

# Visualization-based Multi-Criterion Decision Making with NIMBUS Method Using PaletteViz

COIN Report Number 2021015

Kalyanmoy Deb

Department of Electrical and Computer Engineering  
Michigan State University  
East Lansing, Michigan, MI 48864, USA  
Email: kdeb@egr.msu.edu

AKM Khaled A. Talukder

Department of Computer Science and Engineering  
Michigan State University  
East Lansing, Michigan, MI 48864, USA  
Email: talukde1@msu.edu

**Abstract**—After many efficient evolutionary many-objective optimization (EMaO) algorithms have been proposed and demonstrated to find multiple well-distributed near Pareto-optimal solutions, one main emphasis now is to combine them with suitable multi-criterion decision-making (MCDM) approaches to choose a single preferred solution. For integrating MCDM approaches with EMaO algorithms, there have been growing interests in implementing MCDM concepts algorithmically within EMaOs, but in this paper, we propose a visualization-based MCDM-EMaO integration implementing the well-known NIMBUS method using a recently proposed PaletteViz visualization technique and demonstrate its working by applying it to two test problems and an engineering design problem. The detailed results show the usefulness of the PaletteViz procedure in assisting decision-makers to implement MCDM approaches with a better understanding of trade-off solutions.

**Index Terms**—Visualization, PaletteViz, NIMBUS, Multi-criterion decision making, Interactive optimization.

## I. INTRODUCTION

For the past two and half decades, most studies in evolutionary multi- and many-objective optimization (EMO and EMaO) have mostly focused on designing and evaluating algorithms for their ability to converge near to the Pareto-optimal (PO) front by distributing their non-dominated (ND) population members. However, finding a set of well-distributed ND solutions on or near the PO front is one aspect of many-objective problem solving; suitably choosing a single preferred solution from the obtained PO set is also an important task. The latter task must involve decision-makers (DMs) and a (desired) preference information. Fortunately, a parallel field – multi-criterion decision-making (MCDM) – focuses on the decision-making aspects by using DM’s preference information. Since the early seventies, MCDM has proposed many different approaches for this purpose. The MCDM literature proposes three broad ways of integrating optimization with decision-making tasks:

- 1) **A priori approach**, in which, a decision-making task is performed first and then multiple objectives are converted into a single-objective problem, which can then be solved by a single-objective optimization algorithm [1]. This

approach entirely depends on the *scalarization* method to be used and can be argued to be an inconvenient approach simply because of the lack of adequate information a priori to provide an informed preference information to scalarize the objectives functions [1], [2].

- 2) **A posteriori approach**, in which optimization is performed first to find a set of well-distributed near Pareto-optimal (PO) solutions, followed by a decision-making approach to choose a single solution [3]. This allows a more informed decision-making, as the knowledge of a number of solutions (the range of objective values, their variation across the set, etc.) is available before any preference information is sought. However, for many-objective optimization problems (MaOP), a posteriori approach requires a large number of PO solutions to be processed, thereby making the approach time consuming in practice for decision-making purposes.
- 3) **Interactive approach**, in which decision-making and optimization are performed in an integrated manner, so that preference information can be provided continuously based on the evolution of PO solutions. The search is directed towards the preferred PO solutions, rather than finding a diverse set of PO solutions [4], [5]. While the computational efforts may be less, this method demands recurrent user intervention. Thus, given the user availability, the interactive approach is well suited.

The approach followed in this paper can be categorized as either the second or the third method (or a mix of both). In Section II, we briefly mention a few existing EMO-MCDM integration studies. To overcome the issue of analyzing a large set of trade-off solutions to arrive at a single solution, we employ a method of *visual aid*, which is based on the recently proposed PaletteViz [6] (described in Section III). PaletteViz can help us visualizing solutions according to different functionalities of MaOP. Such as highlighting large trade-off solutions, discriminating boundary versus core solutions, emphasizing near constraint solutions over inactive solutions, identifying robust solutions rather than sensitive solutions,

etc. Thereafter, we use a well-known MCDM approach – the NIMBUS method [7] – to demonstrate the working of the combined EMaO-MCDM method through PaletteViz in Section IV. While the NIMBUS method comes with a few visualization tools, such as parallel coordinate plots and bar charts, however, they do not provide a clear trade-off information to help choose preferred solutions. Our results on three MaOPs on test and engineering problems demonstrate the usefulness of PaletteViz in Section V. Finally, conclusions are drawn in Section VI.

## II. EXISTING EMO-MCDM APPROACHES

We restrict our discussions only to a posteriori and interactive approaches, as an EMO method is usually not required for solving a scalarized single-objective optimization problem arising from a priori MCDM tasks. This is mainly because a priori approaches give rise to a single final preferred solution, which may be found by a single application of a point-based optimization method [8], [9].

Interestingly, MCDM approaches can be applied as a post-optimality task to choose a single preferred solution from a set of optimized ND solutions. Unfortunately, only a few studies exist to implement such a posteriori approach, taking advantage of both EMaO and MCDM concepts. Most of the existing studies have used a specific MCDM concept, but instead of choosing a single solution, a biased set of ND solutions were computed near the preferred part of the PO front. In the reference point based NSGA-II [10] or R-NSGA-III [11], only points *close* to the supplied reference or aspiration points were found. These methods take a smaller computational time than their original counterparts and can also handle multiple reference points simultaneously, a matter which the original reference point based MCDM method can not achieve. Deb and Kumar [12] used the reference direction based MCDM approach and found only those projected PO points that are closest to a supplied reference direction in the objective space, denoting a trade-off line. Another study [13] implemented the light-beam approach from the MCDM literature [14] with multiple simultaneous light-beams to discover different preferred parts of the PO front. There exist a few recent studies in implementing interactive approaches using other MCDM approaches [4], [5], [15], [16].

## III. PALETTEVIZ VISUALIZATION METHOD

PaletteViz [6] is a special type of visualization method where  $M$ -dimensional data points are mapped onto a two-dimensional space. The method first computes the depth-contour of the point cloud  $\mathcal{M}$ ,<sup>1</sup> where the depth contours are defined with respect to a non-convex hull shape (i.e.  $\alpha$ -shape) [17] of  $\mathcal{M}$ . This way, PaletteViz can capture the topological and neighborhood relationship of the data points in terms of inward-outward and spatial arrangement of  $\mathcal{M}$

<sup>1</sup>A point cloud  $\mathcal{M}$  is a set of data points in  $M$ -dimensional space in a simplicial homology parlance. In our case,  $\mathcal{M}$  is the set of all PO (or ND) points in the objective space found by solving a many-objective optimization (MaOP) problem.

