.10)=	338.852053 (1.00)–	382.089037 (1.00)–	418.362318
WFG5	3	633	0.600868 (0.77)–	0.415929 (1.00)–	0.583300 (1.00)–	0.589150 (1.00)–	0.595596 (0.97)–	0.606123
	5	717	1.697246 (1.00)–	1.701140 (1.00)–	1.692102 (1.00)–	1.691216 (1.00)–	1.706922 (1.00)–	1.749525
	10	789	16.085131 (1.00)–	16.074729 (1.00)–	16.236084 (1.00)–	16.650586 (1.00)–	16.264807 (1.00)–	16.696425
	15	760	399.410308 (1.00)–	400.506042 (1.00)–	406.343691 (0.98)–	410.822146 (-0.38)+	362.483440 (1.00)–	409.473967
WFG6	3	614	0.603869 (0.44)=	0.421247 (1.00)–	0.607255 (0.41)=	0.594191 (0.91)–	0.595956 (0.97)–	0.608386
	5	672	1.703455 (1.00)–	1.707527 (1.00)–	1.743344 (0.73)–	1.703245 (0.99)–	1.721951 (0.97)–	1.760429
	10	741	16.291970 (0.99)–	16.343194 (0.99)–	16.663891 (0.67)–	16.622521 (0.71)–	16.829436 (0.15)=	16.852635
	15	703	406.989910 (0.93)–	407.875956 (0.87)–	412.976392 (0.40)=	413.931473 (0.08)=	379.438909 (1.00)–	416.280079
WFG7	3	647	0.657296 (-0.66)+	0.339178 (1.00)–	0.656412 (0.28)=	0.642412 (1.00)–	0.648228 (1.00)–	0.656254
	5	737	1.819349 (1.00)–	1.822387 (1.00)–	1.841699 (1.00)–	1.790287 (1.00)–	1.834089 (1.00)–	1.852200
	10	745	17.064794 (1.00)–	17.155908 (1.00)–	17.430477 (1.00)–	17.593029 (0.99)–	17.639010 (-0.67)+	17.630655
	15	716	428.681251 (1.00)–	431.090149 (0.97)–	435.875194 (-0.89)+	432.104339 (0.63)–	420.014219 (1.00)–	434.243781
WFG8	3	624	0.551790 (-1.00)+	0.309319 (1.00)–	0.533423 (0.86)–	0.526677 (1.00)–	0.535136 (0.80)–	0.540595
	5	672	1.597465 (1.00)–	1.592356 (1.00)–	1.611899 (0.97)–	1.645570 (-0.73)+	1.526619 (1.00)–	1.627801
	10	727	15.833159 (-1.00)+	15.863656 (-1.00)+	16.691580 (-1.00)+	16.908739 (-1.00)+	12.975072 (-0.08)=	13.504189
	15	704	406.147445 (-1.00)+	420.766557 (-1.00)+	425.120915 (-1.00)+	286.861811 (0.30)=	408.471858 (-1.00)+	300.545664
WFG9	3	589	0.545169 (0.43)=	0.345961 (1.00)–	0.516422 (0.86)–	0.528585 (1.00)–	0.537610 (0.73)–	0.604802
	5	639	1.531336 (1.00)–	1.545268 (0.98)–	1.453269 (1.00)–	1.555159 (1.00)–	1.555592 (0.89)–	1.692196
	10	671	14.387451 (0.75)–	14.453288 (0.70)–	14.198227 (0.81)–	14.809235 (0.32)=	14.516544 (0.72)–	15.398655
	15	653	361.723673 (-0.74)+	361.440471 (-0.78)+	353.472829 (-0.35)=	367.853018 (-0.92)+	348.432690 (-0.34)=	346.132153
MaF2	3	496	0.396830 (-1.00)+	0.396337 (-0.99)+	0.394899 (-0.48)=	0.388965 (1.00)–	0.399100 (-1.00)+	0.394214
	5	487	0.845515 (0.76)–	0.831562 (0.98)–	0.836624 (1.00)–	0.870217 (-1.00)+	0.845183 (0.67)–	0.851315
	10	512	8.070216 (-1.00)+	7.957244 (-0.98)+	7.760665 (-0.97)+	8.249674 (-1.00)+	7.981179 (-1.00)+	7.418336
	15	443	205.521874 (1.00)–	199.414940 (1.00)–	207.742628 (1.00)–	114.187993 (0.79)=	204.368759 (1.00)–	214.641323
MaF3	3	1705	1.248911 (0.50)–	1.248917 (-0.25)=	1.248719 (1.00)–	1.246066 (1.00)–	1.248652 (1.00)–	1.248917
	5	1753	2.161394 (0.29)=	2.161394 (0.87)–	2.161362 (0.89)–	2.161393 (0.87)–	2.161310 (0.93)–	2.161394
