Best Order Sort: A New Algorithm to Non-dominated Sorting for Evolutionary Multi-objective Optimization

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Finding the non-dominated ranking of a given set vectors has applications in Pareto based evolutionary multi-objective optimization (EMO), finding convex hull, linear optimization, nearest neighbor, skyline queries in database and many others. Among these, EMOs use this method for survival selection. Until now, the worst case complexity of this problem found to be $O(N\log^{M-1} N)$ where the number of objectives M is constant and the size of solutions N is varying. But this bound becomes too large when M depends on N. In this paper we have proposed an algorithm with average case complexity $O(MN\log N + MN^2)$. This algorithm can make use of the faster implementation of sorting algorithms. This approach removes unnecessary comparisons among solutions and their objectives which improves the runtime. The proposed algorithm is compared with four other competing algorithms on three different datasets. Experimental results show that our approach, namely, best order sort (BOS) is computationally more efficient than all other compared algorithms with respect to number of comparisons and runtime.

1 Introduction

In many fields of study such as evolutionary multi-objective optimization, computational geometry, economics, game theory and databases, the concept of non-dominated ranking

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or Pareto set is used. By definition, a vector valued point or solution $A = (a_1, a_2, \ldots, a_M)$ dominates another solution $B = (b_1, b_2, \ldots, b_M)$ by Pareto-dominance relation if A is better or equal in each dimension or objective than that of B. In other words, $a_i \geq b_i \ \ \forall i = 1, \ldots, M$. Here M is the number of objectives or the dimension of a vector. Given a set of solutions P, finding the solutions which are not dominated by any other solutions in that set is called the problem of finding Pareto set and these solutions are denoted as rank 1 solutions. If we remove rank 1 solutions from S and find the Pareto set again we will end up finding rank 2 solutions. We can perform this process repeatedly until all the solutions are ranked. This process is called non-dominated sorting or ranking. Each solution having rank $r > 1$, is dominated by at least one solution of rank $\{(r-1), \ldots, 1\}$. In Fig. 1 there are four non-dominated fronts: $\{a, b, c, f\}, \{h, e\}, \{g\}$ and $\{d\}$ of rank 1, 2, 3 and 4 respectively.

Some of the most important EMO algorithms use the idea of non-dominated ranking for their survival selection. Among them, we can refer NSGA [23], NSGA-II [7], SPEA2 [28], MOGA [21], NPGA [14], PAES [17], MOPSO [5] and recently published NSGA-III [6], DM1 [1] and EPCS [22]. Non-dominated ranking takes most of the time of these optimization algorithms. So it is very important to find efficient algorithm for ranking.

The rest of the paper is organized as follows. Section 2 discusses about some of the approaches for finding Pareto set and non-dominated sorting. Time complexities of those approaches are also discussed. In Section 3, we talk about the main idea behind the proposed algorithm. Its proof of correctness, best case and worst case time complexity and space complexity is also presented in that section. The comparison with four well known algorithms are presented in Section 4 and results are discussed in Section 5. Section 6 concludes the paper with overall remarks and future work.

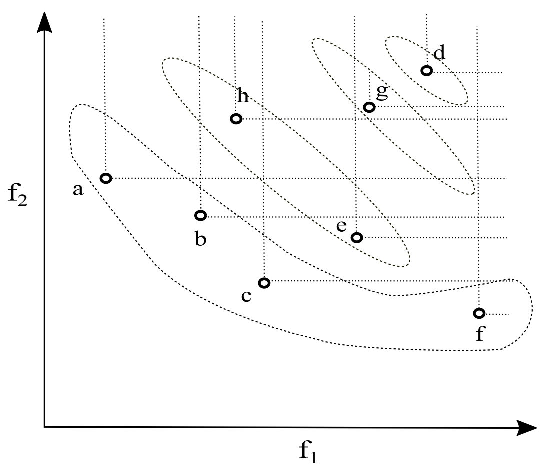


Figure 1: An example with eight points in a two dimensional minimization problem. It has four fronts $\{a, b, c, f\}$, $\{h, e\}$, $\{g\}$ and $\{d\}$ which should be found by a non-dominated sorting algorithm.

2 RELATED WORK

The methods for finding Pareto set and non-dominated sorting can be divided into two categories– sequential and divide-and-conquer approach. Given a set of vectors, sequential brute force method for finding the Pareto set is to compare each solution to every other solution to check whether they dominate each other. If either of them is dominated, then that is removed from current set. It can be used to find the non-dominated ranking by repeating the process and removing the ranked solutions or points from the set [23]. The algorithm has $O(MN^3)$ complexity because of repeated comparisons. Due to high computational complexity of [23], Deb *et al.* [7] described a computationally faster version, termed fast non-dominated sort, whose complexity is $O(MN^2)$. It uses the fact that pairwise comparisons can be saved and used later to find the rank of solutions other than the Pareto set. However, space complexity is $O(N^2)$ because it saves pairwise comparisons.

