Investigating the Equivalence Between PBI and AASF Scalarization 
for Multi-objective Optimization

COIN Lab Report 2019004

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February 28, 2019

Abstract
Scalarization refers to a generic class of methods to combine multiple conflicting objectives into a parametrized single function. Subsequently, Pareto-optimal solutions to the original multi-objective problem can be found by solving the scalarized problems along different reference vectors (or by varying parameter values). Augmented achievement scalarizing function (AASF) is one such scalarization method used popularly in the multi-criterion decision-making (MCDM) field. In evolutionary multi-objective optimization (EMO) literature, scalarization function is commonly used to compare similar solutions within a population. The penalty boundary intersection (PBI) is one such function, proposed first with the MOEA/D framework. Both AASF and PBI methods involve one parameter each and a reference direction, which makes one specific Pareto-optimal solution to be the target of the resulting parametrized single-objective optimization problem. In this paper, we aim to analytically derive and understand the commonalities between these two metrics developed and used in two contemporary fields, as well as gain insights into the limitations of their parametric forms known to respective researchers. We show that for two-objective problems, it is possible to find equivalent AASF functional form for a given PBI parameter, and vice versa. Numerical experiments are presented within the framework of NSGA-III in order to validate the theory developed. The study connects the two contemporary fields their philosophies of solving multi-objective optimization problems and provides a mean to gain a deeper understanding of both these measures, as well as expand their parametric range to provide more flexibility of controlling the search behavior of the evolutionary multi-objective optimization algorithms. An extension to higher objective problems is also suggested. We hope that unified studies such as this would help bring the two contemporary fields close together.

1 Introduction
Practical optimization problems often involve optimization of two or more conflicting performance objectives [8]. Such problems are referred to as multi-objective optimization problems (MOP). Given the conflicting nature of the objectives, the optimum to such problems consists of not one unique solution, but multiple solutions that represent the best possible trade-off in the objective space, known as the Pareto-optimal Front (PF). The corresponding solutions in the decision space are referred to as the Pareto-optimal Set (PS). Given the
multitude of real-world applications such as engineering design, finance, operations research, etc., MOPs are of significant interest to both researchers and practitioners.

The fundamental research in MOPs has often been studied through two different lenses: Evolutionary Multi-objective Optimization (EMO) and Multi-criteria Decision Making (MCDM). In EMO, the main focus is on developing algorithms that could obtain the best possible approximation of the PF of a given MOP, that can then be analyzed further for understanding the trade-offs, decision-making etc. The quality of the PF approximation delivered by an EMO algorithm is measured in terms of its convergence, that is, how close the obtained solutions are to the true PF, and its diversity, that is, how well spread the solutions are on the PF [8]. Correspondingly, there are unary metrics such as hypervolume, generational distance, spread, etc. [5] [8] that can quantify one or both of these properties and compared between different algorithms for benchmarking [6]. On the other hand, the main target in MCDM is to come up with a Pareto-optimal solution that is eventually implemented by the decision-maker. Obtaining the approximation of the entire PF is not of primary significance in this regard. The focus is on designing preference structures and scalarized measures that can combine multiple objectives and directly/interactively search for or select a final solution. Although not a focus of this study, recent interactive EMO-MCDM methods uses MCDM methods during the optimization through an EMO algorithm [12], thereby making them more pragmatic from computation and decision-making points of views.

Needless to say, the above two research directions (EMO and MCDM) are not mutually exclusive, and useful methods from each one are adopted into another frequently. A prominent example is the use of scalarization functions, which are predominantly rooted in the MCDM literature. Generally speaking, scalarization refers to a way of combining multiple objectives into a scalar function, optimizing which will produce one solution to the original MOP [14] [18]. Weighted sum [13] of the form \( s = \sum_{i=1}^{M} w_i f_i \) is a simple and intuitive form of scalarization that assigns different priorities to different objectives. However, mathematically it has a fundamental limitation that if used for optimizing problems with non-convex PF, it can only yield the points lying on the convex part of the front, irrespective of the weight values used. In order to overcome this issue a number of other measures have been developed in the literature [23] [24] [29]. In [24], the behavior of 15 scalarizing functions was illustrated, including STEM, STOM, GUESS, NIMBUS, etc.

In the early years of evolutionary computation (EC) research, scalarizing functions were used to convert a multi-objective problem to single-objective and to find one solution at a time [19] [26]. However, with the development of EMO algorithms that could handle more than one objectives using dominance relations or indicators directly [11] [34] [35], the use of a scalarization approach became less frequent. The interest in scalarization functions surged again when the research into problems containing more than three objectives proliferated, which are now colloquially referred to as many-objective optimization problems (MaOP) [17]. Given the observation that the Pareto-based approaches suffer a loss of selection pressure for MaOPs, an approach that is now popularly followed uses decomposition of the objective space. It involves converting the the given MOP/MaOP into several single-objective problems scalarized using an appropriate measure along certain uniformly distributed reference vectors/points [2] [4] [28] [32]. For example, Tchebycheff is used in [32], achievement scalarizing function (ASF) and augmented ASF (AASF) in [3] [9] [31] and localized weighted sum in [28]. A number of contemporary approaches also combine the advantages of non-dominance and decomposition [10] [21]. More importantly, this has even given rise to some new scalarization functions that have originated in recent EMO literature; primarily geared to work well with reference point/reference vector based methods. Notable ones include, for example, the Penalized Boundary Intersection (PBI) [32] and Angle Penalized Distance (APD) [4] and others [29].

The study presented herein attempts to establish a connection and equivalence conditions between two often-used scalarizing measures - AASF (with its origins in MCDM) and PBI (with its origins in EMO). Both these metrics contain a user-defined parameter, the range of which has been traditionally set based on certain notion inherent in their inception. For example, PBI uses a parameter \( \theta \) to impose a penalty term, and therefore is always assumed to be greater than zero. Similarly, AASF uses a parameter \( \rho \) which attempts to change the ASF contours slightly in order to avoid weakly non-dominated solutions; and hence it is usually set to a very small positive value close to zero. In this study, we show the limits of the dominance relations that these parameter bounds impose during selection. We also show how when we relax these bounds the behavior of PBI and AASF could be made identical to each other for bi-objective problems. We derive the equivalent \( \theta \) for a given \( \rho \), and vice versa theoretically to illustrate their connection. Thereafter, we present numerical experiments
on a range of problems to demonstrate the validity of the proposed reformulations.

The remainder of this paper is organized as follows. In Section 2, a background on the relevant scalarizing functions of interest, in particular PBI, ASF and AASF are presented and the motivation of the study is outlined. The detailed derivations for equivalence conditions and the associated bounds are discussed in Section 3, followed by numerical validation in Section 4. Concluding remarks are presented in Section 5.

2 Background and Motivation

For calculating the scalar measures of interest in this study (e.g. ASF, PBI, AASF), two quantities are required: a reference vector (or weight vector; sometimes used interchangeably) and a reference point, which is the origin or a supplied point on the reference vector. In essence, all these methods attempt to find a solution on the PF that is ‘closest’ to the ideal point along its corresponding reference vector.