with respect to the simplicial boundary [18]. The computed depth contours induce an equivalence class of  $\mathcal{M}$  and we separate them according to the equivalence class. Next, the points that belong to their corresponding equivalence class are mapped onto a two-dimensional space using a standard method like Star-coordinate plot (SC-plot). In our case, we apply SC-plot and each SC-plot is arranged layer by layer in a three-dimensional space – according to their distance (i.e. depth contour) from the core of  $\mathcal{M}$  to its boundary. The points on the outer-most boundary (points with smallest depth contour) of  $\mathcal{M}$  are placed on the top of PaletteViz and the points on the next boundary (points with the next smallest depth contour) are placed below the previous layer, and so on. Therefore, the bottom-most layer contains the points from the *core* of  $\mathcal{M}$ . The PaletteViz method is capable of providing a number of functional attributes of a trade-off solution (say,  $\mathbf{f}$ ) in  $M$ -dimensional space, but projected on a two-dimensional space:

- 1) If a point lies near the core of  $\mathcal{M}$ , the closeness property will ensure the DM that there exists sufficient supporting points around it. On the other hand, for a near-boundary point in  $\mathcal{M}$ , there do not exist many neighboring points trading-off the current point.
- 2) Closeness of the point  $\mathbf{f}$  near one or more *constraint* boundaries. DMs may be interested in choosing a solution which is sufficiently away from any constraint boundary, if possible.
- 3) A point  $\mathbf{f}$  having a large *trade-off* value (defined in Equation 4) represented by dark red color and marker size proportional to the trade-off value. A large trade-off point might be preferable to a DM.
- 4) A point  $\mathbf{f}$  exhibiting *robustness* and *reliability* against failures due to uncertainties in variables and problem parameters.

## IV. MULTI-CRITERION DECISION MAKING APPROACHES

Besides generic scalarization methods, such as weighted-sum, epsilon-constraint, normal constraint, achievement scalarization function (ASF), cone domination approaches [1], there exist specific MCDM step-by-step procedures for combining many-objective optimization and decision-making together. These procedures suggest a different scalarization method at every iteration depending on the progress and properties of the current PO solution. The scalarized problem is then solved using a single-objective optimization algorithm and the new solution is analyzed by the decision-makers to propose a new scalarized problem. We discuss one such MCDM procedure here called NIMBUS.

### A. PaletteViz-NIMBUS Method

The NIMBUS method [7] starts with a neutral PO solution (can be obtained by solving a weighted-sum approach with equal weight for each objective, for example) or a user-supplied solution  $\mathbf{x}^{(0)}$ . Then, new and improved preferred solutions ( $\mathbf{x}^{(t)}$ ,  $t = 1, 2, \dots$ ) are found by solving the following problem (slightly modified from [7]) at every iteration  $t$ :

$$\begin{aligned}
& \text{Minimize} \quad \max \left[ \max_{i \in I^<} \frac{f_i(\mathbf{x}) - z_i^*}{z_i^{\text{nad}} - z_i^{**}}, \max_{j \in I^{\leq}} \frac{f_j(\mathbf{x}) - \hat{z}_j}{z_j^{\text{nad}} - z_j^{**}} \right] + \\
& \quad \rho \sum_{i=1}^M \frac{f_i(\mathbf{x})}{z_i^{\text{nad}} - z_i^{**}}, \quad (1) \\
& \text{Subject to} \quad f_i(\mathbf{x}) \leq f_i(\mathbf{x}^{(t)}), \quad \text{for } i \in I^< \cup I^{\leq} \cup I^=, \\
& \quad f_i(\mathbf{x}) \geq \hat{z}_i, \quad \text{for } i \in I^{\leq}, \\
& \quad f_i(\mathbf{x}) \leq \epsilon_i, \quad \text{for } i \in I^{\geq}, \\
& \quad \mathbf{x} \in \mathcal{S}, \quad (2)
\end{aligned}$$

where  $z_i^*$ ,  $z_i^{**}$  and  $z_i^{\text{nad}}$  are the ideal, utopian and nadir point value of the  $i$ -th objective, respectively,  $\mathcal{S}$  is the feasible variable space, and  $\rho$  is a small number (usually, 0.001). Different indices and  $\epsilon_i$  values are described below:

- The index set  $I^<$  indicates the objectives ( $i$ ) whose values are desired to be improved (that is, decreased for minimization objectives) from the current level ( $f_i(\mathbf{x}^{(t)})$ ).
- The index set  $I^{\leq}$  indicates the objectives ( $i$ ) whose values are desired to be improved till some desired *aspiration* level ( $\hat{z}_i$ ), that is,  $f_i(\mathbf{x}) > \hat{z}_i$ .
- The index set  $I^=$  indicates the objectives ( $i$ ) whose values are acceptable in the current solution  $\mathbf{x}^{(t)}$ .
- The index set  $I^{\geq}$  indicates the objectives ( $i$ ) whose values may be impaired (i.e., increased) till some *upper* bound ( $\epsilon_i$ ), that is,  $f_i(\mathbf{x}) < \epsilon_i$ .
- The index set  $I^{<>}$  indicates the objectives ( $i$ ) whose values are temporarily allowed to change freely.

We integrate the above iterative NIMBUS method with our PaletteViz visualization approach in the following step-by-step procedure. Notice that instead of optimizing problem 2 in each iteration, we choose the best solution from the EMO-obtained ND set by solving problem (2).

**Step 1:** Apply an EMO algorithm and present all  $N$  non-dominated solutions ( $\mathcal{F} = \cup_{k=1}^N \mathbf{x}^{(k)}$ ) (the ND set) in a PaletteViz plot for decision-making using the NIMBUS procedure. Identify ideal  $\mathbf{z}^*$  and nadir point  $\mathbf{z}^{\text{nad}}$  by choosing the minimum and maximum objective values from the ND set  $\mathcal{F}$ , as follows:  $z_i^* = \min_{\mathbf{x} \in \mathcal{F}} f_i(\mathbf{x})$ , and  $z_i^{\text{nad}} = \max_{\mathbf{x} \in \mathcal{F}} f_i(\mathbf{x})$ . Then, compute utopian vector as  $z_i^{**} = z_i^* - \delta_i$ , where  $\delta_i$  is a small positive number (0.001, used here). Make a PaletteViz plot with the ND set and investigate for any interesting points to include in an archive  $A$ .

**Step 2:** Choose a specific starting ND point ( $\mathbf{x}^{(0)}$ ) and its objective vector  $\mathbf{f}^{(0)}$ , either by a visual decision-making (as described elsewhere [6]), or by using a standard scalarizing function method with an equal importance to each objective. By using PaletteViz, locate the highest trade-off point ( $\mathbf{f}_p^{(0)}$ ) near it. Set a counter  $t = 0$ . If the DM likes the solution, then it is included in  $A$ .

**Step 3:** Ask the DM to classify all objective functions at  $\mathbf{x}^{(0)}$  and specify aspiration levels ( $\hat{z}_i$ ) and upper bounds ( $\epsilon_i$ ) for respective objective functions in the sets  $I^{\leq}$  and  $I^{\geq}$ , respectively.