	10	1446	17.757727 (-0.67)+	17.757727 (-0.13)=	17.357847 (1.00)–	17.757727 (-0.78)+	17.757329 (0.78)–	17.757727
	15	1351	0.000000 (1.00)–	401.613386 (0.98)–	437.432268 (1.00)–	437.893890 (-0.68)+	0.000000 (0.98)–	437.893890
MaF5	3	1409	1.227614 (-0.08)=	1.218986 (0.14)–	1.223588 (-0.08)=	1.229515 (-0.88)+	1.216121 (-0.06)=	1.228144
	5	1054	2.161073 (0.41)–	2.154727 (0.85)–	2.152519 (0.87)–	2.161201 (-1.00)+	2.149864 (0.87)–	2.161100
	10	959	17.757727 (-0.70)=	17.727347 (1.00)–	17.756355 (0.65)–	17.757727 (-0.76)+	17.743400 (0.96)–	17.757464
	15	876	437.893890 (-1.00)+	437.855913 (-0.33)=	435.083872 (0.98)–	437.893890 (-1.00)+	437.304041 (0.68)–	437.864261
MaF7	3	1055	0.375745 (-0.75)+	0.369014 (0.88)–	0.369240 (0.78)–	0.360847 (1.00)–	0.379549 (-0.20)+	0.370910
	5	1165	0.798497 (-1.00)+	0.747202 (-0.55)+	0.787275 (-1.00)+	0.756993 (-0.98)+	0.836058 (-0.95)+	0.735514
	10	1698	7.393350 (-1.00)+	7.307299 (-1.00)+	7.160214 (-1.00)+	7.329445 (-1.00)+	7.431150 (-1.00)+	3.548237
	15	1902	178.646697 (-1.00)+	186.034426 (-1.00)+	180.014258 (-1.00)+	177.118064 (-1.00)+	189.502239 (-1.00)+	46.903837
MaF8	3	1759	0.464387 (-1.00)+	0.418595 (1.00)–	0.441880 (1.00)–	0.000000 (1.00)–	0.478298 (-1.00)+	0.458366
	5	1686	0.373270 (0.73)–	0.292557 (0.87)–	0.306309 (0.87)–	0.000000 (0.99)–	0.416144 (-1.00)+	0.398941
	10	1639	0.532874 (-0.25)=	0.349641 (1.00)–	0.407705 (1.00)–	0.000000 (1.00)–	0.726878 (-1.00)+	0.525740
	15	1601	3.266574 (-0.99)+	1.771260 (0.99)–	1.873965 (0.99)–	0.000000 (1.00)–	5.010604 (-1.00)+	2.376523
MaF9	3	2315	1.161602 (-1.00)+	1.161571 (-1.00)+	0.639281 (1.00)–	1.106581 (1.00)–	1.159400 (0.96)–	1.160561
	5	1756	0.957174 (0.24)=	0.476627 (1.00)–	0.797604 (1.00)–	0.703660 (1.00)–	1.019492 (-1.00)+	0.963087
	10	1332	1.189539 (0.92)–	0.587103 (1.00)–	0.869853 (1.00)–	0.280217 (1.00)–	1.600564 (-1.00)+	1.335795
	15	1128	6.534426 (1.00)–	2.318372 (1.00)–	3.738496 (1.00)–	1.552108 (1.00)–	11.062204 (-0.87)+	7.598667
MaF13	3	905	0.368660 (0.59)–	0.373696 (0.25)=	0.369581 (0.55)–	0.406742 (-0.03)=	0.394780 (-0.12)=	0.382698
	5	2089	0.180694 (0.43)–	0.168261 (-0.05)=	0.160573 (0.53)–	0.129572 (0.65)–	0.222126 (-0.44)+	0.210289
	10	2469	0.116582 (0.43)=	0.133353 (0.26)=	0.142977 (0.19)=	0.063590 (0.16)=	0.274674 (-0.55)+	0.242508
	15	2686	0.497255 (0.74)=	0.682328 (0.12)=	0.665291 (0.28)=	0.000577 (0.79)–	1.800052 (-0.80)+	1.497184
MaF14	3	1132	0.695884 (-0.71)+	0.238525 (0.73)–	0.000000 (0.98)–	0.203383 (0.83)–	0.439938 (0.25)=	0.586607
	5	1118	0.000000 (0.75)–	0.584860 (-0.47)+	0.000000 (0.60)=	0.000000 (0.51)=	1.347015 (-1.00)+	0.052228
	10	1046	0.000000 (1.00)–	0.000000 (1.00)–	0.000000 (1.00)–	0.246613 (0.99)–	14.454238 (-1.00)+	9.988891
	15	926	0.000000 (1.00)–	0.000000 (1.00)–	0.000000 (1.00)–	0.000000 (1.00)–	0.000000 (1.00)–	263.348208
	−/=/+ ⟶		**43**/14/23	**51**/11/18	**48**/13/19	**43**/12/25	**37**/08/35	of 80 probs.