Consider a bi-objective optimization problem, and a candidate solution \( P = (p_1, p_2) \) in the objective space. Let the direction of the reference vector be denoted using \( w = (w_1, w_2) \), such that \( w_1 + w_2 = 1 \), and reference point be denoted as \( Z = (z_1, z_2) \). The equation of the reference vector is thus \( f_2 - z_2 = f_1 - z_1 = c \). Since the two objectives can be in different orders of magnitude, it is a common practice to normalize the objective space before conducting any distance based calculations. The linear form of normalization is preferred as it is straightforward and maintains all the dominance relations exactly between the points. This normalization is executed as shown in Eq. 1

\[
f_i^n = \frac{f_i - f_{\min,i}}{f_{\max,i} - f_{\min,i}} \quad i = 1, 2 \ldots M,
\]

where \( f_{\min,i} \) and \( f_{\max,i} \) are the lowest and highest values, respectively, of the \( i \)th objective \( f_i \), and \( M \) is the number of objectives (=2 for our discussions here).

Following from above, the ideal point consisting of the best (lowest) value of each objective in a minimization sense maps to (0,0) in the normalized objective space. In the discussions that follow, we assume always to be operating in the normalized objective space. Thus, if we set the reference point \( Z \) to be the ideal point, then we have \( z_1 = z_2 = 0 \). The equation of the reference vector thus simply becomes \( \frac{f_2}{w_2} = \frac{f_1}{w_1} = c \) in parametric form, or \( f_2 = w_2 f_1 / w_1 \) in the Cartesian form. The calculation of PBI, ASF and AASF then proceeds as follows.

2.1 ASF Scalarization

For a given set of objective functions defined by \( f \), the commonly used form of ASF [29] is calculated as shown in Eq. 2

\[
ASF(f, w, z) = \max_{i=1}^{M} \left( \frac{f_i - z_i}{w_i} \right), \quad i = 1, 2 \ldots M,
\]

The behavior of the ASF function can be understood from Fig 1(a) by considering the calculation for a sample point \( P \). On the horizontal line (parallel to \( f_1 \) axis) passing through \( P \), \( f_2 = p_2 \) and hence \( f_2 / w_2 = p_2 / w_2 \) is constant value. Also, since this point lies above the line whose equation is \( f_2 = w_2 / w_1 f_1 \), we have \( p_2 > w_2 / w_1 f_1 \), and therefore \( \max(p_1, w_1, p_2, w_2) = p_2 / w_2 \). The same holds true for all points on the horizontal line. Lets denote the point where the horizontal line intersects the reference vector as \( K \). Now, extending the same logic along a vertical line (parallel to \( f_2 \) axis) passing through \( K \), we can infer that \( \max(f_1 / w_1, f_2 / w_2) = f_1 / w_1 \) for each point on the line. At the point \( K \) both the terms are equal \( f_1 / w_1 = f_2 / w_2 \).

Thus, the isolines for ASF run parallel to the two objective axes. This creates an undesirable ranking scenario in the cases where one solution weakly dominates another. For example, if we consider the two solutions \( A \) and \( B \) in Fig 1(a) then \( B \) weakly-dominates \( A \), but ASF would assign the same rank to these two solutions.

Following from above, the angular range established by ASF is asymmetric about the reference vector. The angle made by the contour lines above the reference vector with it is \( \phi_a = \tan^{-1}(w_2 / w_1) \), whereas the angle
made by the contour lines below the reference vector is \( \phi_b = \tan^{-1}(w_1/w_2) \). Evidently, the symmetry \( \phi_a = \phi_b \) will apply only in the special case of when \( w_1 = w_2 \).

2.2 Augmented ASF (AASF) Scalarization

In order to avoid the aforementioned issue in handling weakly-dominated points, an augmented version of ASF was proposed in [23]. AASF is calculated as shown in Eq. [3] by adding a term to the original ASF function. Here, \( \rho \) is taken as a small constant (e.g. \( 10^{-5} \)) [1]. This update slightly expands the angles made by the isolines with the reference vector, so that in case a solution weakly dominates another, it gets a better (lower) value. Referring again to the solutions \( A \) and \( B \) discussed in Fig. 1(a), \( B \) will now lie below the AASF isoline shown in Fig. 1(b) and hence will have a lower AASF value than \( A \).

\[
AASF(f, w, z, \rho) = \max_{i=1}^M \left( \frac{f_i - z_i}{w_i} \right) + \rho \sum_{i=1}^M \left( \frac{f_i - z_i}{w_i} \right),
\]

\[
= \max_{i=1}^M \left( \frac{f_i}{w_i} \right) + \rho \sum_{i=1}^M \left( \frac{f_i}{w_i} \right), \quad \text{if} \ z_i = 0, \ \forall i.
\]

Like ASF, the angular range established by AASF is also asymmetric about the reference vector. The angle made by the contour lines above the reference vector with it is \( \phi_a = \tan^{-1}(w_1/w_2) + \tan^{-1}(\rho w_1/(1+\rho) w_2) \), whereas the angle made by the contour lines below the reference vector is \( \phi_b = \tan^{-1}(w_1/w_2) + \tan^{-1}(\rho w_1/(1+\rho) w_2) \) (the calculations are discussed later in Section 3). Once again, the symmetry \( \phi_a = \phi_b \) will apply only in the special case of when \( w_1 = w_2 \).

2.3 PBI Scalarization

Penalized boundary intersection (PBI) method was proposed in [32] for its use within the multi-objective evolutionary algorithm based on decomposition (MOEA/D). The key idea was to improve the contemporary boundary intersection (BI) techniques [7] by removing an equality constraint and replacing it with a penalty term. The calculation of PBI can be understood from Fig 1(c). For a given point \( P \), a perpendicular is drawn from it to the reference vector, the intersection point is denoted here as \( T \). Thereafter, the distance \( ZT \) is calculated as \( d_1 \) and the distance \( PT \) is calculated as \( d_2 \). A penalty parameter \( \theta \) is taken as a user input to quantify the penalty applied to the solution for misalignment of \( P \) with the reference vector. The PBI is
calculated as shown in Eq. 4:

\[ PBI(f, w, z, \theta) = d_1 + \theta d_2, \]

where \( d_2 = |PT| = \frac{|w_1 f_2 - w_2 f_1|}{\sqrt{w_1^2 + w_2^2}}, \)

\[ d_1 = |ZT| = \sqrt{|P|^2 - d_2^2}. \]

As evident from the above formula, a solution with \( d_2 = 0 \) and least possible \( d_1 \) would be the desired solution on the PF along the direction \( w \). However, as is the case generally with penalty based methods, the right value of \( \theta \) is problem dependent and the obtained results could vary significantly depending on this user-defined choice. Established through various empirical studies, \( \theta = 5 \) is the commonly used value which seems to work well across (at least) the range of test problems in the standard benchmark problems such as DTLZ [13] and WFG series [16].

![Contour plots](image)

Figure 2: Contour plots obtained by (from the top row) ASF, AASF and PBI. Here, \( w_1 = (0.25, 0.75) \), \( w_2 = (0.5, 0.5) \). Darker colors represent lower/better values on the isolines.