**Step 4:** Select the best solution  $\mathbf{x}^{(t+1)}$  from the ND set by

restricting the min and max operations in problem (2) within the ND set  $\mathbf{x} \in \mathcal{F}$ . By PaletteViz, locate highest trade-off point ( $\mathbf{f}^{(t+1)}$ ) near it. Set  $A = A \cup \mathbf{f}^{(t+1)}$ .

**Step 5:** Ask the decision-maker to select the maximum number of alternative solutions (say,  $K$ , between one and four) to be chosen uniformly along the line joining the two objective vectors  $\mathbf{f}^{(t)}$  and  $\mathbf{f}^{(t+1)}$  in the  $M$ -dimensional space and finding nearest points in the PaletteViz plot (see Section IV-B for a detail description). Present new solutions to the DM.

**Step 6:** If the DM wants to save one or more of the new solutions to the archive  $A$ , add it (them) to  $A$  for a further analysis.

**Step 7:** Ask the decision-maker to choose the most preferred solution among the new and/or the intermediate solutions or the solutions in  $A$ . Denote it as the current solution  $\mathbf{x}^{(t+1)}$  with objective vector  $\mathbf{f}_p^{(t+1)}$ . If the decision maker wants to continue, set  $t \leftarrow t + 1$  and go to Step 3. Otherwise, report the archive  $A$  as the chosen set of preferred solutions.

Since the choice of one or more new preferred solutions at every iteration becomes a selection from the already-obtained ND set  $\mathcal{F}$ , the proposed a-posteriori approach makes a computationally faster and more visually intuitive decision-making process than the original NIMBUS procedure.

### B. Finding Intermediate Solutions in Step 5

Consider Figure 1, in which a part of the ND set in  $M > 2$  objective space is shown as circles. Two chosen

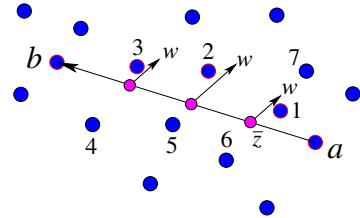


Fig. 1: Procedure for choosing intermediate points.

points  $a = \mathbf{f}(\mathbf{x}^{(t)})$  and  $b = \mathbf{f}(\mathbf{x}^{(t+1)})$  are also shown. For three ( $K = 3$ ) uniformly distributed intermediate points along line  $ab$ , the respective ND points are identified by using the minimum achievement scalarizing function (ASF) [1] value with equal weights. For an intermediate point  $\bar{\mathbf{z}}$ , the chosen point is obtained as follows:

$$\mathbf{x}_{\bar{\mathbf{z}}} = \operatorname{argmin}_{\mathbf{x} \in \mathcal{F}} \max_{i=1}^M \left( \frac{f_i(\mathbf{x}) - \bar{z}_i}{z_i^{\text{nad}} - z_i^*} \right). \quad (3)$$

In the figure, the chosen objective points are marked as 1, 2 and 3 for an equal weight vector ( $\mathbf{w}$ ) shown. Alternatively, the normalized Euclidean distance can also be used to choose the nearest ND points for the intermediate points.

## V. RESULTS

We consider two test problems and one engineering design problem to illustrate the working of the proposed approach.

#### A. Four-objective DEBMDK-Knee Problem

First, we choose a 4-objective problem [19] having knee-like points at the core of the Pareto front. Figure 2a shows 2,000 ND points obtained using NSGA-III [20] with standard parameter settings. Two red-colored points on the bottom-most layer (representing the core of the front) are clearly visible on the PaletteViz plot. The ideal and nadir points of the ND points are as follows:

$$\mathbf{z}^* = [0, 0, 0, 0], \quad \mathbf{z}^{\text{nad}} = [9.5, 9.5, 9.5, 9.5].$$

Let us say, our DM is interested in finding a large trade-off solution at the core of the PO front with equal importance to all four objectives. A search for such a point with  $\mathbf{w}^d = [1, 1, 1, 1]/\sqrt{4}$  with ASF minimization approach with ideal point and equal weight vector finds the following target vector from the ND set:  $\mathbf{z}^d = [1.5068, 1.5068, 1.8081, 1.8081]$  (Figure 2a). To find a large trade-off point  $\mathbf{x}$ , near this ASF solution, we use the following definition of trade-off value [21]:

$$\mathcal{T}(\mathbf{x}) = - \frac{\min_{\mathbf{z} \in B(\mathbf{x})} \min(f(\mathbf{x}) - f(\mathbf{z}), 0)}{\max_{\mathbf{z} \in B(\mathbf{x})} \max(f(\mathbf{x}) - f(\mathbf{z}), 0)}. \quad (4)$$

It computes the minimum loss versus maximum gain for a ND point ( $\mathbf{x}$ ) in moving to any neighboring point within set  $B(\mathbf{x})$ . Here, we define the nearest  $P$  ( $|B(\mathbf{x})|=20$ ) points as the neighborhood size. Larger the trade-off value, the more preferred is the solution. With this definition, we are able to find the following target solution ( $\mathbf{z}^T$ ) which has an improved trade-off of 0.3241 from 0.0795 for  $\mathbf{z}^d$ :  $\mathbf{z}^T = [1.0470, 1.0470, 1.5705, 2.0940]$ .

To begin the NIMBUS procedure, let us say, we start our PaletteViz-NIMBUS procedure with a solution that has a large weight for the first two objectives ( $w_i = 0.5$  for  $i = 1, 2$  and  $w_i = 0.01$  for  $i \geq 3$ ) resulting in:  $\mathbf{f}_p^{(0)} = \mathbf{f}^{(0)} = [5.1854, 5.1854, 0, 0]$ . Set  $A = [f_p^{(0)}]$ . Using the DM principle set out, in Step 3, the DM provides the following classification of objectives with  $\hat{z}_1 = \hat{z}_2 = 1.0$  and  $\epsilon_3 = \epsilon_4 = 2.0$ :

$$I^< = \emptyset, \quad I^{\leq} = \{1, 2\}, \quad I^= = \emptyset, \quad I^{\geq} = \{3, 4\}, \quad I^{>} = \emptyset,$$