Table VI: LHFiD/LHFiD-n versus other MaOEAs, on DTLZ, WFG and MaF problems: comparative summary on three performance metrics (HV, $d^{\parallel}$ and $d^{\perp}$), under three different settings. The entries below each MaOEA are formatted as $\{B/E/W\}$, where B, E, W mark the number of instances where LHFiD/LHFiD-n performs statistically better than, equivalent to, or worse than the MaOEA, respectively. Since multiGPO is not an RV-based MaOEA, there are (–)s marked against $d^{\parallel}$ and $d^{\perp}$.

Setting			Indicator metric	Base MaOEA	NSGA-III	θ-DEA	RPD-NSGA-II	MOEA/D-LWS		multiGPO	Details in
No.	t_{TM}	UEA									
I	on-the-fly (80 instances)	No	HV	LHFiD	43/14/23	51/11/18	48/13/19	43/12/25		37/08/35	Table V, Paper
				LHFiD-n	41/18/21	55/08/17	50/13/17	45/17/18		39/07/34	Table S1, S.D.
			$d^{\parallel}$	LHFiD	43/13/24	42/07/31	48/08/24	37/16/27		–	Table S2, S.D.
				LHFiD-n	44/15/21	40/08/32	50/08/22	44/09/27			Table S3, S.D.
			$d^{\perp}$	LHFiD	20/11/49	38/12/30	41/13/26	61/11/08		–	Table S4, S.D.
				LHFiD-n	13/12/55	33/11/36	40/15/25	61/11/08			Table S5, S.D.
					UEA1	UEA1	UEA1	UEA1	UEA2	UEA1	
II	500 (36 instances)	Yes	HV	LHFiD	22/07/07	19/07/10	18/05/13	18/06/12	10/06/20	17/09/10	Table S6, S.D.
			$d^{\parallel}$	LHFiD	23/08/05	21/02/13	22/04/10	07/02/27	14/09/13	–	Table S7, S.D.
			$d^{\perp}$	LHFiD	12/06/18	08/05/23	09/09/18	32/02/02	22/07/07	–	Table S8, S.D.

runs. This reported t_{TM} value is used for each of the 31 runs, of the other MaOEAs.

It is interesting to note that *LHFiD performs statistically better on the HV metric, in about* 56% *comparisons* (222 out of 400, split over five other MaOEAs); and *statistically better or equivalent in about* 70% (280 out of 400) *comparisons*.

The above trends bear more insights when read in conjunction with the performance of LHFiD-n, which, as per Table VI, performs statistically better in about 58% comparisons (230 out of 400); and statistically better or equivalent in about 73% (293 out of 400) comparisons. Here, two aspects deserve special attention:

- the fact that LHFiD-n reports significantly better performance over other MaOEAs, is a testimony to the efficacy of *hf*-dominance based environmental selection, regardless of how frequently Z^N is updated.
- the earlier claimed utility for intermittent Z^N update is also verified by the fact that LHFiD offers a faster convergence (lower t_{TM}) than LHFiD-n in about 94% instances (Table V, and Table S1 in the S.D.).

Interestingly, the above HV trends can be interpreted in terms of the $d^{\parallel}$-$d^{\perp}$ tradeoff. In that, wherever LHFiD/LHFiD-n offers better HV than another MaOEA, while being poorer in either $d^{\parallel}$ or $d^{\perp}$, the gain in the other metric, $d^{\perp}$ or $d^{\parallel}$, overcomes that loss. For example, as per Table VI, LHFiD performs worse than NSGA-III in $d^{\perp}$ in 49 out of 80 test instances. Yet, LHFiD manages to offer better HV in 43 instances, and this could be attributed to the fact that it offers better $d^{\parallel}$ in 43 instances. The above observation can be explained by how NSGA-III operates. Notably, NSGA-III replies on PD for convergence (whose limitations on MaOPs are known) and has a dedicated preference for solutions closest to the RVs. This explains why compared to LHFiD, NSGA-III fares better in $d^{\perp}$ and worse in $d^{\parallel}$.