Unlike in AASF, the angular range of dominance established by \( PBI \) is symmetric around the reference vector, regardless of the reference vector being considered. The PBI contours on either side of the reference vector make a constant angle of \( \phi = \angle PKT = \tan^{-1}(d_2/\theta d_2) = \tan^{-1}(1/\theta) \) with it. Contour plot for each of the functions is given in Fig. 2 with respect to two different directions: \( w_1 = (0.25, 0.75) \) and \( w_2 = (0.5, 0.5) \). For these figures, \( \rho \) and \( \theta \) for AASF and PBI are set to 0.1 and 5, respectively.

The above concepts are extendable to higher objective problems as well, but the nature of contours will differ. For PBI, the points with equal value form a symmetric cone around the reference vector, whereas for
AASF will constitute a pyramid which may not be symmetrical about the reference vector. In view of the above discussions and a review of relevant literature, a few issues are noteworthy:

- There exist some works in the literature investigating the use of different values of $\theta$ to vary the selection pressure to deal with different types of problems \[30\]. However, for AASF, the parameter $\rho$ has been predominantly used as a mean to remove the possibility of removing weakly-dominated solutions. Its potential as an explicit parameter to exert more control over the search behavior has not been explored, although some other parametrized versions of ASF do exist \[22, 25\].

- The relationship, if any, between two of these widely used measures has not been studied in the literature yet. The origins of ASF/AASF are rooted in the MCDM domain \[23, 29\]. With the increasing interest in decomposition/reference point based methods in the EMO domain, these measures have been widely adopted to search collaboratively along a set of uniformly distributed reference vectors. PBI is a relatively recent proposal compared to the above two and originated in the EMO domain to work within a decomposition based optimization framework \[32\].

In this paper, we attempt to bridge the above gaps in these two methods with different origins and show that it is possible to consider them equivalent measures for multi-objective search under certain scenarios for bi-objective problems. More specifically, we derive the relationship between the parameters of PBI and AASF such that for a given PBI function, one can construct an equivalent AASF function; and vice versa.

3 Derivations of Equivalence of PBI and AASF

3.1 Equivalent PBI for a Given AASF

In order to make the PBI measure to be equivalent to AASF, we need to work out a $\theta$ corresponding to the $\rho$ parameter of the AASF. For ease of reference, we call this setting of PBI as Equivalent PBI (EPBI). For the two measures to yield the same selection outcome during the search, the key requirement is that the contours formed by them are the same (i.e., make the same angle(s) with the reference vector). Note that the raw value itself of the two measures need not be the same at all given points, just a matching trend among the isolines is sufficient to drive the search in exactly the same way.

Consider the diagram shown in Fig. 3, where the foot of the perpendicular from the point $P$ to the reference vector is denoted as $T = (tw_1, tw_2)$ in the parametric form. Similarly, the point where the isoline of AASF
intersects the reference vector is denoted as $K = (kw_1, kw_2)$. Once the values of $k$ and $t$ are determined, the length of segment KT can be calculated as shown in Eq. 5.

$$|KT| = \delta_{KT} = \sqrt{(kw_1 - tw_1)^2 + (kw_2 - tw_2)^2}$$

$$= (k - t)\sqrt{w_1^2 + w_2^2}$$

(5)

For the isolines of PBI to make the same angle with the reference vector as the AASF, we can then simply make sure that the point K also has the same PBI value as point P (as it already has the same AASF value as P). This can done by setting the distance between K and T to be equal to $\theta d_2$; since the remainder of the PBI term ($d_1$) is equal to $|ZT|$ at point P. Thus, we have the required relationship established as shown in Eq. 6.

$$|KT| = (k - t)\sqrt{w_1^2 + w_2^2} = \theta d_2$$

$$\Rightarrow \theta = \frac{(k - t)\sqrt{w_1^2 + w_2^2}}{d_2}$$

(6)

In the above formula, if we can find and substitute the values of $k$ and $t$ in terms of the known quantities, we would obtain an explicit relation between $\theta$ and $\rho$. These calculations are shown below.

### 3.1.1 Calculation of $t$:

The value of $t$ can be calculated by utilizing the orthogonality of the segments PT with the reference vector, which implies that the dot product of these vectors should be 0. This is shown in Eq. 7 where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along $f_1$ and $f_2$ axis respectively.

$$\mathbf{P}T \cdot \mathbf{w} = 0,$$

$$\Rightarrow [(tw_1 - p_1)\mathbf{i} + (tw_1 - p_2)\mathbf{j}] \cdot [w_1\mathbf{i} + w_2\mathbf{j}] = 0,$$

$$\Rightarrow tw_1^2 - p_1w_1 + tw_2^2 - p_2w_2 = 0,$$

$$\Rightarrow t = \frac{p_1w_1 + p_2w_2}{w_1^2 + w_2^2},$$

$$\Rightarrow T = \left( \frac{p_1w_1 + p_2w_2}{w_1^2 + w_2^2} w_1, \frac{p_1w_1 + p_2w_2}{w_1^2 + w_2^2} w_2 \right).$$

(7)

### 3.1.2 Calculation of $k$:

For the calculation of $k$, we need to equate the AASF at the points P and K. Consider first the condition where the point P lies on or above the reference vector, which implies $f_2 \geq w_2f_1/w_2$, i.e., $f_2/w_2 \geq f_1/w_1$. The
calculations will then proceed as presented in Eq. [7]

\[
AASF(P, w, \rho) = \max \left( \frac{p_1}{w_1}, \frac{p_2}{w_2} \right) + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right),
\]

\[
= \frac{p_2}{w_2} + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right) \quad \text{if } f_2 \geq \frac{w_2}{w_1} f_1,
\]

\[
AASF(K, w, \rho) = \max \left( \frac{k w_1}{w_1}, \frac{k w_2}{w_2} \right) + \rho \left( \frac{k w_1}{w_1} + \frac{k w_2}{w_2} \right),
\]

\[
= k(1 + 2\rho).
\]  

AASF \(= AASF(K, w, \rho)\) (isoline),

\[
\Rightarrow \frac{p_2}{w_2} + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right) = k(1 + 2\rho),
\]

\[
\Rightarrow k = \left( \frac{1}{1 + 2\rho} \right) \left( \frac{p_2}{w_2} + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right) \right).
\]  

The expressions of \(t\) obtained from Eq. [7] and \(k\) obtained from Eq. [8] can then be substituted back into Eq. [5]. The subsequent calculations are shown in Eq. [9]

\[
\theta d_2 = (k - t) \sqrt{w_1^2 + w_2^2},
\]

\[
= \left( \frac{\frac{p_2}{w_2} + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right)}{1 + 2\rho} - \frac{p_1 w_1 + p_2 w_2}{w_1^2 + w_2^2} \right) \sqrt{w_1^2 + w_2^2},
\]

\[
\Rightarrow \theta = \frac{|w_1 f_2 - w_2 f_1|}{w_1^2 + w_2^2} = \left( \frac{\frac{p_2}{w_2} + \rho \left( \frac{p_1}{w_1} + \frac{p_2}{w_2} \right)}{1 + 2\rho} - \frac{p_1 w_1 + p_2 w_2}{w_1^2 + w_2^2} \right).
\]  