In Step 4, the obtained solution from ND set is  $\mathbf{f}^{(1)} = [1.1140, 1.1140, 1.9496, 1.9496]$ , with a trade-off value  $\mathcal{T} = 0.0906$ . However, the PaletteViz plot in Figure 3 helps to identify that there is a better trade-off solution ( $\mathcal{T} = 0.3241$ ) close to it. The DM chooses this solution:  $\mathbf{f}_p^{(1)} = \mathbf{f}^{(1)} = [1.0470, 1.0470, 1.5705, 2.0940]$ . This is the advantage of using PaletteViz in the NIMBUS process, as it allows the DM to corroborate the analytically-derived solution to a better neighboring trade-off solution through a visual aid. In Step 5, the DM plans to find two ( $K = 2$ ) intermediate points between  $\mathbf{f}^{(0)}$  and  $\mathbf{f}^{(1)}$ , marked as  $\mathbf{z}^{(1)}$  and  $\mathbf{z}^{(2)}$ . These two points are visualized using PaletteViz in Figure 2c. The DM observes that  $\mathbf{z}^{(1)}$  has no large trade-off solution (red colored point) near it, but  $\mathbf{z}^{(2)}$  has such a point. Thus, solution  $\mathbf{z}^{(1)}$  is retained and  $\mathbf{z}^{(2)}$  is replaced with its nearest large trade-off solution

obtained from the PaletteViz plot:

$$\begin{aligned} \bar{\mathbf{z}}^{(1)} &= [4.9216, 4.9216, 0.4922, 0.4922], \quad \mathcal{T}(\bar{\mathbf{z}}^{(1)}) = 0.1725, \\ \bar{\mathbf{z}}^{(2)} &= [2.384, 2.384, 3.9734, 0], \quad \mathcal{T}(\bar{\mathbf{z}}^{(2)}) = 0.3081. \end{aligned}$$

In Step 6, we update  $A = [f_p^{(0)}, f_p^{(1)}, \bar{\mathbf{z}}^{(1)}, \bar{\mathbf{z}}^{(2)}]$ . Then, in Step 7, we choose the best solution using DM's set-out principle – closest point to  $\mathbf{z}^d$ :  $\mathbf{f}_p^{(1)} = [1.0470, 1.0470, 1.5705, 2.0940]$  with  $\mathcal{T}(\mathbf{f}_p^{(1)}) = 0.3241$ . The archive is replaced with this solution ( $A = [f_p^{(1)}]$ ) alone. Note that this solution is identical to  $\mathbf{z}^T$ . The decision-making process is then terminated. Note how PaletteViz allows the DM to pick better trade-off solutions in the neighborhood of NIMBUS-obtained solutions. The use of original NIMBUS method without any visualization procedure would not have resulted in discovery of  $\mathbf{f}_p^{(1)}$ .

#### B. Five-objective DTLZ2 Problem

To further demonstrate the working of PaletteViz-NIMBUS, next, we choose the 5-objective DTLZ2 problem. In Step 1, we employ NSGA-III to obtain 1,000 ND points as shown in Figure 4a.

The respective ideal and nadir points are computed from the ND set as

$$\mathbf{z}^* = [0, 0, 0, 0, 0], \quad \mathbf{z}^{\text{nad}} = [1.0015, 1.0001, 1, 1.0077, 1.0023].$$

Notice that the boundary (top) layer in Figure 4a, has five large trade-off points (shown in bigger circles proportional to the trade-off value and in red color). Due to small number of large trade-off points for this problem, the DM does not look for a large trade-off point in the neighborhood of a NIMBUS point. Points in lower levels belong to core regions of the five-dimensional PO front. At Step 2, we choose an initial starting point from the ND set:

$$\mathbf{f}_p^{(0)} = \mathbf{f}^{(0)} = [0.4473, 0.4446, 0.4461, 0.4467, 0.4515]$$

corresponding to the minimum ASF value with an equal weight vector. Notice that this point (the green point in Figure 4a) lies at the bottom layer, indicating that  $\mathbf{f}_p^{(0)}$  is at the core of  $\mathcal{M}$ . Since a DM is required to guide the decision-making process, we choose a mathematical function as a decision-maker, which is not revealed to NIMBUS. Say, a specific objective vector  $\mathbf{z}^d = [0.10, 0.21, 0.31, 0.41, 0.83]$  is desired at the end, and the DM, in their mind, always compares any point from  $\mathbf{z}^d$  to provide the preference information at every iteration. The closest ND point to  $\mathbf{z}^d$  is  $\mathbf{f}^d = [0.0000, 0.1696, 0.3456, 0.3368, 0.8594]$ , which is marked as a target for the NIMBUS method in Figure 4a.

The DM does not like the chosen initial point ( $\mathbf{f}_p^{(0)}$ ) as it is far away from the desired point. Thus, in Step 2, it is not included in  $A$ , which is still an empty set. Based on the above decision-making principle, in Step 3, the DM makes the following classification of five objectives:

$$I^< = \{1, 2, 3\}, \quad I^{\leq} = \emptyset, \quad I^= = \emptyset, \quad I^{\geq} = \{5\}, \quad I^{>} = \{4\},$$

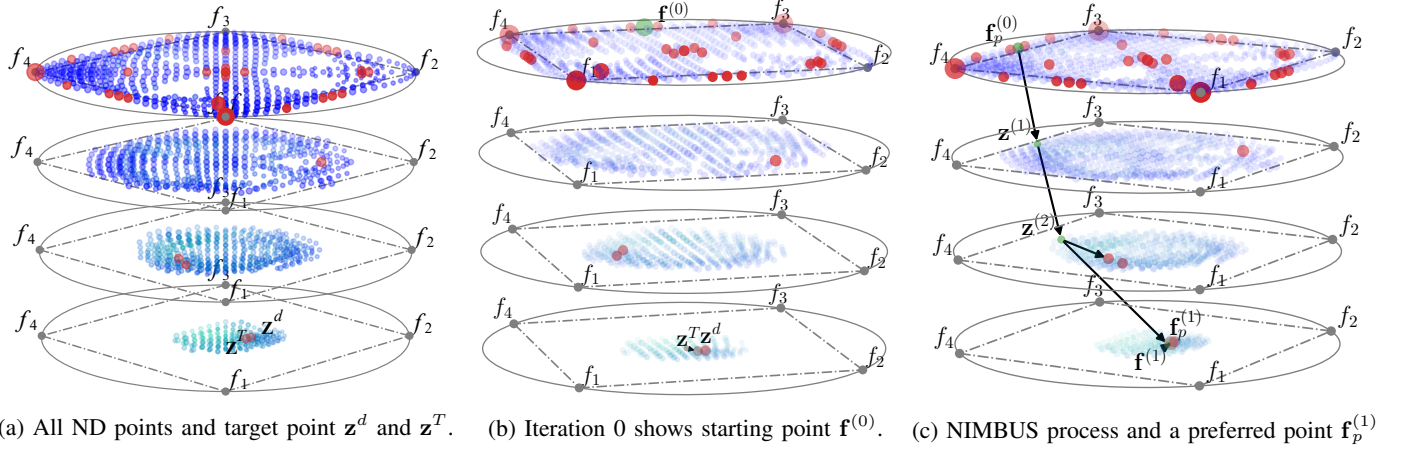


Fig. 2: PaletteViz-NIMBUS on DEBMDK problem.