Finally, it is worth re-iterating that LHFiD-n was introduced primarily to testify the efficacy of *hf*-dominance alone (while isolating the impact of intermittent Z^N update). Interestingly, the significantly superior performance of LHFiD-n over the other MaOEAs, evident above, bears this testimony. Hence, the experiments ahead (LHFiD with UEA, and LHFiD-i for inverted problems) involve the intermittent Z^N update, since it promises faster convergence.

2) Setting II: a priori set $t_{TM} = 500$; with UEA: It may be noted that once the solutions in the UEA are clustered, each cluster's representative needs to be identified. This can be done in different ways, including:

- UEA1 approach: by applying the MaOEA's underlying environmental selection.
- UEA2 approach: picking the solution, that is nearest to the underlying RV (as suggested in [44]).

Notably, the UEA2 approach is favorable for MOEA/D-LWS, since (i) it dedicatedly pursues LWS-guided convergence over the generations, even at the cost of diversity (departure from the RV), and (ii) selecting the solution closest to the RV, finally from the archive, helps improve diversity. However, the advantage of this approach will be largely restricted for other MaOEAs whose environmental selection strategies are devised to balance convergence and diversity (as opposed to pursuing either one over the generations, and compensating for the other, at the end). In this light: (i) the UEA1 approach is adopted to compare all the considered MOEAs, and (ii) additional results with the UEA2 approach are presented only for MOEA/D-LWS, to assess the maximum advantage possible.

The summary Table VI reveals that with UEA1: LHFiD performs statistically better on the HV metric, in about 52% comparisons; and statistically better or equivalent in about 71% comparisons. In particular, LHFiD is statistically better than or equivalent to MOEA/D-LWS in 24 out of 36 instances. However, when MOEA/D-LWS is assisted by UEA2, expectedly it's HV performance improves significantly. The net effect is that the comparative advantage offered by LHFiD diminishes, and it is found to be statistically better than or equivalent to MOEA/D-LWS in just 16 out of 36 instances.

Table VII: HV based comparison of LHFiD-i and other MaOEAs (all without UEA), on DTLZ^{-1}, WFG^{-1} and MaF problems, with t_{TM} determined *on-th-fly* for LHFiD-i: the entries are formatted as $med(r_b)$, followed by "–" or "=" or "+". Here, *med* and r_b represent the median and rank biserial effect size for HV over 31 runs, while –/=/+ inform that the underlying performance is statistically worse than, comparable to, or better than LHFiD-i, respectively.