Since \(f_2 \geq w_2 f_1/w_1\), \(|w_1 f_2 - w_2 f_1| = w_1 f_2 - w_2 f_1\),

\[
\Rightarrow \theta = \frac{(w_1 f_2 - w_2 f_1)(\rho(w_1^2 - w_2^2) + w_2^2)}{w_1 w_2 (1 + 2\rho)(w_1^2 + w_2^2)},
\]

\[
= \frac{\rho(w_1^2 - w_2^2) + w_2^2}{w_1 w_2 (1 + 2\rho)}.
\]  

The expression for \(\theta\) derived in Eq. [9] is valid for all the points lying above/on the reference vector. The expression for \(\theta\) for the points that lie below the reference vector can be calculated following the exact same steps, and considering \(f_2 < w_2 f_1/w_1\), which implies \(\max \left( \frac{f_1}{w_1}, \frac{f_2}{w_2} \right) = \frac{f_1}{w_1}\) and \(|w_1 f_2 - w_2 f_1| = w_2 f_1 - w_1 f_2\). Therefore, the overall expression for \(\theta\) can be written as shown in Eq. [10], where \(\theta_a\) and \(\theta_b\) are used to refer to the expressions above and below the reference vector, respectively.

\[
\theta(w, \rho) = \begin{cases} 
\theta_a = \frac{\rho(w_1^2 - w_2^2) + w_2^2}{w_1 w_2 (1 + 2\rho)}, & \text{if } f_2 \geq w_2 f_1/w_1, \\
\theta_b = \frac{\rho(w_1^2 - w_2^2) + w_2^2}{w_1 w_2 (1 + 2\rho)}, & \text{if } f_2 < w_2 f_1/w_1,
\end{cases}
\]  

\[
EPBI(f, w, z, \rho) = d_1 + \theta(w, \rho)d_2.
\]  

In order to validate the expressions in Eq. [10] AASF values are computed on a grid of uniformly distributed points along \(f_1\) and \(f_2\), with a resolution of 0.001 (i.e. 1001 points along each objective) for the reference vector \(w_1 = (0.25, 0.75)\). Correspondingly, the equivalent PBI (EPBI) of these points is computed by setting the value of \(\theta\) as derived above in the PBI expression. The contour plots obtained by the two methods are
Two values of $\rho$ are chosen, one positive (0.2) to show contours with higher angles (overall obtuse) compared to the right-angled contours of ASF, and one negative (-0.2) to show acute angled contours. The contours are also identical when calculated along other reference vectors, and are not shown here for brevity.

Figure 4: Left column of subfigures shows the contours corresponding to AASF using $\rho = 0.2$ and $-0.2$ with respect to the reference vector $\mathbf{w}_1 = (0.25, 0.75)$. The right column shows the contours (identical) obtained using the equivalent PBI (EPBI) formulated by setting the $\theta$ as shown in Eq. 10. Darker colors represent lower/better values on the isolines.

### 3.2 Equivalent AASF for a Given PBI

Having derived the equivalent PBI for a given AASF, we now move on to the reverse, i.e., given a $\theta$ value for the PBI, the equivalent $\rho$ values need to be found for generating the equivalent AASF (EAASF) function. In-fact, the process to calculate the expression for $\rho$ follows directly from the previous subsection. Given the expression of $\theta$ as a function of $\rho$, the reverse ($\rho$ as a function of $\theta$) can be calculated by simply rearranging the terms. These expressions are shown in Eq. 11, where $\rho_a$ and $\rho_b$ refer to the calculations of the points above and
below the reference vector, respectively, for $z_i = 0, \forall i$.

$$
\rho(w, \theta) = \begin{cases} 
\rho_a = \frac{w_1 w_2 \theta - w_2^2}{w_1^2 - w_1^2 - 2w_1 w_2 \theta}, & \text{if } f_2 \geq w_2 f_1 / w_1, \\
\rho_b = \frac{w_1 w_2 \theta - w_2^2}{w_1^2 - w_1^2 - 2w_1 w_2 \theta}, & \text{if } f_2 < w_2 f_1 / w_1,
\end{cases}
$$

(11)

$$
EAASF(f, w, z, \theta) = M \max_{i=1}^{M} \left( \frac{f_i}{w_i} \right) + \rho(w, \theta) \sum_{i=1}^{M} \left( \frac{f_i}{w_i} \right).
$$

However, although the above deduction of $\rho$ formula seems straightforward, it cannot be directly used for calculating EAASF (unlike in the case of EPBI). The reason is that for any given $w$ and $\theta$, the values of $\rho_a$ and $\rho_b$ will be unequal. This creates a discontinuity in the value of AASF calculated on the line approaching from above vs approaching from below, because the EAASF contour lines with the matching slopes for given $\theta$ intersect at different points with the reference vector. This implies that even though the orientation of the contours are matched, their values are not. This can be visualized from Fig. 5(a)-5(b), where the contour lines are plotted for $w_1 = (0.25, 0.75)$ using PBI with $\theta = 3$ and the corresponding EAASF. It can be seen that the isolines of PBI are symmetric in angle as well as magnitude about the reference vector. The corresponding EAASF on the other hand has the angles correctly matched to the PBI, but the values at each of the isolines differ in magnitude above and below the line. This is evident from the difference in colors of the lines as well as mismatch in the intersection points with the reference vector approaching from above and below it (in Fig. 5(b)). The exception to this is obviously when $w_1 = w_2$, i.e., $w = (0.5, 0.5)$, as this results in equal $\rho$ values approaching from both sides and consequently the isolines are continuous, as shown in Fig. 5(c)-5(d).

### 3.2.1 Proposed Updated AASF

The mismatch between the contour lines from top and bottom of a reference line is easy to amend and a new updated AASF can be proposed to make symmetric contour lines. Since the slopes of the isolines are already the desired ones, an appropriate scaling on the EAASF value on either side can translate the isolines to the required continuous value. In order to do this, consider a point on the reference vector as $K_0 = (k_0 w_1, k_0 w_2)$ which is the intersection point of the isolines with the reference vector using $\rho_a$ from above it, and $\rho_b$ below it. The EAASF at point $K_0$ computed using $\rho_a$ (approaching from above) will be $EAASF_a = \max \left( \frac{k_0 w_1}{w_1}, \frac{k_0 w_2}{w_2} \right) + \rho_a \left( \frac{k_0 w_1}{w_1} + \frac{k_0 w_2}{w_2} \right) = k_0 + 2k_0 \rho_a$. Similarly, approaching from below, it will be $EAASF_b = \max \left( \frac{k_0 w_1}{w_1}, \frac{k_0 w_2}{w_2} \right) + \rho_b \left( \frac{k_0 w_1}{w_1} + \frac{k_0 w_2}{w_2} \right) = k_0 + 2k_0 \rho_b$. Therefore, to make $EAASF_b$ equal to $EAASF_a$, we can simply multiply it by $\frac{1 + 2 \rho_a}{1 + 2 \rho_b}$, i.e., $EAASF_b(modified) = \frac{1 + 2 \rho_a}{1 + 2 \rho_b} EAASF_b$. Alternatively, we can update the $EAASF_a$ in a similar way. The overall updated formulation is summarized in Eq.(12).
Figure 5: Left column of subfigures shows the contours corresponding to PBI using $\theta = 3$ for the reference vectors $w_1 = (0.25, 0.75)$ and $w_2 = (0.5, 0.5)$ respectively. The right column shows the contours obtained using the equivalent AASF (EAASF) formulated by setting the $\rho$ as shown in Eq. 11. The EAASF contours show discontinuity for $w_1$ but not for $w_2$. Darker colors represent lower/better values on the isolines.