McClymont and Keedwell [19] described a set of algorithms which improves space complexity to $O(N)$ and has time complexity $\emptyset(MN^2)$. Among them deductive sort is reported to work best. The algorithm consists of multiple passes and one pass is completed by removing dominated solutions with an arbitrary unranked solution. Corner sort [25] is a new approach for finding non-dominated ranking of vectors which works similar to deductive sort. But instead of choosing an arbitrary vector for checking dominance, it always chooses a vector which is guaranteed to be in current rank. So it works better than deductive sort in some cases but has the same worst case complexity. Recently, an efficient approach of non-dominated ranking called ENS [27] has been described. This method uses the idea of sorting the solutions by the values of first objective with in-place heap sort. In case of tie, the authors use lexicographic ordering. This algorithm can achieve best case complexity $O(MN \log N)$ although it has $O(MN^2)$ in worst cases. Other methods e.g. dominance tree based non-dominated sorting [10], non-dominated rank sort or omni-optimizer [9] and arena's principle [24] can also be used to improve the best case time complexity upto $O(MN\sqrt{N})$. However, the worst case time complexity remains the same as $O(MN^2)$. Recently, parallel GPU based NSGA-II algorithm [13] has been proposed to speed up the non-dominated sorting and other steps of the evolutionary algorithm.

Unlike the sequential algorithms, the set of divide-and-conquer algorithms work by repeatedly dividing the data using objective values. These methods are asymptotically faster than sequential ones in the worst case for fixed number of objectives. The first divide-and-conquer method was proposed by Kung *et al.* [18] for finding the Pareto set. This method is later analyzed by Bentley [2]. This algorithm divides the data and reduces dimensions recursively. At first, it divides input set into two halves by the median of their first objective values. If size of the set is still more than one and if there is unused dimension left, then the set is divided again with unused dimension. The division goes on until data can no longer be divided or unused dimensions are reduced to two. When the dimension becomes less than 3, special-case algorithm is applied for ranking which has complexity $O(N \log N)$. If the dimension is not fixed then its complexity is bounded by $O(MN^2)$ [4, 20]. These algorithms exhibit many unnecessary comparisons which increases with the number of objectives [12]. The space complexity is said to be $O(N)$ for Kung's algorithm. Bentley [3] improved the average case of this divide-and-conquer algorithm. It assumes the fact that size of Pareto set of vectors is equal to $O(\log^{M-1} N)$ on average. One can find the non-dominated ranking by repeating Kung's algorithm the number of times equal to number of ranks which gives complexity $O(N^2 \log^{M-2} N)$. By removing the repeated comparisons, Jensen [16] and Yukish [26] both extended Kung's algorithm to find the Pareto set of vectors and perform non-dominated ranking in time $O(N \log^{M-1} N)$. Jensen's algorithm assumes that, for any objective, no two vectors have the same objective value [10]. Because of this assumption it generates different Pareto ranking from the baseline algorithm of NSGA-II [7]. It was corrected later in [4, 11]. Buzdalov *et al.* proved the time complexity of non-dominated ranking to be $O(N \log^{M-1} N)$ for fixed dimension. Our aim is to find better algorithms in terms of N and M both.

3 PROPOSED METHOD

3.1 Basic Idea

In this section we propose an algorithm named best order sort (BOS) which reduces the number of comparisons in the worst case. The main idea of the algorithm is described in Fig. 2. For each solution s, we can get a set for each objective that denotes the solutions which are not worse than s in that objective. So there will be M sets as there are M objectives. To find the rank of s, only one set T is sufficient to be considered. Some members $t \in T$ of the set dominate s while others are non-dominated with s. Suppose highest rank of T is r. The rank of s will then be $(r + 1)$. Although any of the M sets can be considered to find rank of s, our method finds the smallest set by sorting the population with their objective values.

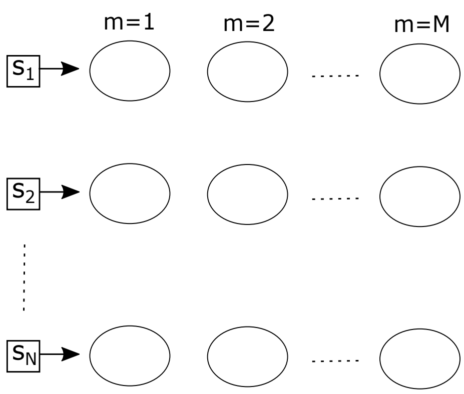


Figure 2: The basic idea of the proposed method is that there are M sets for each solution which denote the 'not-worse' solutions in corresponding objective. The algorithm finds the smallest set to compare and finds their ranks.

3.2 The Algorithm

The algorithm starts with initializing $N \times M$ empty sets denoted by L_i^j (see Algorithm 1). Here N is the size of population and M is the number of objectives. L_j^r denotes the set of solutions which has rank r and they are found in j-th objective. The algorithm saves sorted population in Q_j which denotes that j-th objective value is used for sorting. It maintains a objective list $C_u = \{1, 2, \ldots, M\}$ for each solution u. It signifies that, if we want to check whether a solution s is dominated by u then only the objectives in C_u needs to be compared. The variable $isRanked(s) = false$ denotes that solution s is not ranked yet. We initialize number of solutions done SC to zero, ranks of solutions $R(s)$ equal to zero for all $s \in P$ and fronts found so far $RC = 1$.