Problem	M	t_{TM}	NSGA-III-i	θ-DEA-i	RPD-NSGA-II-i	MOEA/D-LWS-i	multiGPO	MOEA/AD	LHFiD-i
DTLZ1^{-1}	3	661	5.321032 (0.98)–	5.185823 (1.00)–	5.465229 (-0.30)=	5.514807 (-1.00)+	5.430422 (0.55)–	5.448132 (0.14)=	5.453909
	5	765	8.778678 (1.00)–	8.613118 (1.00)–	10.820458 (-0.77)+	10.805669 (-0.95)+	10.454348 (0.97)–	10.864578 (-0.94)+	10.711868
	10	1003	12.033475 (1.00)–	27.469490 (0.59)–	28.691564 (-0.52)+	22.008506 (1.00)–	28.576375 (-0.67)+	28.915690 (-0.91)+	28.301796
	15	1123	2.474729 (1.00)–	40.173968 (0.64)–	35.214119 (1.00)–	16.677023 (1.00)–	41.733507 (-0.95)+	41.494341 (-0.91)+	40.774528
DTLZ2^{-1}	3	733	6.737638 (-1.00)+	6.546082 (1.00)–	6.700307 (1.00)–	6.694632 (1.00)–	6.475378 (1.00)–	6.708475 (0.99)–	6.716809
	5	677	17.244680 (1.00)–	10.968503 (1.00)–	16.888077 (1.00)–	17.220836 (1.00)–	16.731706 (1.00)–	17.731302 (0.18)=	17.732970
	10	854	44.793118 (1.00)–	74.677363 (1.00)–	58.556445 (1.00)–	80.929615 (1.00)–	77.079043 (1.00)–	88.077195 (-0.98)+	87.533146
	15	1102	8.064056 (1.00)–	76.701183 (1.00)–	60.082022 (1.00)–	179.000019 (-0.28)=	148.055863 (1.00)–	135.625282 (1.00)–	177.008774
DTLZ3^{-1}	3	624	6.639462 (0.89)–	6.329778 (1.00)–	6.631357 (1.00)–	6.690538 (0.99)–	6.474463 (1.00)–	6.623381 (1.00)–	6.705548
	5	708	14.716498 (1.00)–	10.939546 (1.00)–	16.566652 (1.00)–	17.220905 (1.00)–	16.239973 (1.00)–	17.314315 (0.67)–	17.473273
	10	921	32.879445 (1.00)–	61.555774 (1.00)–	57.255817 (1.00)–	81.092136 (-0.42)=	72.195231 (1.00)–	84.903286 (-1.00)+	80.547151
	15	1027	5.297406 (1.00)–	68.291984 (1.00)–	56.350901 (1.00)–	179.251574 (-1.00)+	129.025343 (1.00)–	135.182552 (1.00)–	165.007927
DTLZ4^{-1}	3	1102	6.737063 (-1.00)+	6.578854 (0.87)–	4.274406 (0.98)–	6.698110 (0.87)–	6.563383 (0.87)–	6.710198 (0.85)–	6.719137
	5	914	17.260881 (1.00)–	14.637911 (1.00)–	14.770470 (1.00)–	17.220233 (1.00)–	16.878278 (1.00)–	17.762318 (-0.90)+	17.730603
	10	900	14.435002 (1.00)–	73.081088 (0.76)–	15.435077 (1.00)–	79.318075 (0.41)–	77.326657 (0.41)–	87.483619 (-0.99)+	86.274794
	15	732	5.190236 (1.00)–	72.235429 (0.81)–	25.512287 (0.98)–	176.872902 (-0.90)+	136.052126 (-0.37)=	136.112002 (0.21)=	123.238152
WFG1^{-1}	3	1014	4.613506 (1.00)–	4.684888 (1.00)–	4.825751 (0.95)–	4.894926 (-0.94)+	4.879445 (-0.98)+	4.692581 (0.87)–	4.862926
	5	1161	5.717205 (1.00)–	7.271357 (-0.75)+	7.681976 (-1.00)+	7.833274 (-1.00)+	8.259899 (-1.00)+	7.212628 (-0.93)+	6.687868
	10	2575	2.359665 (-1.00)+	17.304644 (-1.00)+	15.319700 (-1.00)+	13.368477 (-1.00)+	17.685255 (-1.00)+	12.253176 (-1.00)+	1.418229
	15	1597	1.385766 (0.46)–	22.528708 (-1.00)+	14.447235 (-1.00)+	14.848118 (-1.00)+	25.043095 (-1.00)+	6.864025 (-1.00)+	1.464636
WFG2^{-1}	3	1004	6.120650 (0.89)–	6.153553 (-0.89)+	6.098785 (1.00)–	6.148926 (-0.68)+	6.066885 (0.99)–	6.160325 (-0.95)+	6.140188
	5	1082	8.952006 (1.00)–	10.722476 (-0.69)+	9.611421 (0.90)–	11.194892 (-1.00)+	11.129135 (-1.00)+	11.183493 (-1.00)+	10.170628
	10	1537	3.088956 (1.00)–	9.141235 (0.99)–	4.989753 (1.00)–	22.658846 (-0.99)+	24.536144 (-1.00)+	24.444951 (-1.00)+	14.670346
	15	1374	0.978073 (1.00)–	8.624583 (-1.00)+	2.999568 (-0.07)=	25.970925 (-0.83)+	33.129479 (-1.00)+	36.938056 (-1.00)+	3.311394
WFG4^{-1}	3	793	6.686995 (1.00)–	6.522317 (1.00)–	6.685487 (1.00)–	6.691651 (1.00)–	6.552838 (1.00)–	6.704290 (1.00)–	6.716685
	5	731	13.005645 (1.00)–	14.546994 (1.00)–	15.775947 (1.00)–	17.263048 (1.00)–	16.437628 (1.00)–	17.472220 (0.90)–	17.588024
	10	797	29.282349 (1.00)–	30.471417 (1.00)–	27.239884 (1.00)–	81.792435 (-1.00)+	78.642683 (-1.00)+	81.375526 (-1.00)+	66.963585
	15	937	21.862387 (1.00)–	19.424221 (1.00)–	18.986000 (1.00)–	189.225425 (-1.00)+	162.467462 (-1.00)+	129.652503 (-0.92)+	119.310140