\[
\rho(w, \theta) = \begin{cases} 
\rho_a = \frac{w_1 w_2 \theta - w_1^2}{w_1^2 w_2^2 - 2 w_1 w_2 \theta}, & \text{if } f_2 \geq w_2 f_1 / w_1, \\
\rho_b = \frac{w_1 w_2 \theta - w_2^2}{w_1^2 w_2^2 - 2 w_1 w_2 \theta}, & \text{if } f_2 < w_2 f_1 / w_1,
\end{cases}
\]

\[
EAASF_a(f, w, z, \theta) = \max_{i=1}^{M} \left( \frac{f_i}{w_i} \right) + \rho_a(w, \theta) \sum_{i=1}^{M} \left( \frac{f_i}{w_i} \right),
\]

\[
EAASF_b(f, w, z, \theta) = \max_{i=1}^{M} \left( \frac{f_i}{w_i} \right) + \rho_b(w, \theta) \sum_{i=1}^{M} \left( \frac{f_i}{w_i} \right) \times \frac{1 + 2 \rho_a}{1 + 2 \rho_b},
\]

or,

\[
EAASF_b(f, w, z, \theta) = \max_{i=1}^{M} \left( \frac{f_i}{w_i} \right) + \rho_b(w, \theta) \sum_{i=1}^{M} \left( \frac{f_i}{w_i} \right),
\]

\[
EAASF_a(f, w, z, \theta) = \max_{i=1}^{M} \left( \frac{f_i}{w_i} \right) + \rho_a(w, \theta) \sum_{i=1}^{M} \left( \frac{f_i}{w_i} \right) \times \frac{1 + 2 \rho_b}{1 + 2 \rho_a}.
\]
The contour lines or the updated formulation of EAASF are shown in Fig. 6. It can be seen that the EAASF contours now exactly match those obtained from PBI with $\theta = 3$. The subplots for $w = (0.5, 0.5)$ are identical to those in Fig. 5 and hence not shown again for brevity.

Notably, an alternate method to achieve the above matching could be to use a slightly modified (normalized) version of AASF where the Eq. 3 is divided by $1 + 2\rho$.

$$U_{AASF}(f, w, z, \rho) = \frac{AASF(f, w, z, \rho)}{1 + 2\rho}.$$  

(13)

As this factor is a constant value for a given $\rho$, it will not change the nature of contours, but at the weight vector the AASF value will become $k$ regardless of $\rho$. The expressions for $\rho$ as a function of $\theta$ will remain the same as shown in Eq. 11; as the above update does not change the contour angles of AASF.

![Figure 6](image)

(a) PBI, $w_1, \theta = 3$

(b) EAASF, $w_1, \rho_a = 0.3077, \rho_b = 0$

Figure 6: Left: Contours for PBI using $\theta = 3$ for the reference vectors $w_1 = (0.25, 0.75)$. Right: contours obtained using the updated EAASF formulated by setting the $\rho$ as shown in Eq. [12] as the above update does not change the contour angles of AASF.

### 3.3 Bounds and Singularities

From the original formulation of PBI discussed earlier in Eq. 4 (and Fig. [1(c)]), we know that the angle made by the isolines with the reference vector is $\phi = \tan^{-1}(1/\theta)$. Therefore, when $\theta = 0$, $\phi = \pi/2$ and the contour lines will be perpendicular to the reference vector. On the other extreme, when $\theta = \infty$, $\phi = 0$, and the contour lines will be perpendicular to the reference vector. Thus, $\theta \in [0, \infty)$ and correspondingly, $\phi \in [\pi/2, 0)$

However, as discussed before, the angles formed by AASF contours are asymmetric for a given $\rho$ value. Based on Fig. [1(b)] the angles $\phi_a$ and $\phi_b$ formed by the contours can be calculated as follows.

Let $\beta_a$ be the acute angle formed by the segment PK with the horizontal $(f_1)$ axis; such that $\phi_a = \alpha_a + \beta_a$, where $\alpha_a$ is simply the angle made by the reference vector with the horizontal axis, i.e., $\alpha_a = \tan^{-1}(w_2/w_1)$.

The angle $\beta_a$ can be calculated as shown in Eq. [14]
\[
\tan \beta = \frac{f_2 - kw_2}{f_1 - kw_1},
\]

\[
k = \left(\frac{1}{1 + 2\rho}\right) \left(\frac{p_2}{w_2} + \rho \left(\frac{p_1}{w_1} + \frac{p_2}{w_2}\right)\right), \text{ from Eq. 8}
\]

\[
\Rightarrow \tan \beta_a = -\frac{f_2 - w_2\left(\frac{p_2}{w_2} + \rho\left(\frac{p_1}{w_1} + \frac{p_2}{w_2}\right)\right)}{f_1 - w_1\left(\frac{k w_2}{2} + \rho\left(\frac{p_1}{w_1} + \frac{p_2}{w_2}\right)\right) + 2\rho} = \frac{\rho w_2}{(1 + \rho)w_1},
\]

\[
\Rightarrow \beta_a = \tan^{-1}\left(\frac{\rho w_2}{(1 + \rho)w_1}\right).
\]

Similarly,
\[
\beta_b = \tan^{-1}\left(\frac{\rho w_1}{(1 + \rho)w_2}\right),
\]

\[
\Rightarrow \phi_a = \tan^{-1}(w_2/w_1) + \tan^{-1}\left(\frac{\rho w_2}{(1 + \rho)w_1}\right), \text{ and}
\]

\[
\phi_b = \tan^{-1}(w_1/w_2) + \tan^{-1}\left(\frac{\rho w_1}{(1 + \rho)w_2}\right).
\]

Now, when the AASF contours form one continuous line segment (without change in slope at the intersection with \(w\), then \(\beta_a + \beta_b + \pi/2 = \pi \Rightarrow \beta_a + \beta_b = \pi/2 \Rightarrow \tan \beta_a = \cot \beta_b = 1/\tan \beta_b\). Substituting the values of \(\tan \beta_a\) and \(\tan \beta_b\) from Eq. 14 and solving through, we obtain \(\beta^2 = 1\). There are two possible solutions to this equation, \(\rho = \infty\) and \(\rho = -1/2\). For \(\rho = \infty\), the resulting contours have the slope of \(-w_2/w_1\) from the \(f_1\) axis, and hence are not perpendicular to the given reference vector \(w\) (the product of slopes is not equal to -1). In this case, \(\phi_a = 2 \tan^{-1}(w_2/w_1)\) and \(\phi_b = 2 \tan^{-1}(w_1/w_2)\). The exception again is when \(w_1 = w_2\), in which case \(\phi_a = \phi_b\) and the resulting contours will be perpendicular to \(w\), as seen in Fig. 7.

For \(\rho = -1/2\), the contour lines will be parallel to the reference vector (regardless of its direction), and hence \(\phi_a = \phi_b = 0\), as shown in Fig. 8.