At first, we sort the solutions according to each objective j and put those into sorted list Q_j (see line 8 of Algorithm 1). We use lexicographic order if two objective values are same. In that case, if the first objective values are same then sorting will be based on the second objective value. Note that we just need to perform single lexicographic ordering for the first objective. We can then use the information of first objective to find lexicographic order of other objectives.

Algorithm 1: INITIALIZATION

Data: Population P of size N and objective M
Result: Sorted set of solutions in Q_j

```
1 L_j^i ← ∅,  ∀ j = 1,2,...,M,  ∀ i = 1,2,...,N// global variable
2 C_i ← {1,2,...,M}  ∀ i = 1,2,...,N// comparison set, global variable
3 isRanked(P) ← false// solutions ranked or not, global variable
4 SC ← 0// number of solutions already ranked, global variable
5 RC ← 1// number of fronts discovered so far, global variable
6 R(P) ← 0// Rank of solutions, global variable
7 for j = 1 to M do
8   | Q_j ← Sort P by j-th objective value, use lexicographic order in case of tie;
    | // lexicographic sort
9 end
```

Algorithm 2 describes the main procedure for finding the non-dominated ranking. It takes the lexicographically sorted population Q_j for each objective j. The algorithm starts taking the first element s from the sorted list Q_1 of first objective, which can be denoted by $Q_1(1)$. Then it excludes objective $\{1\}$ from the list C_s. This is because, if other solution t is compared with s later, t is already dominated in objective 1. Next, the algorithm checks whether s is already ranked or not. If it is ranked then it will be included to the corresponding list $L_1^{R(s)}$. For instance, if s has to be included in L_2^5, then s's rank is 5 and it is found in second objective. L_j is the set of all solutions that is needed to find rank of s (see Algorithm 3) because they are not worse than s in objective j. At the end of this algorithm, each solution should appear in every objective set L_j only once, if line 14 of Algorithm 2 is not executed.

If the solution s is not ranked then FindRank(s, j) procedure is called. It finds the rank of s in $R(s)$. After returning from this method, we assign $isRanked(s)$ to be true so that it never gets ranked again. We increment the number of solutions done (SC) by 1. The algorithm then goes through the next objective to find the next solution of corresponding list Q_j. After finishing loop at line 2-12, the algorithm then checks if number of solutions done is equal to total population N. If it is not, then it takes the next element from lists Q_j. Once all solutions are ranked it breaks out of the loop. Note that, a solution s is ranked when it is observed first time in any list Q_j. Therefore, it guarantees to compare to the smallest set of solutions which are in L_j^r for all ranks r discovered so far in that particular objective j.

Algorithm 2: Main Loop

Data: Sorted Population, Q
Result: Rank of each solution, R
1 **for** $i = 1$ *to* N **do** `// for all solutions`
2   **for** $j = 1$ *to* M **do** `// for all sorted set`
3     $s \leftarrow Q_j(i)$ `// Take` i`-th element from` Q_j
4     $C_s \leftarrow C_s - \{j\}$`// reduce comparison set`
5   **if** $isRanked(s) = True$ **then**
6       $L_j^{R(s)} = L_j^{R(s)} \cup \{s\}$`// Include s to` $L_j^{R(s)}$
7   **else**
8   FindRank(s, j)`// Find` $R(s)$
9       $isRanked(s) \leftarrow True$`// non-dominated ranking done`
10       $SC \leftarrow SC + 1$`// total done`
11   **end**
12   **end**
13   **if** $SC = N$ **then** `// if all solutions are done`
14   break`// ranking done`
15   **end**
16 **end**
17 **return** R`// return`

Given the set of solutions already discovered in objective j in sets L_j^r, we can use Algorithm 3 to find out the rank of the solution found in objective j. Suppose s is discovered first time in objective j. The algorithm starts by comparing s with all the solutions t of first rank $L_j^k (k = 1)$. If s is not better in the objectives defined in C_t then it is dominated by t (see Algorithm 4). Then s will be compared to the next rank solutions L_j^{k+1}. If s is not dominated by any solution of some rank k, then its rank will be k. If s is dominated by at least one solution of all ranks then s is discovering a new front and its rank will be $RC + 1$. At the end of this procedure we will find rank of s, update rank count RC and update set $L_j^{R(s)}$. Algorithm 4 describes the procedure of checking Pareto-domination with sets C. The algorithm finishes execution as soon as all the solutions are ranked. In the next section, we will see an example describing this algorithm.