WFG5^{-1}	3	737	6.677232 (0.92)–	6.419721 (1.00)–	6.660999 (1.00)–	6.673056 (0.97)–	6.524039 (1.00)–	6.692463 (-0.52)+	6.688513
	5	682	15.091225 (1.00)–	14.561953 (1.00)–	16.795677 (1.00)–	17.136072 (1.00)–	16.293712 (1.00)–	17.654624 (-1.00)+	17.506101
	10	737	30.225807 (1.00)–	31.819932 (1.00)–	30.281606 (1.00)–	80.419168 (0.93)–	78.942710 (0.99)–	84.922020 (-0.51)=	83.842360
	15	784	20.512023 (1.00)–	19.864822 (1.00)–	17.181476 (1.00)–	184.937006 (-1.00)+	161.147739 (-0.57)=	132.906965 (1.00)–	155.307854
WFG6^{-1}	3	755	6.689858 (1.00)–	6.537745 (1.00)–	6.687755 (1.00)–	6.693246 (1.00)–	6.559841 (1.00)–	6.700972 (1.00)–	6.714569
	5	675	16.057535 (1.00)–	10.223359 (1.00)–	16.897332 (1.00)–	17.198322 (1.00)–	16.410546 (1.00)–	17.655623 (0.65)–	17.696316
	10	725	34.319902 (1.00)–	37.296790 (1.00)–	33.923977 (1.00)–	81.240208 (0.99)–	78.496642 (1.00)–	85.393086 (-0.11)=	85.199281
	15	746	20.438187 (1.00)–	20.616824 (1.00)–	16.963551 (1.00)–	185.808124 (-1.00)+	151.827754 (1.00)–	133.650205 (1.00)–	177.842403
WFG7^{-1}	3	717	6.628204 (1.00)–	6.511455 (1.00)–	6.641952 (1.00)–	6.690431 (1.00)–	6.538538 (1.00)–	6.698498 (1.00)–	6.712521
	5	706	12.460419 (1.00)–	15.303710 (1.00)–	13.535354 (1.00)–	17.229634 (1.00)–	16.420354 (1.00)–	17.554375 (-0.18)=	17.540271
	10	760	28.071851 (1.00)–	29.199420 (1.00)–	24.100871 (1.00)–	81.033687 (-0.28)=	78.676529 (0.75)–	81.312820 (-0.51)=	80.367310
	15	841	19.561503 (1.00)–	17.971074 (1.00)–	15.077915 (1.00)–	190.303027 (-1.00)+	153.651938 (-0.54)=	133.148905 (0.99)–	150.370896
WFG8^{-1}	3	692	6.697174 (0.14)–	6.559608 (1.00)–	6.682289 (0.55)–	6.518453 (1.00)–	6.572401 (1.00)–	6.705086 (-0.05)=	6.707523
	5	675	16.900910 (0.99)–	15.118666 (1.00)–	16.924690 (0.99)–	16.561326 (1.00)–	16.698889 (1.00)–	17.729539 (-1.00)+	17.269182
	10	709	50.232813 (1.00)–	55.999719 (1.00)–	52.672862 (1.00)–	82.090024 (-1.00)+	81.768372 (-1.00)+	87.065442 (-1.00)+	79.297774
	15	771	36.351548 (1.00)–	47.374427 (1.00)–	38.446933 (1.00)–	186.790778 (-0.67)+	162.018922 (0.03)=	138.919209 (0.28)–	183.254371
WFG9^{-1}	3	671	6.614398 (0.99)–	6.516501 (1.00)–	6.543756 (1.00)–	6.660548 (0.78)–	6.548784 (1.00)–	6.671372 (0.60)–	6.681920
	5	723	15.373047 (1.00)–	14.923000 (1.00)–	15.062583 (1.00)–	17.106736 (0.69)–	16.561851 (0.98)–	17.454025 (0.03)=	17.497049
	10	757	30.770525 (1.00)–	33.574203 (1.00)–	25.203809 (1.00)–	80.902649 (-1.00)+	81.926875 (-1.00)+	85.398248 (-1.00)+	68.564610
	15	778	27.270281 (1.00)–	27.235223 (1.00)–	13.610507 (1.00)–	191.298702 (-1.00)+	167.198142 (-0.23)=	143.631805 (0.98)–	165.421021
MaF1	3	1155	5.390622 (0.89)–	5.215092 (1.00)–	5.328821 (1.00)–	4.513541 (1.00)–	0.252315 (1.00)–	6.973434 (-1.00)+	5.436903
	5	2470	9.397905 (1.00)–	8.267729 (1.00)–	6.336189 (1.00)–	5.961321 (1.00)–	0.055023 (1.00)–	19.287283 (-1.00)+	10.406663
	10	3735	14.113148 (-1.00)+	9.594485 (-0.67)+	0.003216 (1.00)–	0.002968 (1.00)–	0.004749 (1.00)–	138.686618 (-1.00)+	8.099483
	15	3729	0.070184 (0.88)–	0.000000 (1.00)–	0.000000 (1.00)–	16.002663 (-0.75)+	0.004137 (1.00)–	665.759826 (-1.00)+	0.756027
MaF4	3	2921	6.594352 (-0.58)=	6.476611 (-0.48)=	4.160507 (-0.40)=	0.000000 (1.00)–	1.235924 (0.59)=	6.713035 (-0.89)+	6.440258
	5	1573	13.889034 (-0.79)+	0.000000 (1.00)=	0.000000 (1.00)=	0.000000 (1.00)=	2.073794 (-1.00)+	17.346694 (-1.00)+	0.000000
	10	3475	0.000000 (0.68)=	0.000000 (1.00)–	0.000000 (1.00)–	0.000000 (1.00)–	15.225913 (0.53)=	83.285239 (-1.00)+	31.395244
	15	4133	0.000000 (1.00)=	0.000000 (1.00)=	0.000000 (1.00)=	0.000000 (1.00)=	309.068504 (-1.00)+	130.347952 (-1.00)+	0.000000
	–/=/+ ⟶		**48**/03/05	**46**/03/07	**46**/05/05	**29**/05/22	**34**/07/15	18/09/**29**	of 56 probs.