Thus, the ranges of contour angles covered using the above limits of \(\theta\) and \(\rho\) can be visualized in Fig. 10.

If the traditional ranges of \(\theta\) and \(\rho\) being non-negative are considered, then AASF the angles below 90\(^\circ\) (total) will not be achieved, as shown in Fig. 10(a), whereas if \(\rho\) is allowed to go up to -1/2, these smaller angles can be achieved, as shown in Fig. 10(b).

The variation of the \(\theta_a\), \(\theta_b\), (of EPBI) as a function of \(\rho\) (of AASF) is shown in Fig. 11 for the reference vector \(w = (0.25, 0.75)\). On closer look, it can be observed that while \(\theta_b > 0\) for all values of \(\rho\), which is in-line with the range of \(\theta\) in \([0, \infty)\) discussed above. However, \(\theta_a\) spans both positive and negative values. The reason \(\theta_a\) needs to go beyond this expected range can be explained by first observing the value of \(\rho\) at which the sign of \(\theta_a\) switches from positive to negative. For this, we refer back to Eq. 10.

The denominator of \(\theta_a\) is \(w_1 w_2(1 + 2\rho)\), which is positive for all \(\rho > -1/2\). Hence, for \(\theta_a < 0\), the numerator, i.e., \(\rho (w_1^2 - w_2^2) + w_1^2 < 0 \Rightarrow \rho > \frac{w_1^2}{w_2^2 - w_1^2}\), assuming \(w_2 > w_1\) (such as the case of \(w = (0.25, 0.75)\)). Let’s refer this cut-off value of \(\rho\) as \(\rho_c\). Thus, for this \(w\), the \(\theta_a\) values will be negative for \(\rho > 0.125\). Now, at \(\rho_c\), we can calculate \(\tan \beta_a\) (for AASF) from Eq. 14 as \(\frac{\rho w_2}{(1 + \rho)w_1} = w_1/w_2\). Hence the slope of the contour lines above the reference vector is \(-w_1/w_2\). The product of this slope with the slope of the reference vector \(w_2/w_1\) is -1, and hence these two are perpendicular.

Revisiting the range of angles formed by \(\theta\), we know that the angle between the contour line and the reference vector, \(\phi \leq \pi/2\), which occurs at \(\theta = 0\). Therefore, when \(\rho > \rho_c\), an angle greater than \(\pi/2\) is required to match the EPBI contour with AASF, for which \(\theta\) needs to go below 0. Similarly, when we consider the cases with \(w_1 > w_2\), \(\theta_a\) will be positive but \(\theta_b\) will have negative values beyond the corresponding cutoff value of \(\rho\).

The lower limits to which \(\theta_a\) and \(\theta_b\) curves asymptotically converge can be calculated by evaluating Eq. 10 at
Figure 7: Left column of subfigures shows the contours corresponding to PBI using θ ≈ 0 for the reference vectors $w_1 = (0.25, 0.75)$ and $w_2 = (0.5, 0.5)$ respectively. The right column shows the contours obtained using AASF for $ρ ≈ ∞$. AASF contours are normal to the reference vector only when $w_1 = w_2$. $ρ → ∞$, which comes out as $θ_a(\text{min}) = (w_1 - w_2)/2w_1w_2 = -4/3$ and $θ_b(\text{min}) = (w_2 - w_1)/2w_1w_2 = 4/3$.

Next, we look at the variation of $ρ_a$ and $ρ_b$ (of EAASF) as a function of $θ$ (of PBI) for $w = (0.25, 0.75)$, shown in Fig. 12. This time it can be seen that while $ρ_a$ for all $θ$, $ρ_b$ takes very large positive and negative values (theoretically $+∞$ and $−∞$) close to a certain value of $θ$ (let’s refer to this as $θ_c$). Once again, to understand these different behaviors, we refer back to the expressions of $ρ_a$ and $ρ_b$ in Eq. 11. A singularity will occur when the denominator term goes to 0. Let’s consider again first the case where $w_1 < w_2$. Now, when $w_1 < w_2$ it would imply that $θ_c < 0$. However, the assumed range of $θ$ is $[0, ∞)$, and therefore this situation will not arise. However, the same when applied to $ρ_b$ will yield $θ_c = w_2 - w_1/2w_1w_2$, which is a positive value in the assumed range of $[0, ∞)$. Therefore, as $θ$ approaches $θ_c$ from the left side, $ρ_b$ assumes a large negative value, whereas when approaches $θ_c$ from the right side $ρ_b$ assumes a large positive value. At $θ = 0$, from Eq 11 we have $ρ_a = 0.125$, $ρ_b = 1.125$. As $θ → ∞$ both $ρ_a$ and $ρ_b$ asymptotically go to $-1/2$. 
Figure 8: Left column of subfigures shows the contours corresponding to PBI using $\theta \approx \infty$ for the reference vectors $w_1 = (0.25, 0.75)$ and $w_2 = (0.5, 0.5)$ respectively. The right column shows the contours obtained using AASF for $\rho \approx -0.5$. The contours are identical in both cases.

4 Experimental Validation

4.1 Baseline Framework

In order to demonstrate the analytical formulae developed in the previous section, we conduct experiments using the Non-dominated Sorting Genetic Algorithm III (NSGA-III) \[10\]. The canonical framework of NSGA-III is similar to its precursor NSGA-II \[11\], but the key difference lies in the selection of points from the last candidate front for survival. At any given generation $t$, parent population $P_{t+1}$ of size $N$ that undergoes recombination is first identified using binary tournament selection. Then, $P_t$ is used to generate the child population $C_{t+1}$ through the crossover and mutation operators. This yields a total of $2N$ solutions which undergo non-dominated sorting, which assigns a rank $j$ to each solution where $j = 1, 2, \ldots, J$, where $J$ is the total number of fronts. Let’s say the number of solutions in $j^{th}$ front is denoted by $n_j$. Then, the front $j_{\text{min}}$ is identified such that $S = \sum_{j=1}^{j_{\text{min}}} n_j \geq N$. If $S = N$, then all the solutions in the first $j_{\text{min}}$ fronts are selected to move on to the new generation, and the process repeats. However, if $S > N$, then $N - (S - n_{j_{\text{min}}})$ solutions need to be chosen from $j_{\text{min}}$ front to constitute the next generation population with $N$ members. In NSGA-II, this is accomplished by sorting the solutions in this front using crowding distance; whereas in NSGA-III, a reference point based strategy is adopted. A uniformly distributed set of reference points are generated on a hyperplane using the systematic sampling \[7\]. Then, the points are associated to the reference lines formed by joining
Figure 9: (a) Variation of $\phi$ (holds regardless of $w$) for PBI (b) Variation of $\phi_a$, $\phi_b$ for $w = (0.25, 0.75)$ for AASF.

Figure 10: Range of achievable contour angles using PBI and AASF.

the above reference points with the ideal point based on their proximity (normal distance $d_2$) to these lines. This creates niches of points associated to certain reference directions; while some other directions may end up being “empty”. The solutions assigned to any given niche are ranked based on the their closest perpendicular distance to the associated reference direction, and the one with the least $d_2$ is considered the selected point for that direction. Using this ranking process, the required number of surviving solutions are selected. The detailed discussion of each component and considerations around their algorithmic design is discussed in detail in [10] and hence not repeated here for brevity.