Algorithm 3: FindRank

```
Data: Solution s, List number j
Result: Rank of s
done = False// done bit
for k = 1 to RC do // for all discovered ranks
    check = False// check bit
    for t ∈ L_j^k do // for all solutions in L_j^k
        check ← dominationCheck(s,t)// domination
        if check = True then // if dominated
            break// break the loop
        end
    end
    if check = False then // rank found
        R(s) ← k// update rank
        done = True// update done bit
        L_j^{R(s)} = L_j^{R(s)} ∪ {s}// Include s to L_j^{R(s)}
        break// break the loop
    end
end
if done = False then // if not done
    RC ← RC + 1// update fronts count
    R(s) ← RC// update rank
    L_j^{R(s)} = L_j^{R(s)} ∪ {s}// Include s to L_j^{R(s)}
end
return
```

Algorithm 4: dominationCheck

```
Data: Solution s and t
Result: True if t dominates s, false otherwise
for j ∈ C_t do // for all objectives in C_t
    if s is better than t in objective j then
        return False// t cannot dominate s
    end
end
return True// t dominates s
```

3.3 Illustrative Example

In Fig. 1, population with eight individuals (namely a, b, c, d, e, f, g and h) are shown graphically in a two objective minimization problem. The algorithm begins sorting the solutions by objective f_1 and f_2 and put them into Q sets (Fig. 3). After sorting, the algorithm takes the elements in this order $a, f, b, c, h, e, c, b, e, a, g, h, d$ (see Fig. 3) to find their ranks. Solutions are ranked only when they are discovered for the first time in line 3 of MAIN LOOP. The ranked solutions are distributed in different sets L_1^1, L_1^2, L_2^1 etc. For solution d, it will be compared with a, b, h, c, e and g which belongs to L_1^1, L_1^2 and L_1^3 (see Fig. 4). By FINDRANK method, d will be compared to L_1^1 first and then L_1^2 and finally L_1^3. L_1^4 will be discovered by d. While going over the solutions, the algorithm drops the corresponding objective from the comparison list (C) of that solution. C entries in four successive steps are shown in Fig. 5. After first step, objective 1 from a and objective 2 from f is dropped. At the end of 4th step, C_b and C_c becomes empty. It means that any solution which is discovered after this step will be considered dominated by b and c.

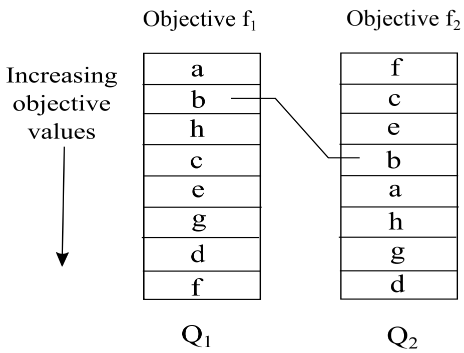


Figure 3: Sorted list of population in different objectives. For solution b, it will be compared to only $\{a\}$, if we use objective f_1. The set becomes larger with $\{f, c, e\}$ when considering f_2. The proposed algorithm will use $\{a\}$ set because b will be found in f_1 for the first time.

3.4 Correctness

Lemma 1. *The proposed algorithm finds the correct rank of a solution $s \in P$.*

Proof. From the definition of non-dominated sorting it is clear that if a solution s is dominated by a set of points $u \in U$ then the rank of s is one plus maximum rank of U. When a solution is ranked (line 8, Algorithm 2) then it is only compared to L_j sets of solutions. L_j sets contain only those solutions which are not worse than s in objective j. Because we know that only L_j solutions can dominate s, the algorithm FINDRANK ensures that s gets the rank one plus the highest rank of L_j. Inside FINDRANK algorithm, if we find that s is dominated by a solution of rank $k = 1$ then the algorithm moves to higher ranks ($k > 1$) to check for domination. Once a rank is found where no other

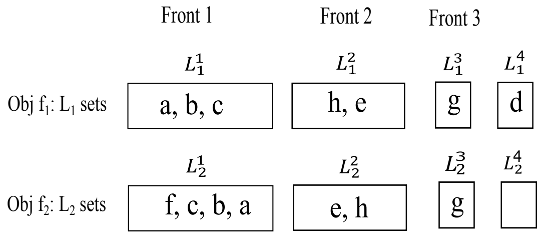


Figure 4: Four different fronts are discovered by the algorithm which are again distributed in four different fronts. Rank 1 solutions are given by union of L_1^1 and L_2^1 sets etc. The algorithm exits as soon as all the solutions are ranked, otherwise it would have been true that $L_1^1 = L_2^1$, $L_1^2 = L_2^2$ and so on.