B. Results on Inverted Problems

The results in the Table VII, reveal that LHFiD-i performs statistically better on the HV metric, in about 66% comparisons (221 out of 336, split over 6 MaOEAs); and statistically better or equivalent in about 75% comparisons (253 out of 336). In that, LHFiD-i performs better than all MaOEAs, except MOEA/AD, where LHFiD-i performs statistically better in only 18 (out of 56) instances but statistically worse in 29. This could be attributed to the MOEA/AD's co-evolution of two populations (with respect to each of Z^I and Z^N), that rely on two different scalarization functions, individually emphasizing on convergence and diversity [50]. However, the MOEA/AD's advantages come at the cost of additional run-time, since in each generation: (i) the offspring are associated with two sets of RVs and paired together (with $\mathcal{O}(N^2 \log N)$ complexity), and (ii) two populations need to be maintained simultaneously.

VII. Car Side-impact Problem

This section aims to demonstrate on a real-world problem, how *hf*-dominance based LHFiD, can incorporate DM's preferences for complete PF approximation. Towards it, a three-objective, car side-impact problem [8] has been considered, and LHFiD's performance is compared for two different settings, as below:

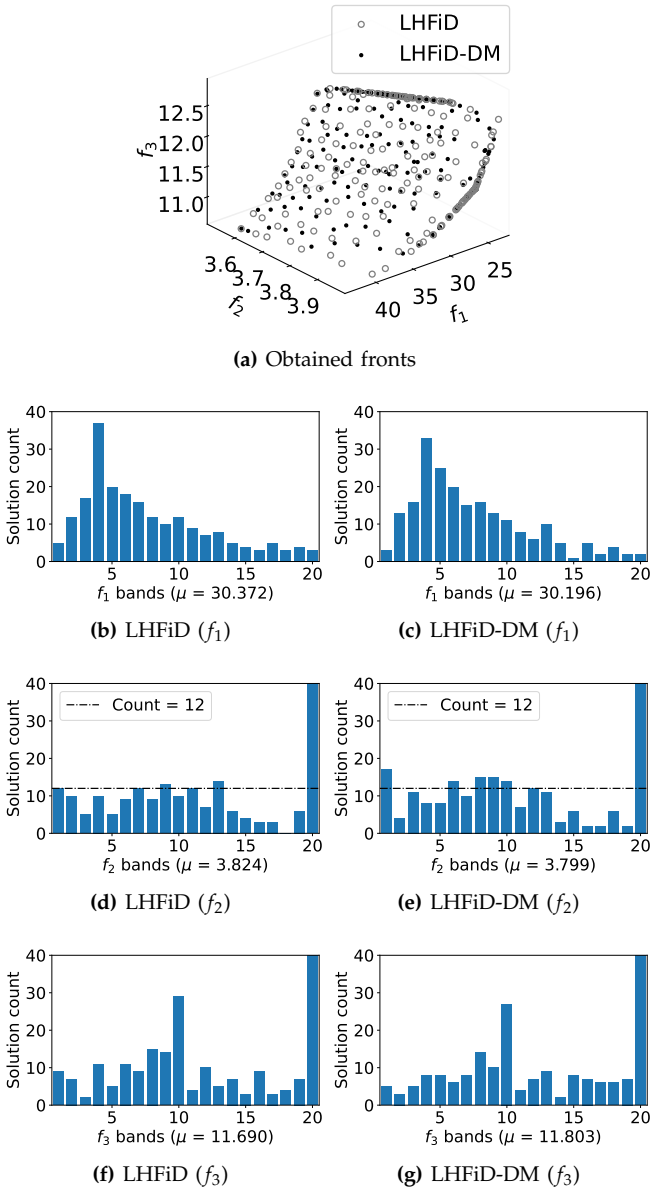
- as per LHFiD's recommendation in the absence of DM's preferences, $p_m = 1/w_m$ has been used.
- the DM's preferences are emulated by $p \equiv \{0.1, 0.8, 0.1\}$. For ease of reference, the corresponding performance is denoted by LHFiD-DM.