In order to demonstrate the ideas discussed in Section 3, very simple changes are required in the above framework to create the following variants:

- **NSGA-III_{PBI}**: In this variant, PBI measure is used instead of $d_2$ for the niche preservation operation described above.
- **NSGA-III_{EAASF}**: In this variant, AASF is used with the $\rho$ values calculated to match the prescribed $\theta$ in NSGA-III_{PBI}.
- **NSGA-III_{AASF}**: In this variant, AASF measure is used instead of $d_2$ for the niche preservation operation
Figure 11: The variation of $\theta_a, \theta_b$ as a function of $\rho$ for the reference vector $w = (0.25, 0.75)$.

Figure 12: The variation of $\rho_a, \rho_b$ as a function of $\theta$ for the reference vector $w = (0.25, 0.75)$. Singularity occurs at $\theta = 4/3$.

- NSGA-III_{EPBI}: In this variant, PBI is used with the $\theta$ values calculated to match the prescribed $\rho$ in NSGA-III_{PBI}.

### 4.2 Experimental Setup and Results

All the algorithm variants have been set up in the PlatEMO framework [27]. The crossover and mutation probabilities are set to 1 and $1/n$ respectively, where $n$ denotes the number of variables in the problem. The SBX crossover index and polynomial mutation index are both set to 20. A population size of 40 is evolved for a maximum of 20,000 function evaluations in each run. The statistics for each variant are computed across 21 independent runs, and the performance is quantified using the hypervolume [34] metric. The reference point for HV computation is taken as $1.1^M$ in the normalized space as per the common practice. A total of twelve commonly used multi-objective problems are selected from the literature to conduct the experiments, which include ZDT1-3 [33], bi-objective DTLZ1-3 [13] and bi-objective WFG4-9 [16]. Two separate sets of experiments are performed:

- **Experiment 1**: In this experiment, NSGA-III_{PBI} is run on all problems with $\theta$ set to the often used value of 5. Thereafter, corresponding to each run, NSGA-III_{EAASF} is run with the same initialization.
• **Experiment 2:** In this experiment, NSGA-III_{AASF} is run on all problems with \( \rho \) set to a small value of 0.1. Thereafter, corresponding to each run, NSGA-III_{EPBI} is run with the same initialization.

The results for the first experiment are shown in Table [1]. It can be observed that the results of baseline NSGA-III shows different statistics compared to NSGA-III_{PBI}. However, between NSGA-III_{PBI} and NSGA-III_{EAASF}, the statistics are almost identical. The last column denotes the number of times the exact same final population was obtained using the two approaches. It can be seen that in most cases, all runs are identical. We observed a few exceptions where during one or two runs the results deviated very slightly. This observation is attributed to precision of calculations; where the ranks of the solutions changed due to a very small difference, e.g., around 10\(^{th}\) decimal place. A visual comparison of the final populations from a typical run of baseline NSGA-III, NSGA-III_{PBI} and NSGA-III_{EAASF} is shown in Fig. [13] which reiterates the observations made numerically. These experiments confirm that the equivalent formulation of AASF works the same as PBI. The best median hypervolume obtained among the three variants is indicated in bold. The baseline NSGA-III shows better median values in majority of instances. However, this observation is indicated is for information only, and the experiment itself is not meant to compare the different scalarization metrics in terms of how efficient they are for the given parameter values.
<table>
<thead>
<tr>
<th>Prob.</th>
<th>Baseline NSGA-III</th>
<th>NSGA-III with PBI, $\theta = 5$</th>
<th>NSGA-III with EAASF</th>
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Similarly, the results from the second experiment are shown in Table 2. The observations are much the same, with most of the runs being identical using NSGA-III_AASF vs. NSGA-III_EBI. The corresponding final populations are shown in Fig. 14 for visual comparisons. These experiments confirm that the equivalent formulation of PBI works the same as the use of AASF.
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Table 2: Hypervolume statistics over 21 runs obtained using Baseline NSGA-III, NSGA-III with AASF with $\rho = 0.1$ and NSGA-III with EPBI.
4.3 Further Discussion and Extensions to Larger Number of Objectives

The numerical experiments above clearly demonstrate that it is possible to find equivalent AASF for a given PBI and vice versa, and the expressions are dependent only on the vector $w$ (apart from $\theta$ or $\rho$). The implication of this for algorithmic design is that either of these metrics could be easily adapted to control the search behavior and exploit each other’s advantages; beyond what they have traditionally used for. For example, typically $\theta$ is considered as a penalty parameter in PBI and hence always considered to be positive. But as shown in Section 3, using a negative $\theta$ value (up to a limit) can expand its contour angles further to include more solutions in comparison. This is likely to be beneficial for obtaining the extreme solutions of the Pareto front,
as very small contour angles may rank such solutions poorly. This aspect has never crossed EMO researchers’ minds so far. Similarly for AASF, its large contour angles with positive $\rho$ are likely to fare better with regards to the extreme solutions, but to make the selection pressure stronger in other parts of the front, the parameter $\rho$ can be modified to create smaller angles. This aspect was also not suggested or used anywhere by the MCDM researchers.

Unfortunately the equivalence derived in this study is difficult to extend to three or more objectives exactly as was possible to achieve for two objectives, but perhaps approximate relations could still be established. In a two-objective space, both measures have contours that comprise straight lines, which makes it possible to find relevant parameters and conditions of equivalence, as demonstrated in Section 3. However, in higher dimensions (say three-objective), the values of PBI will be constant over a cone with the reference vector as its central axis. On the other hand, the AASF will be an intersection of multiple planes, asymmetric around the reference line, except for when weights corresponding to each objective are the same. The contour shapes for AASF with $\rho = 0.1$ and PBI with $\theta = 5$ are shown in Fig. 15.

Figure 15: The contour shapes for AASF ($\rho = 0.1$) and PBI ($\theta = 5$) for 3-objective case for reference vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Evidently, no values of $\theta$ and $\rho$ will be able to match these two exactly. However, it may still be possible to find equivalent formulations along certain directions. Given that PBI contours are always symmetric, we derive below the equivalence for the unique case where AASF is also symmetric, i.e., when the reference vector is $(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$. We aim to match the contours along the center-lines of the planes enclosed between the point and the axis system, shown as dotted lines in Fig 15. For ease of reference, we label the vertices of the AASF surface as follows: $P_1, P_2, P_3$ are the points on the $f_1, f_2, f_3$ axes, respectively; $P_{12}, P_{23}, P_{31}$ are the points along the $f_1-f_2, f_2-f_3, f_3-f_1$ planes respectively, and $K$ is the point along the reference vector. For a given AASF value $a$ and a given $\rho$, the co-ordinates of these points can be calculated by substituting the known values in Eq. 3 as: $K = \left(\frac{a/3}{1+3\rho}, \frac{a/3}{1+3\rho}, \ldots, \frac{a/3}{1+3\rho}\right)$, $P_1 = \left(\frac{a/3}{1+\rho}, 0, 0\right)$, $P_2 = \left(0, \frac{a/3}{1+\rho}, 0\right)$, $P_3 = \left(0, 0, \frac{a/3}{1+\rho}\right)$, $P_{12} = \left(\frac{a/3}{1+2\rho}, \frac{a/3}{1+2\rho}, 0\right)$, $P_{23} = \left(0, \frac{a/3}{1+2\rho}, \frac{a/3}{1+2\rho}\right)$, $P_{31} = \left(\frac{a/3}{1+2\rho}, 0, \frac{a/3}{1+2\rho}\right)$.