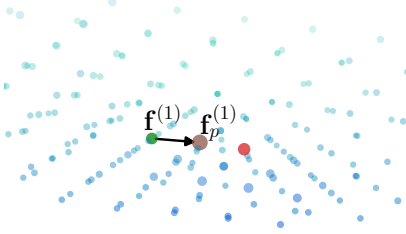


Fig. 3: PaletteViz helps find  $f_p^{(1)}$ , a neighboring solution having a larger trade-off than  $f^{(1)}$  (a part of bottom-most PaletteViz plot) for the DEBMDK problem.

with  $\epsilon_5 = 0.9$ . This way, the DM wants to improve  $f_1, f_2$ , and  $f_3$  as much as possible. Also, the DM is willing to achieve the above at the expense of worsening  $f_5$  until a limit ( $\epsilon_5$ ), and have no restriction on  $f_4$ . Step 4 finds that the point:

$$f^{(1)} = [0.0000, 0.0000, 0.0000, 1.0077, 0.0000]$$

as the best solution of the problem (2) in the ND set. Notice from the PaletteViz plot in Figure 4b that this point lies on the boundary of the PO front. One iteration of the NIMBUS approach with DM's supplied preferences takes the initial point  $f^{(0)}$  (core) to  $f^{(1)}$  (boundary). Without any visualization technique, the DM will not know this fact. The DM now moves to Step 5, as the DM wants to choose three uniformly distributed intermediate points between  $f^{(0)}$  and  $f^{(1)}$  to find how this transition takes place. The DM identifies following three points from the ND set, having minimum ASF value, (shown in Figure 4b):

$$\begin{aligned} \bar{z}^{(1)} &= [0.2109, 0.4234, 0.4271, 0.6429, 0.4250], \\ \bar{z}^{(2)} &= [0.1740, 0.0000, 0.3478, 0.8541, 0.3455], \\ \bar{z}^{(3)} &= [0.1589, 0.1575, 0.1625, 0.9483, 0.1569]. \end{aligned}$$

It is interesting to note that  $\bar{z}^{(1)}$  stays at the core, but closer towards  $f_1$  anchor point. With a big jump, the next point  $\bar{z}^{(2)}$  moves close to the boundary layer and closer to  $f_2$  anchor point. Since the point is outside the pentagon and maximally away from  $f_4$  anchor point, it indicates that at this point  $f_4$  has a poor objective value. The third point  $\bar{z}^{(3)}$  migrates to

the boundary layer and also with a poor  $f_4$  value. In Step 6, the DM decides to choose  $\bar{z}^{(1)}$  among the above three points and  $f^{(1)}$  based on the decision-making principle (minimum Euclidean distance from  $z^d$ ) and adds this point to the archive ( $A = [f_p^{(1)}]$ ):

$$f_p^{(1)} = [0.2109, 0.4234, 0.4271, 0.6429, 0.4250].$$

In Step 7, the DM is not entirely happy with this solution and moves to Step 3 for the second iteration ( $t = 1$ ). In the second iteration, the DM then provides following classification: improve  $f_1$  and  $f_2$ , and keep  $f_3$  at the same level:

$$I^< = \{1, 2\}, I^{\leq} = \emptyset, I^= = \{3\}, I^{\geq} = \emptyset, I^{>} = \{4, 5\}.$$

Step 4 finds the following point to be best according to problem (2), shown in Figure 4c:  $f^{(2)} = [0.0000, 0.0000, 0.0000, 0.0000, 1.0023]$ . This point is far away from  $f_p^{(1)}$ , hence two ( $K = 2$ ) more intermediate solutions between  $f^{(2)}$  to  $f_p^{(1)}$  are looked for. The DM obtains the following two points in Step 5:

$$\begin{aligned} \bar{z}^{(1)} &= [0, 0.1815, 0.3668, 0.5412, 0.7345], \\ \bar{z}^{(2)} &= [0.1548, 0, 0.1527, 0.3136, 0.9245]. \end{aligned}$$

In Step 6, the DM chooses both solutions to be included in the archive, but chooses the following point ( $\bar{z}^{(1)}$ ) from the set of above two and  $f^{(2)}$  in Step 7:

$$f_p^{(2)} = [0.0000, 0.1815, 0.3668, 0.5412, 0.7345].$$

In the third iteration, the DM notices that the first three objectives are almost close to desired, but  $f_4$  still needs to be improved to at least 0.3 and the  $f_5$  can be worsened to a limit of 0.9. The DM then provided the following classification:  $I^< = \emptyset, I^{\leq} = \{4\}, I^= = \{1, 2, 3\}, I^{\geq} = \{5\}, I^{>} = \emptyset$ , with  $\hat{z}_4 = 0.3$  and  $\epsilon_5 = 0.9$ . Following point (shown in Figure 5) is found by solving problem (2):

$$f^{(3)} = [0.0000, 0.1696, 0.3456, 0.3368, 0.8594].$$

Getting  $K = 2$  more solutions between  $f^{(3)}$  to  $f^{(2)}$ , the DM

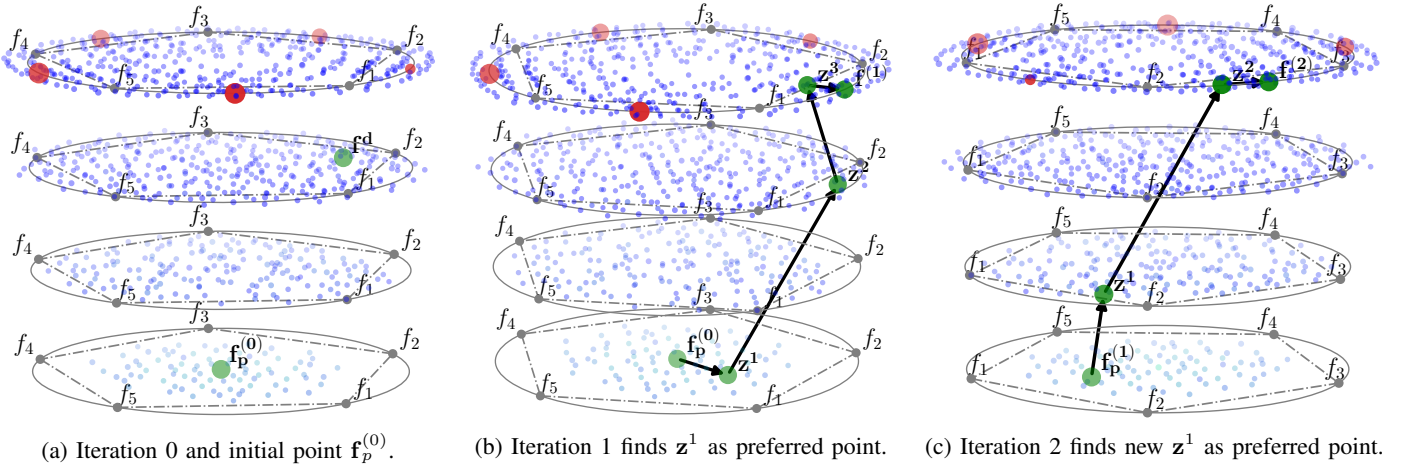


Fig. 4: PaletteViz-NIMBUS on DTLZ2 problem.