Notably, 31 runs for each, LHFiD and LHFiD-DM, are done with a population size of $N = 210$, where t_{TM} is determined on-the-fly. Their PF approximations corresponding to their respective median runs, are shown in Figure 3a. While, it may be difficult to distinguish the two PFs visually, it is worth noting that the difference in their respective HV measures is statistically significant[3].

Fore more insights on how the distribution of solutions across the PF are impact by DM's preferences, the objective-wise density plots for LHFiD and LHFiD-DM are shown in Figures 3b–3g. To obtain these density plots: (i) lower and upper bounds of each objective were computed from the combined fronts of LHFiD and LHFiD-DM, (ii) 20 interval bands were created separately for each objective, within their respective bounds (marked on horizontal axis), and (iii) the solutions lying within each band were counted (marked on vertical axis).

Given the DM's preferences as $\{0.1, 0.8, 0.1\}$, it is fair to expect that LHFiD-DM may offer lower values of f_2, compared to LHFiD (where the relative preferences for each objective, vary from one RV to another). Interestingly, a comparison between Figures 3d and 3e, reveals that the density of solutions in the lower-to-medium range of f_2 is higher, when DM's preferences are used. This shift towards lower f_2 values is further endorsed by LHFiD-DM's lower mean value (μ) of f_2, than LHFiD. This improvement in the mean value accounts for about 6% of the total range of f_2.

The authors, believe that the above illustration has practical bearings. In that, the scope to incorporate the DM's preferences within the dominance principle, reinforce the confidence in the DM that his/her preferences have been accounted for, over the entire PF.



(a) Obtained fronts

(b) LHFiD (f_1) **(c)** LHFiD-DM (f_1)

(d) LHFiD (f_2) **(e)** LHFiD-DM (f_2)

(f) LHFiD (f_3) **(g)** LHFiD-DM (f_3)

Figure 3: Final obtained fronts and objective-wise density plots for LHFiD and LHFiD-DM on car side-impact problem.

VIII. Conclusions

The core contribution of this paper lies in its proposition of a novel *hf*-dominance principle, and its implementation in a reference vector based framework, leading up to a computationally efficient MaOEA, namely, LHFiD. The *hf*-dominance principle is the first ever to unify the key human decision-making elements, towards an efficient environmental selection. It helps to discriminate between two solutions, by explicit and simultaneous accounting of the number of better objectives; the degree of betterment; and the objectives' relative preferences, without requiring tuning of any parameter. This paper has also hired from the earlier developed proof-of-concept on stabilization tracking of an MOEA's population, to equip LHFiD with the capability to determine, on-the-fly, the timing for - intermittent nadir point update, and final termination. The efficacy of LHFiD has been demonstrated on a wide range of problems, through its comparison with popular and/or recently proposed MaOEAs that cover different categories of dominance principles and also different MaOEA architectures. The efficacy of *hf*-dominance as a selection criterion on problems with inverted PFs, has also been established through LHFiD-i (a minor adaptation of LHFiD), and its comparison with other MaOEAs. Recognizing that such problems bridge an important research gap in the literature, the authors' next focus will be on periodic

[3]The median HV for LHFiD and LHFiD-DM are 0.641544 and 0.640514, respectively, with an effect size of 0.58. However, in the current context, which one is better is irrelevant, but whether or not altering the p_ms could lead to statistically significant difference, is intriguing.

adjustment of weight vectors to fit the *PF* shapes (regular or irregular), or developing a unified approach where multiple populations around RVs emanating from both ideal and nadir points may co-evolve. The authors hope that the novelty of *hf*-dominance, and the simplicity and computational efficiency of LHFiD will find wider applications, particularly on real-world problems where decision maker's preferences may be available.

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