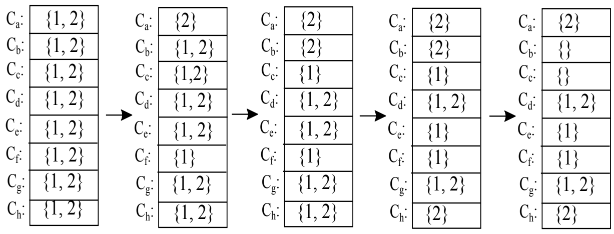


Figure 5: Four steps execution of the loop at line 1 in Algorithm 2 is shown. Dropping some objectives from the list indicates that a solution discovered later is already dominated on those objectives.

solution of that rank dominates s, the correct rank of s is identified as the same rank as of those. If s is found to be dominated by at least one solution of each rank found so far, a new rank $(RC + 1)$ is introduced with the solution s. In each case, comparing solutions of s have smaller set of objectives to compare. The objectives in which s is already dominated, are dropped from the objective list. Therefore each comparison is correct. Thus each solution finds its rank correctly. □

3.5 Best Case Complexity

Best case for this algorithm happens when the population has N fronts, each front having only one solution. In this case, we will get the similar order in all Q_j after sorting by Algorithm 1. While executing line 2 of Algorithm 2 with $j = 1$ up to $j = M$, all the objectives will be deleted (see line 4) from the objective list C_s. So there will be no objective value comparison. Total execution time for line 1 and 2 of Algorithm 2 is $O(MN)$. Therefore we get the best case time complexity $O(MN \log N)$.

3.6 Average Case Time and Space Complexity

Theorem 1. *Under independence assumption, in average case, best order sort has time complexity is $O(MN \log N + MN^2)$. Its space complexity is $O(MN)$.*

First we assume that solutions are independently distributed along the objectives. The assumption is given below.

Definition 3.1 (Independence Assumption)**.** *Distribution of one objective values of the population is statistically independent of the distribution of other objective values.*

Proof. Average case complexity depends on number of solutions to compare and total number of objectives in objective lists C. We will do amortized analysis here. A solution is ranked when it is found first time in Algorithm 2 (line 3). In one row (row is found by line 2, for same i and all $j = \{1, 2, \dots, M\}$), only unique solutions, those previously not found, are ranked by Algorithm 3. Suppose, on average, p unique solutions are found in one row. As the size of M grows, there will be increase of unique solutions according to independence assumption. So p is a function of M i.e. $p = F(M)$. In each row, M objectives will be deleted. Deletion is done in $O(1)$ time with the help of $M \times N$ direct access pointers to the members of the lists. Suppose r denotes the row number (value of i) when Algorithm 2 terminates first loop (line 1) and breaks out of the loop at line 14. It means that, all the solutions are already discovered (appears first time) by row r from the sorted lists $Q_j,\ j \in \{1, 2, \dots, M\}$. In this case, $pr = N$ is a constraint that needs to be maintained. As we have seen from the algorithm, solution s is only compared to all the solutions found in that objective previously, s will be compared to $(r-1)$ solutions if it is found in row r. Total remaining objectives for p solutions will be $(Mp-M)$ after executing one row. So number of objective value comparisons is $((Mp-M)/p) \times (r-1)$. It should be multiplied by p as we will compare p solutions in a row. Total number of comparisons can be found if we execute the algorithm up to r row where $r = N/p$.

$$
\begin{aligned}
T(M, N) &= (Mp - M) \sum_{r=1}^{r=N/p} (r-1) \\
&= O(Mp) \times O\left(\frac{N^2}{p^2}\right) = O(MN^2/p)
\end{aligned}
$$

The equation holds for $2 \le p \le M$ and p is an integer. This is because if $p = 1$, then all the objectives in the list will be deleted per each row and best case of the algorithm is achieved. For $p = \sqrt{M}$ we get the complexity $O(N^2\sqrt{M})$. Worst case is achieved when $p = 2$. Then only two solutions are found in each row (Fig. 6) and $r = N/2$. The complexity becomes $O(MN^2)$ where independence assumption is violated. When p is a linear function of M then we achieve complexity $O(N^2)$ for ranking. If the Algorithm 2 finds unique solution each time line 3 is executed, then $p = M$ and $r = N/M$. The complexity becomes $O(N^2)$. Lexicographical sort (Algorithm 1) takes $O(MN \log N)$ time. So the time complexity of this algorithm is $O\left(MN \log N + MN^2\right)$ under independence assumption. If M number of processors are used for lexicographic sort, then

Table 1: Comparison of best and worst case time and space complexities of seven different non-dominated sorting algorithms

Algorithm	Best Case	Worst Case	Domination	Space Complexity	Parallelism
Best Order Sort	$O(MN \log N)$	$O\left(MN \log N + MN^2\right)$	One way	$O(MN)$	Yes
ENS-BS [27]	$O(MN \log N)$	$O(MN^2)$	One way	$O(N)$	No
ENS-SS [27]	$O(MN\sqrt{N})$	$O(MN^2)$	One way	$O(N)$	No
Corner Sort [25]	$O(MN\sqrt{N})$	$O(MN^2)$	One way	$O(N)$	No
Deductive Sort [19]	$O(MN\sqrt{N})$	$O(MN^2)$	Two way	$O(N)$	No
Jensen's algorithm [16]	$O(MN \log N)$	$O(MN^2)$	One way	$O(MN)$	No
Fast Non-dominated Sort [7]	$O(MN^2)$	$O(MN^2)$	Two way	$O(N^2)$	No

time complexity becomes $O(N \log N)$ for sorting and total time becomes $O(N^2)$. For sorted lists Q_j and objective lists C, size of memory is $M \times N$ in each case, thus its space complexity is $O(MN)$. Table 1 shows the algorithmic complexity of different methods. In the table, one way comparison means that, when two solution a and b is compared for ranking, it compares whether a dominates b or b dominates a. On the other hand, two way domination check needs to detect whether a dominates b and b dominates a. On average, one way domination check needs less number of comparisons than two way domination. $\square$