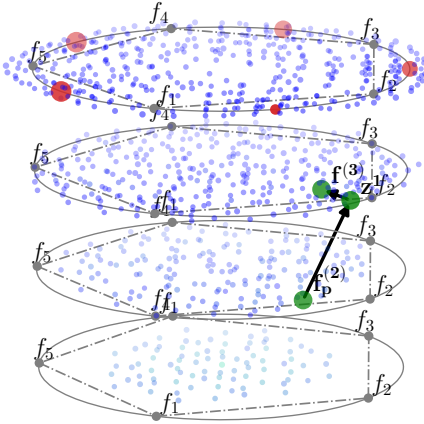


Fig. 5: Iteration 3:  $f^{(3)}$  as preferred point for DTLZ2.

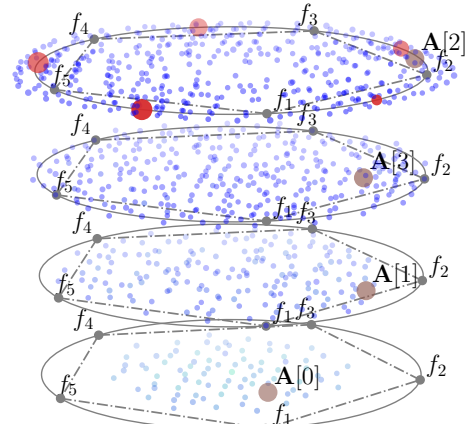


Fig. 6: Archive ( $A$ ) with four final preferred points of DTLZ2. The point  $A[3]$  on second layer is finally chosen.

obtains the following two points in Step 5:

$$\begin{aligned}\bar{z}^{(1)} &= [0, 0, 0.3227, 0.4845, 0.8131], \\ \bar{z}^{(2)} &= [0, 0.1696, 0.3456, 0.3368, 0.8594].\end{aligned}$$

The PaletteViz plot shows that these two solutions are close to each other in two consecutive levels compared to that in Iteration 2, indicating that the decision-making process followed a convergent behavior. In comparison, PCP or other visualization methods cannot provide such an useful information. In Step 6, the DM chooses only  $\bar{z}^{(2)}$  to enter the archive and also as the next preferred point in Step 7:

$$f_p^{(3)} = [0.0000, 0.1696, 0.3456, 0.3368, 0.8594].$$

At this point, if the DM is satisfied with above solution being close to the original specified target solution  $z^d$ , terminates the process. With  $z^2$ , the archive  $A$  has the following four solutions for further processing:

$$A = \begin{pmatrix} [0.2109, 0.4234, 0.4271, 0.6429, 0.4250] \\ [0.0000, 0.1815, 0.3668, 0.5412, 0.7345] \\ [0.1548, 0.0000, 0.1527, 0.3136, 0.9245] \\ [0.0000, 0.1696, 0.3456, 0.3368, 0.8594] \end{pmatrix}.$$

Archived points are shown in Figure 6 to indicate several

preferred solutions discovered along the way of decision-making, including  $A[3] = f_p^{(3)}$ .

Interestingly, finding the shortest normalized point ( $f^d$ ) from the ND set to  $z^d$  turns out to be the same solution  $f_p^{(3)}$ . Thus, the above iterative PaletteViz-NIMBUS illustrates how a combination of the NIMBUS approach and the PaletteViz visualization helps to find a desired solution by reducing the focus to a few preferred solutions from the original 1,000 ND points, and in the process provides a better understanding of several alternatives and interesting Pareto-optimal points.

### C. 10-Objective GAA problem

Finally, we choose an engineering design problem called the *General Aviation Design (GAA)* problem – which has 10 objectives and 16 constraints [22]. Figure 7a shows a PaletteViz plot of 3,000 ND points obtained using NSGA-III procedure (Step 1). It is interesting to note that the boundary layer has many large trade-off points, however larger trade-off points are sparse around the core region. Here, in this color scheme, the blue points are farther away from constraints and pink points are comparatively closer to at least one constraint boundary. The ideal and nadir points of the ND set are:

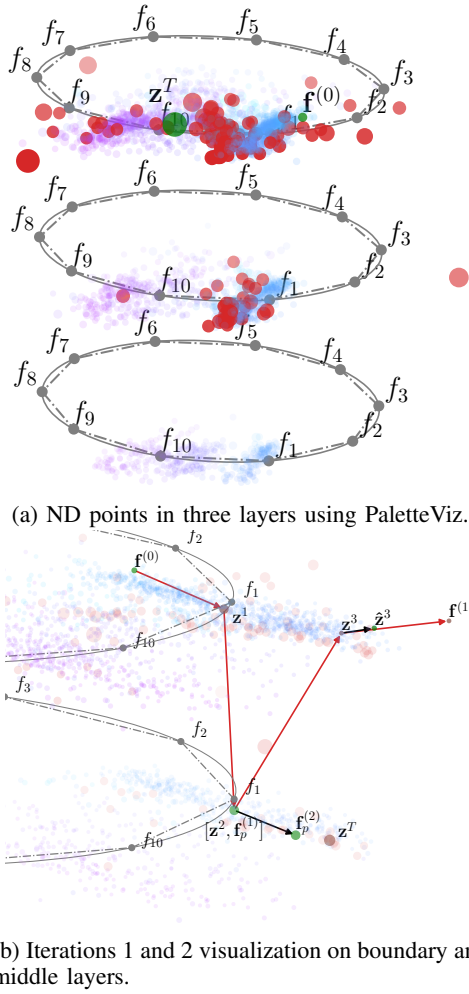


Fig. 7: PaletteViz-NIMBUS method is illustrated for 10-objective GAA problem. Iteration 1:  $\mathbf{f}^{(0)} \rightarrow \mathbf{z}^{(1)} \rightarrow \mathbf{z}^{(2)} \rightarrow \mathbf{z}^{(3)} \rightarrow \mathbf{f}^{(1)}$ . The chosen solution  $\mathbf{f}_p^{(1)} = \mathbf{z}^{(2)}$ .  $\mathbf{f}_p^{(2)}$  is the final solution of Iteration 2 obtained from  $\mathbf{f}_p^{(1)}$ .

$$\begin{aligned} \mathbf{z}^* &= [73.25, 1882.74, 57.70, 1.86, 376.36, 42206.65, \\ &\quad -2547.43, -16.79, -202.90, 0.00], \\ \mathbf{z}^{\text{nad}} &= [74.44, 2038.54, 82.54, 2.00, 500.00, 44945.04, \\ &\quad -1999.96, -14.46, -186.15, 3.70]. \end{aligned}$$

We assume here that the DM's goal is to find a solution which emphasizes objectives  $f_1$  to  $f_{10}$  in a linearly increasing manner. To find the respective target ND point from the entire NSGA-III set, we compute pseudo-weight vector [23] of all solutions and chose the one having smallest  $\ell_2$  norm from the target unit vector:  $d^T = [1, 2, 3, \dots, 10] / (\sum_{i=1}^{10} i)$ . The resulting target solution is

$$\mathbf{z}^T = [73.48, 1894.34, 60.28, 1.99, 450.00, 42713.74, -2088.43, -16.02, -201.02, 0.71]$$

with a pseudo-weight vector

$$\mathbf{w}^T = [0.151, 0.139, 0.135, 0.011, 0.061, 0.123, 0.024, 0.101, 0.134, 0.122].$$

This solution has a trade-off  $\mathcal{T} = 0.5638$ . For deciding among intermediate solutions, we shall choose the solution having a pseudo-weight vector closest to  $\mathbf{w}^T$  in terms of the  $\ell_2$  norm difference, as if the DM's preference is to get a solution having a pseudo-weight vector close to  $\mathbf{w}^T$  through the NIMBUS decision-making method.