Now, let us consider the direction $P_3K$, along which we want to have equivalent PBI to AASF. The angle $\phi$ of this direction with $w$ can be calculated as shown in Eq. 15.
\[
\cos \phi = \frac{\mathbf{P}_3 \mathbf{K} \cdot \mathbf{w}}{||\mathbf{P}_3 \mathbf{K}|| \times ||\mathbf{w}||},
\]
where
\[
\mathbf{P}_3 \mathbf{K} = \left( \frac{a/3}{1 + 3\rho} (\mathbf{i} + \mathbf{j} + \mathbf{k}) \right) - \left( 0\mathbf{i} + 0\mathbf{j} + \frac{a/3}{1 + \rho} \mathbf{k} \right)
\]

(\(\mathbf{i}, \mathbf{j}, \mathbf{k}\) are unit vectors along \(f_1, f_2, f_3\) axis respectively)

\[
\Rightarrow \cos \phi = \frac{2}{\sqrt{3(6\rho^2 + 4\rho + 2)}} \Rightarrow \tan \phi = \frac{1 - \cos^2 \phi}{\cos^2 \phi} = \frac{1 + 3\rho}{\sqrt{2}}.
\]

Also, from the discussion in Section [1], we know that \(\tan \phi = 1/\theta\). Therefore, equating the two, we can find the relation between \(\theta\) and \(\rho\) as shown in Eq. (16). For \(\rho = 0.1\), the equation yields \(\theta = 1.0879\), corresponding to which the contour surfaces can be seen in Fig. 16(a). Similarly, for a smaller angle with the reference vector, \(\rho = -0.1\) is used, with corresponding \(\theta\) calculated as 2.0203, the contours for which are shown in Fig. 16(b).

\[
\tan \phi = \frac{1 + 3\rho}{\sqrt{2}} \Rightarrow \theta = \frac{\sqrt{2}}{1 + 3\rho}.
\]

Figure 16: Equivalent \(\theta\) to match the contours along center-lines of AASF surfaces for symmetric reference vector \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\).

It can be also noted that in this case \(\theta = \infty\), i.e., the contours parallel to the reference vector, will correspond to \(\rho = -1/3\) (it was \(\rho = -1/2\) for 2-objective case). In fact, by following the exact same process for an \(M\)-objective problem, Eqs. (15-16) can be generalized as shown in Eq. (17). Correspondingly, \(\phi = -1/M\) gives the contours parallel to the reference vector, forming a prism with reference vector as its central axis. This corresponds to \(\theta = \infty\) in a general case, whose contour surface will be cylindrical with reference vector as its central axis. Also, \(\rho = \infty\) corresponds to the contour places perpendicular to the reference vector, i.e., equivalent to PBI with \(\theta = 0\).
\[
\cos \phi = \frac{M - 1}{\sqrt{M(M - 1)(\rho^2 M + 2 \rho + 1)}},
\]
\[
\tan \phi = \sqrt{\frac{1 - \cos^2 \phi}{\cos^2 \phi}} = \frac{1 + M \rho}{\sqrt{M - 1}} = \frac{1}{\theta},
\]
\[\Rightarrow \theta = \frac{\sqrt{M - 1}}{1 + M \rho}, \quad \text{and } \rho \text{ can also be back-calculated as:}
\]
\[
\rho = \frac{1}{M} \left( \frac{\sqrt{M - 1}}{\theta} - 1 \right).
\]

The variation of \(\theta\) vs \(M\) for different \(\rho\) values and vice versa can be visualized as shown in Fig. 17. It can be observed that for a fixed low or negative values of \(\rho\), the value of equivalent \(\theta\) must monotonically increase with an increase in number of objectives \(M\). For achieving a similar AASF-based selection effect, PBI-based EMO researchers must use this principle. For high values of \(\rho\) (often not recommended or used in MCDM literature), the behavior is reversed, and \(\theta\) must be closer to zero, and interestingly must be reduced with an increase in \(M\).

For the case of \(\rho\) variation with \(M\) for a given \(\theta\) (right plot), it can be seen that equivalent \(\rho\) must be chosen differently. For usual \(\theta\) values (1 to 5), the equivalent \(\rho\) must be increased with an increase in \(M\). For achieving a similar PBI-based selection effect, MCDM researchers and AASF users of EMO field must use such a \(\rho\) variation. Note that for the symmetric reference direction, \(\theta\) does not need to assume negative values to match the PBI contours with AASF, but the \(\rho\) needs to become negative to match the PBI with \(\theta\) values (i.e. small cone angles).

![Graphs showing \(\theta\) vs \(M\) and \(\rho\) vs \(M\) for different values of \(\theta\) and \(\rho\).](image)

**Figure 17: Variation of \(\theta\) with \(\rho\) and vice versa with respect to the number of objectives \(M\) for the reference vector \(w = \left(\frac{1}{M}, \frac{1}{M}, \ldots, \frac{1}{M}\right)\).**

Lastly, given that the AASF isosurfaces for a generic reference vector (where all weights are not equal) are asymmetric about it. Therefore, if they are to be made symmetric, then the AASF formulation itself will need to be modified such that the angles made by the different isoplanes are the same from the reference vector. This would require modification of the original form of AASF itself. This investigation is left as a future work.

## 5 Concluding Remarks

In this paper, we have investigated behaviors of two scalarization measures from different but related origins, namely, AASF (with roots in MCDM) and PBI (with roots in EMO). Both of these measures involve a user-
defined parameter each and a reference direction introduced in order to affect the search behavior. However, these parameters have been used in limited ranges, which in-turn also prevents from realizing the full potential of their use on the search that they are capable of. Moreover, for bi-objective problems, for certain parameter settings, they can also be considered equivalent. In this paper, we have derived these relations analytically which resulted in developing exact expressions for an equivalent AASF for a given PBI, and vice versa. An interesting outcome of this study is that the equivalence relations have clearly indicated that these parameters could be used beyond their conventional bounds, and these new bounds have also been studied here. Using the updated formulation has provided more flexibility and control over the search behavior, and has provided insights for designing future algorithms to solve MOPs. Numerical experiments on two-objective test problems have been conducted on a range of problems to validate the theory developed. While the studies presented herein have been conducted on two-objective problems, we have derived a relationship between AASF and PBI parameters for higher dimensions for a specific reference direction. Its further generalization to other reference directions and related research will be explored by the authors in the future extensions of this study. Such unified studies connecting different algorithmic aspects of two contemporary fields should stay as a beneficial contribution to both fields.

Acknowledgment

The authors would like to acknowledge Endeavour Fellowship from Australian Department of Education and Training and Discovery Project grant DP190102591 from the Australian Research Council.

References

References