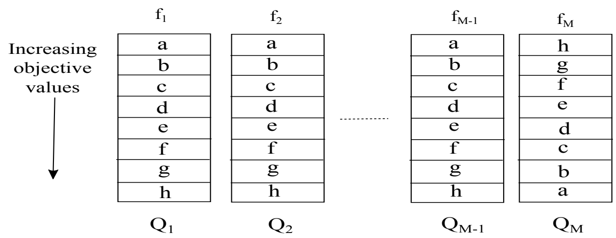
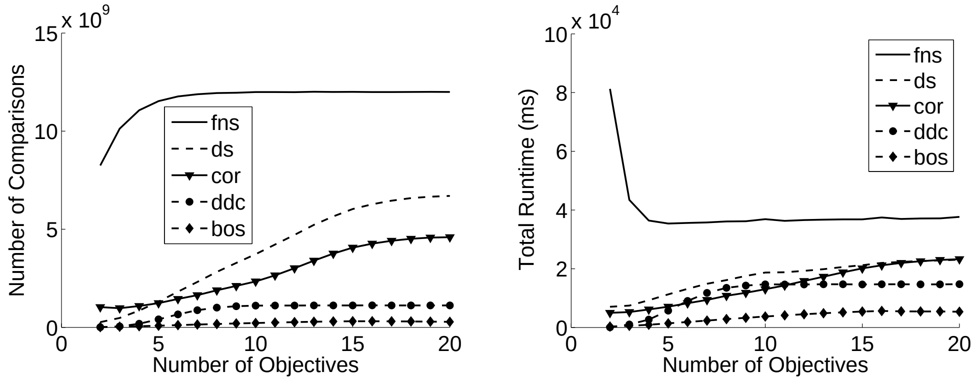


Figure 6: After sorting by Algorithm 1, the solutions get the same order in each objective except objective M. A solution (d) can be compared with at most $N/2 - 1$ other solutions ($\{a, b, c\}$).

4 Experimental Results

We compared the proposed algorithm with four different algorithms– fast non-dominated sort [7], deductive sort [19], corner sort [25] and divide-and-conquer algorithm [4]. These algorithms are compared in cloud dataset, fixed front dataset and dataset obtained from multi-objective evolutionary algorithm (MOEA). Cloud dataset is a uniform random data generated by Java Development Kit 1.8. Fixed front data is the dataset where number of fronts is controlled. We have used the procedure described in [25] for generating cloud and fixed front datasets. We vary size of population N from 500 to 10,000



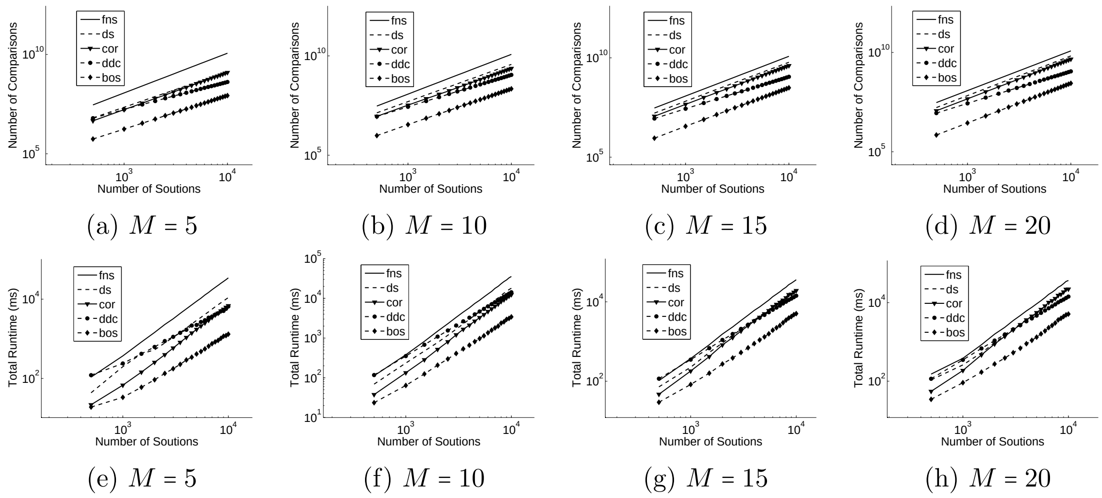
(a) Comparison (b) Time

Figure 7: Number of comparisons and runtime (in milliseconds) for cloud dataset of size 10,000 for increasing number of objectives. Results for fast non-dominated sort (fns), deductive sort (ds), corner sort (cor), divide-and-corner sort (ddc) and best order sort (bos) is shown.