We start the NIMBUS method with an initial neutral point, obtained by minimizing ASF with an ideal point and equal weights:

$$\begin{aligned} \mathbf{f}_p^{(0)} &= \mathbf{f}^{(0)} = [73.82, 1895.35, 59.33, 1.97, 450.00, \\ &\quad 42565.30, -2145.27, -15.33, -197.96, 1.08]. \end{aligned}$$

This solution is marked in panels (a) and (b) of Figure 7b. It has a pseudo-weight vector  $\mathbf{w}^{(0)} = [0.126, 0.143, 0.139, 0.011, 0.063, 0.126, 0.025, 0.104, 0.138, 0.125]$  with  $\ell_2(\mathbf{w}^{(0)}, \mathbf{w}^T) = 0.0604$ . Since  $f_{10}$  is most important objective for a preferred solution, this objective is to be improved, while  $f_1$  can be sacrificed. The other objectives have no preferences. Thus, in Step 3, the following setting is made:

$$I^< = \{10\}, I^= = \emptyset, I^= = \emptyset, I^> = \{1\}, I^{<>} = \{2-9\}$$

with  $\epsilon_1 = 74$ . The respective solution for problem (2), found in Step 4, is

$$\begin{aligned} \mathbf{f}^{(1)} &= [73.50, 1945.10, 62.37, 1.99, 450.00, \\ &\quad 43438.32, -2000.30, -15.29, -195.53, 0.00], \end{aligned}$$

with  $\mathcal{T} = 0.0834$ . Note that, by comparing with  $\mathbf{f}^{(0)}$ , a smaller value of  $f_{10}$  is now achieved with a slightly larger value of  $f_1$ . Both  $\mathbf{f}^{(0)}$  and  $\mathbf{f}^{(1)}$  lie on the boundary (top) layer. The respective pseudo-weight vector is  $\mathbf{w}^{(1)} = [0.187, 0.112, 0.152, 0.013, 0.076, 0.103, 0.000, 0.067, 0.105, 0.187]$  with  $\ell_2(\mathbf{w}^{(1)}, \mathbf{w}^T) = 0.0990$ . In Step 5,  $K = 3$  intermediate points ( $\mathbf{z}^{(1)}$  to  $\mathbf{z}^{(3)}$ ) are chosen between  $\mathbf{f}^{(0)}$  and  $\mathbf{f}^{(1)}$ . They are presented in the PaletteViz plot in Figure 7b. Interestingly,  $\mathbf{z}^{(1)}$  and  $\mathbf{z}^{(3)}$  lie on the boundary layer, while  $\mathbf{z}^{(2)}$  lies on the middle layer. Such insights of intermediate NIMBUS points are not possible to achieve using PCP or other visualization methods used in the NIMBUS software, but PaletteViz plot allows the DM to have. Besides this insight, the the PaletteViz plot also reveals that the following larger trade-off points exist close to these intermediate points:

$$\hat{\mathbf{z}} = \begin{pmatrix} [73.71, 1883.91, 59.11, 1.98, 456.38, 42550.94, -2145.49, \\ \quad -15.67, -200.52, 0.99], & \mathcal{T} = 0.36 \\ [73.77, 1885.33, 58.83, 1.98, 450.15, 42563.51, -2100.18, \\ \quad -15.81, -200.58, 0.75], & \mathcal{T} = 0.40 \\ [73.71, 1922.04, 59.07, 1.98, 449.96, 43237.92, -2000.00, \\ \quad -15.80, -199.65, 0.28], & \mathcal{T} = 0.13 \end{pmatrix}.$$

Thus, among the five solutions ( $\mathbf{f}^{(0)}$ ,  $\mathbf{f}^{(1)}$ ,  $\hat{\mathbf{z}}^{(1)}$ ,  $\hat{\mathbf{z}}^{(2)}$ ,  $\hat{\mathbf{z}}^{(3)}$ ], the DM chooses the solution  $\hat{\mathbf{z}}^{(2)}$  based on the linear emphasis of objectives from  $f_{10}$  down to  $f_1$ . The respective pseudo-weight

vector [23] for these points are calculated below:

$$\tilde{\mathbf{w}} = \begin{pmatrix} [0.151, 0.148, 0.141, 0.021, 0.053, 0.131, 0.040, 0.078, \\ 0.128, 0.110], \quad \ell_2(\tilde{\mathbf{w}}^{(1)}, \mathbf{w}^T) = 0.0362 \\ [0.149, 0.145, 0.141, 0.021, 0.059, 0.128, 0.027, 0.085, \\ 0.127, 0.118], \quad \ell_2(\tilde{\mathbf{w}}^{(2)}, \mathbf{w}^T) = 0.0227 \\ [0.163, 0.121, 0.153, 0.023, 0.066, 0.101, 0.000, 0.093, \\ 0.131, 0.150], \quad \ell_2(\tilde{\mathbf{w}}^{(3)}, \mathbf{w}^T) = 0.0538 \end{pmatrix}.$$

From the five  $\ell_2$  values, it is clear that  $\hat{\mathbf{z}}^{(2)}$  has the smallest  $\ell_2$  norm value (0.0227) with the linearity vector  $\mathbf{w}^T$ . Hence, it is the preferred solution:

$$\mathbf{f}_p^{(1)} = [73.77, 1885.33, 58.83, 1.98, 450.15, \\ 42563.51, -2100.18, -15.81, -200.58, 0.75],$$

with  $\mathcal{T} = 0.3998$ .

In the second iteration, a new classification is provided based on the above solution:

$$I^< = \emptyset, I^{\leq} = \{10\}, I^= = \emptyset, I^{\geq} = \{6\}, I^{>} = \{1-5, 7-9\},$$

with  $\hat{z}_{10} = 0.7$  and  $\epsilon_6 = 42715$ . Following the PaletteViz-NIMBUS method, we obtain a new solution:

$$\mathbf{f}_p^{(2)} = [73.74, 1886.47, 58.30, 2.00, 450.00, \\ 42674.91, -2055.81, -16.05, -201.58, 0.66],$$

with  $\mathcal{T} = 0.3849$ . The respective pseudo-weight vector is

$$\mathbf{w}^{(2)} = [0.150, 0.145, 0.145, 0.0000.060, \\ 0.123, 0.015, 0.102, 0.137, 0.122],$$

with  $\ell_2(\mathbf{w}^{(2)}, \mathbf{w}^T) = 0.0189$ , which is better than that at  $\mathbf{f}_p^{(1)}$ . Figure 7b also shows that this solution is closer to the desired target  $\mathbf{z}^T$ , thereby terminating the decision-making process.

## VI. CONCLUSIONS

In this paper, we have proposed a visualization based multi-criterion decision-making (MCDM) procedure by combining a recently proposed PaletteViz method with the well-known NIMBUS procedure. On three many-objective optimization problems involving up to 10 objectives and starting with thousands of non-dominated solutions obtained by an EMaO procedure, we have demonstrated how the visualization approach has helped to execute the NIMBUS procedure in an informative manner to arrive at a single or a few preferred solutions. Due to the lack of one or more real DMs in this study, we have assumed certain pre-determined preference structures of DMs – close to a pre-defined target objective vector or a pre-defined target pseudo-weight vector, and used them to choose a single preferred solution from a set of alternate solutions. The decision-making tasks are specific to individual problems, but a suitable visualization aid and step-by-step procedures can allow DMs to understand the alternatives better and exercise their preferences better in choosing a single preferred solution. This paper has shown how an EMaO procedure can be useful in finding a huge number of alternate trade-off solutions first, and then utilize a functionally decomposed visualization method to help locate alternate solutions close to large trade-off regions, close to

constraint boundaries, and close to core or edge of Pareto front in implementing a well-known MCDM approach.

A user-friendly code of PaletteViz is available from <https://gitlab.msu.edu/talukdel/pareto-explorer> and we are working on a combined and interactive optimization-cum-decision-making approach to implement the proposed method. While different other existing ideas can be implemented to make the overall approach more practical, this paper has demonstrated the importance of a functionally decomposed visualization method in better executing existing EMaO and MCDM methods in handling many-objective problems.

## REFERENCES

- [1] K. Miettinen, *Nonlinear Multiobjective Optimization*. Kluwer, 1999.
- [2] J. Ma, Z.-P. Fan, and L.-H. Huang, "A subjective and objective integrated approach to determine attribute weights," *European journal of operational research*, vol. 112, no. 2, pp. 397–404, 1999.
- [3] C. A. C. Coello, D. A. VanVeldhuizen, and G. Lamont, *Evolut. Algorithms for Solving Multi-Objective Problems*. Boston: Kluwer, 2002.
- [4] K. Deb, A. Sinha, P. Korhonen, and J. Wallenius, "An interactive evolutionary multi-objective optimization method based on progressively approximated value functions," *IEEE Transactions on Evolutionary Computation*, vol. 14, no. 5, pp. 723–739, 2010.
- [5] J. Branke, S. Greco, R. Slowinski, and P. Zielniewicz, "Interactive evolutionary multiobjective optimization using robust ordinal regression," in *Proceedings of the Fifth International Conference on Evolutionary Multi-Criterion Optimization (EMO-09)*. Berlin: Springer-Verlag, 2009, pp. 554–568.
- [6] A. K. M. Talukder and K. Deb, "Paletteviz: A visualization method for functional understanding of high-dimensional pareto-optimal datasets to aid multi-criteria decision making," *Computational Intelligence Magazine*, 2020.
- [7] K. Miettinen and M. M. Mäkelä, "Interactive bundle-based method for nondifferentiable multiobjective optimization: NIMBUS," *Optimization*, vol. 34, no. 3, pp. 231–246, 1995.
- [8] K. Deb, *Optimization for Engineering Design: Algorithms and Examples*. New Delhi: Prentice-Hall, 1995.
- [9] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear programming: Theory and algorithms*. Singapore: Wiley, 2004.
- [10] K. Deb, J. Sundar, N. Uday, and S. Chaudhuri, "Reference point based multi-objective optimization using evolutionary algorithms," *International Journal of Computational Intelligence Research (IJCIR)*, vol. 2, no. 6, pp. 273–286, 2006.
- [11] Y. Vesikar, K. Deb, and J. Blank, "Reference point based NSGA-III for preferred solutions," in *EEE Symposium Series on Computational Intelligence (SSCI-2018)*. Bengaluru, India: IEEE, Nov. 2018.
- [12] K. Deb and A. Kumar, "Interactive evolutionary multi-objective optimization and decision-making using reference direction method," in *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO-2007)*. New York: The Association of Computing Machinery (ACM), 2007, pp. 781–788.
- [13] —, "Light beam search based multi-objective optimization using evolutionary algorithms," in *Proceedings of the Congress on Evolutionary Computation (CEC-07)*, 2007, pp. 2125–2132.
- [14] A. Jaszkiewicz and R. Slowinski, "The light beam search approach – An overview of methodology and applications," *European Journal of Operation Research*, vol. 113, pp. 300–314, 1999.
- [15] M. Kadziński, M. K. Tomczyk, and R. Slowiński, "Preference-based cone contraction algorithms for interactive evolutionary multiple objective optimization," *Swarm and Evolutionary Computation*, vol. 52, p. 100602, 2020.
- [16] M. K. Tomczyk and M. Kadziński, "Robust indicator-based algorithm for interactive evolutionary multiple objective optimization," in *Proceedings of the Genetic and Evolutionary Computation Conference*, 2019, pp. 629–637.
- [17] H. Edelsbrunner, D. Kirkpatrick, and R. Seidel, "On the shape of a set of points in the plane," *IEEE Transactions on Information Theory*, vol. 29, no. 4, pp. 551–559, 1983.
- [18] J. Munkres, *Elements Of Algebraic Topology*. CRC Press, 1984.

- [19] J. Branke, K. Deb, H. Dierolf, and M. Osswald, "Finding knees in multi-objective optimization," in *Parallel Problem Solving from Nature (PPSN-VIII)*. Heidelberg, Germany: Springer, 2004, pp. 722–731, INCS 3242.
- [20] K. Deb and H. Jain, "An Evolutionary Many-Objective Optimization Algorithm Using Reference-Point-Based Nondominated Sorting Approach, Part I: Solving Problems With Box Constraints," *IEEE Trans. on Evol. Comput.*, vol. 18, no. 4, pp. 577–601, 2014.
- [21] L. Rachmawati and D. Srinivasan, "Multiobjective Evolutionary Algorithm With Controllable Focus on the Knees of the Pareto Front," *IEEE Trans. on Evolutionary Computation*, vol. 13, no. 4, pp. 810–824, 2009.
- [22] T. Simpson, J. Allen, W. Chen, and F. Mistree, "Conceptual design of a family of products through the use of the robust concept exploration method," in *Proc. 6th Symp. Multidisciplinary Analysis and Optimization*, 1996, pp. 1535–1545.
- [23] K. Deb, *Multi-objective optimization using evolutionary algorithms*. Chichester, UK: Wiley, 2001.