with an increment of 500 in cloud dataset. In another test (Fig. 6), number of objectives are varied from 2 to 20 to evaluate performance with population size 10,000. For fixed front dataset, number of front is varied from 1 to 10 where number of solution is kept 10,000 with objectives 5, 10, 15 and 20. MOEA dataset is obtained by running 200 generations of NSGA-II algorithm in DTLZ1 and DTLZ2 [8], WFG1 and WFG2 [15] problems with 5, 10, 15 and 20 objectives. In these cases, all the parameter values are kept as standard ones. For example, simulated binary crossover with polynomial mutation are employed with probabilities 0.80 and (1/number of variables) respectively. Each algorithm is repeated 30 times in 30 different datasets to get the averages. All the algorithms are optimized and implemented in Java Development Kit 1.8 update 65 and run in Dell computer with 3.2 GHz Intel core i7 and 64 bit Windows 7 machine.

5 Discussion

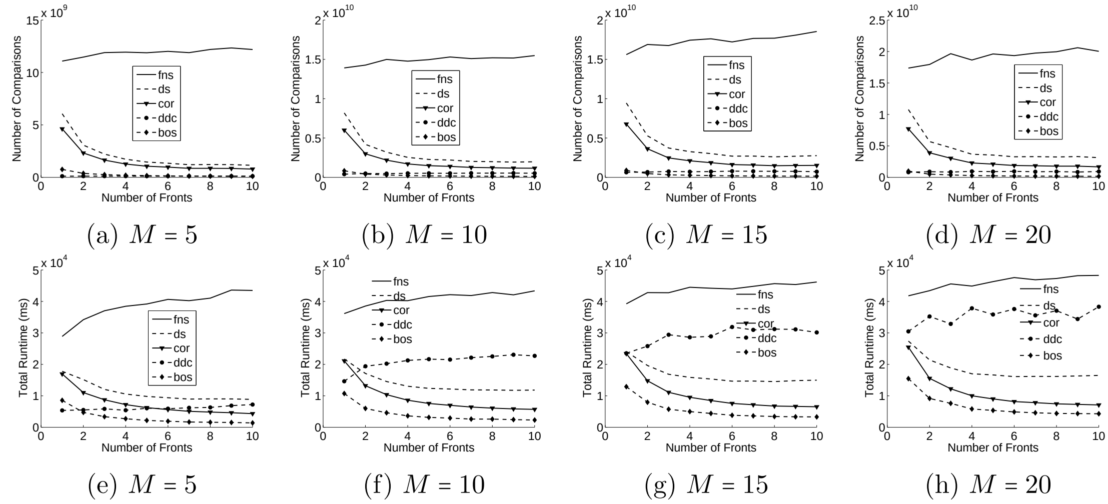
The results describe the average case behavior of the algorithms in three different cases. Fig. 7 shows that with increased number of objectives, number of comparisons and runtime increases for deductive sort, corner sort, divide-and-conquer sort and best order sort. Fast non-dominated sort performs worst in two objectives compare to other number of objectives. This is because, number of fronts is very high (see Fig. 7) in two objective random data and fast non-dominated sort takes most of the time (in milliseconds) just for saving dominated solutions in a list of size $O(N^2)$. This behavior is exhibited because of lower runtime (few hundred milliseconds) and large amount of memory accesses. Best order sort performs the best followed by divide-and-conquer, corner sort and deductive sort. Log-based plots in Fig. 8 show that fast non-dominated has the highest and best order sort has the lowest order in terms of number of comparisons and runtime in



(a) $M = 5$ (b) $M = 10$ (c) $M = 15$ (d) $M = 20$

(e) $M = 5$ (f) $M = 10$ (g) $M = 15$ (h) $M = 20$

Figure 8: Figure describes number of comparisons and runtime (in milliseconds) with increasing population size for cloud dataset in objectives 5, 10, 15 and 20. Results for fast non-dominated sort (fns), deductive sort (ds), corner sort (cor), divide-and-conquer sort (ddc) and best order sort (bos) is shown.



(a) $M = 5$ (b) $M = 10$ (c) $M = 15$ (d) $M = 20$

(e) $M = 5$ (f) $M = 10$ (g) $M = 15$ (h) $M = 20$

Figure 9: Figure describes number of comparisons and runtime (in milliseconds) of 10,000 solutions with increasing number of fronts for fixed front dataset in objectives 5, 10, 15 and 20. Results for fast non-dominated sort (fns), deductive sort (ds), corner sort (cor), divide-and-conquer sort (ddc) and best order sort (bos) is shown.

objectives 5, 10, 15 and 20. Corner sort performs better than deductive sort in terms of comparisons in most of the cases but the runtime performance deteriorates with increasing number of objectives. Divide-and-conquer algorithm performs better than all

Table 2: Total number of comparisons (#cmp) and running time (in milliseconds) of 200 generations of data for DTLZ1, DTLZ2, WFG1 and WFG2 problems in 5, 10, 15 and 20 objectives.

Test Problem	**Obj.**	FNS		DS		COR		DDC		BOS	
		#cmp	time(ms)	#cmp	time(ms)	#cmp	time(ms)	#cmp	time(ms)	#cmp	time(ms)
DTLZ1	5	3.25e+08	1.03e+03	1.02e+08	4.48e+02	7.12e+07	2.18e+02	9.41e+06	3.79e+02	**7.95e+06**	**1.06e+02**
	10	5.13e+08	1.21e+03	2.74e+08	7.33e+02	1.71e+08	4.05e+02	**1.44e+07**	5.03e+02	2.03e+07	**2.53e+02**
	15	7.09e+08	1.41e+03	4.23e+08	1.02e+03	2.67e+08	5.52e+02	**1.50e+07**	5.42e+02	2.86e+07	**3.87e+02**
	20	8.98e+08	1.59e+03	5.67e+08	1.21e+03	3.55e+08	6.56e+02	**1.56e+07**	5.55e+02	3.51e+07	**4.72e+02**
DTLZ2	5	2.97e+08	8.59e+02	1.24e+08	4.77e+02	8.22e+07	2.58e+02	**9.52e+06**	3.52e+02	1.07e+07	**1.17e+02**
	10	4.30e+08	1.11e+03	2.31e+08	6.82e+02	1.59e+08	4.36e+02	**1.55e+07**	5.46e+02	1.80e+07	**2.35e+02**
	15	5.58e+08	1.27e+03	3.31e+08	8.37e+02	2.20e+08	5.41e+02	**1.63e+07**	5.84e+02	2.20e+07	**3.15e+02**
	20	6.95e+08	1.40e+03	4.34e+08	1.02e+03	2.81e+08	6.36e+02	**1.65e+07**	5.97e+02	2.49e+07	**3.73e+02**
WFG1	5	2.67e+08	7.99e+02	1.12e+08	4.38e+02	6.59e+07	2.44e+02	**9.89e+06**	3.53e+02	1.11e+07	**1.18e+02**
	10	2.95e+08	9.30e+02	1.47e+08	5.26e+02	1.03e+08	3.64e+02	**2.19e+07**	7.74e+02	2.09e+07	**2.63e+02**
	15	3.26e+08	9.65e+02	1.75e+08	5.85e+02	1.27e+08	4.47e+02	**2.41e+07**	8.74e+02	2.58e+07	**3.64e+02**
	20	3.57e+08	1.07e+03	2.00e+08	6.34e+02	1.47e+08	5.06e+02	**2.46e+07**	8.99e+02	2.91e+07	**4.50e+02**
WFG2	5	3.00e+08	9.18e+02	1.10e+08	4.69e+02	6.68e+07	2.06e+02	**9.64e+06**	3.55e+02	1.11e+07	**1.25e+02**
	10	5.56e+08	1.16e+03	2.80e+08	7.19e+02	1.78e+08	3.27e+02	**1.53e+07**	5.30e+02	3.02e+07	**3.28e+02**
	15	8.98e+08	1.52e+03	5.03e+08	1.06e+03	3.26e+08	4.71e+02	**1.58e+07**	5.53e+02	5.60e+07	**5.36e+02**
	20	1.26e+09	1.75e+03	7.46e+08	1.47e+03	4.88e+08	6.30e+02	**1.53e+07**	**5.40e+02**	8.30e+07	7.22e+02

sequential type algorithms except the proposed method. The number of comparisons and runtime decreases with the increasing number of fronts (Fig. 9) except fast non-dominated sort and divide-and-conquer sort. In those two cases, runtime and number of comparisons increases with the increased number of fronts. Best order sort performs better than all other algorithms followed by corner sort and deductive sort respectively. In MOEAs, divide-and-conquer algorithm has fewest number of comparisons in most of the cases but running time is slightly worse than best order sort. Best order sort becomes second in terms of comparisons followed by corner sort and deductive sort. Divide-and-conquer algorithm has advantage with small dimension of data in MOEAs. Best order sort outperforms all the comparing algorithms when size of population and number of objectives are increased.

6 Conclusion

In this paper we have proposed a non-dominated sorting algorithm for many objective evolutionary algorithms. Basic idea of the proposed method is to use faster sorting algorithms inside non-dominated ranking. The algorithm contains two distinguishable part– sorting and ranking, which has upper bound $O(MN \log N)$ and $O(MN^2)$ respectively. Average case time complexity analysis of this algorithm is also discussed. The experimental results show that this algorithm has some advantage over other algorithms for many objective problems. This method can also be used to find first layer i.e. maximal vectors of a set of points. One drawback of this method is that, sorting time is increased with the number of objectives and it might exceed the time for ranking. A good balance between these two parts should be identified. In future, the idea of this algorithm can be extended to parallel architecture. We would also like to find a progressive or incremental version of this algorithm.

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