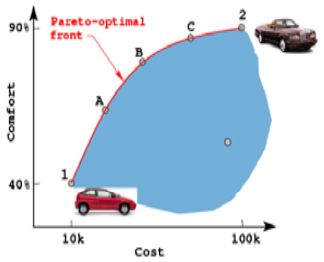
Lecture 7:
Advanced Topics in EMO

- Three areas for applications
  - Better decision-making
  - Unveiling common principles (*Innovization*)
  - Solving other optimization problems
- Handling uncertainties
  - Robust and reliability based EMO

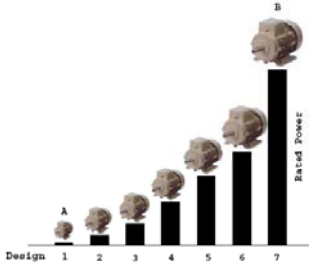
Decision-Making Easier

- Existence of multiple trade-off solutions
  - Provide trade-off information
  - A better idea of the nature of Pareto-optimal front
  - An idea of range of solutions
- Weighted scheme with 70-30
  - Always wonder what if 69-31 or 71-29?



Innovization:
Discovery of Innovative design principles through optimization

- Understand important design principles in a routine design scenario
- Example: Electric motor design with varying ratings, say 1 to 10 kW
  - Each will vary in size and power
  - Armature size, number of turns etc.
- How do solutions vary?
  - Any common principles!

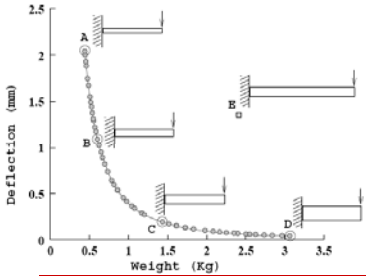


An Engineering Case Study



- All P-O solutions
  - l=200 mm
  - 18.94 ≤ d ≤ 50.00 mm

$$\text{Minimize} \quad f_1(d,l) = \rho\frac{\pi d^2}{4}l,$$
$$\text{Minimize} \quad f_2(d,l) = \delta = \frac{64Pl^3}{3E\pi d^4},$$
$$\text{subject to} \quad \sigma_{max} \le S_y,$$
$$\delta \le \delta_{max},$$
$$10 \le d \le 50 \text{ mm}, \quad 200 \le l \le 1000 \text{ mm}, \quad \sigma_{max} = \frac{32Pl}{\pi d^3}$$



Solution	d (mm)	l (mm)	Weight (kg)	Deflection (mm)
A	18.94	200.00	0.44	2.04
B	21.24	200.00	0.58	1.18
C	34.19	200.00	1.43	0.19
D	50.00	200.00	3.06	0.04
E	33.02	362.49	2.42	1.31

In Support of Common Optimality Properties

Fritz-John Necessary Condition:

Solution x^* satisfy

1. $\sum_{m=1}^{M} \lambda_m \nabla f_m(x^*) - \sum_{j=1}^{J} u_j \nabla g_j(x^*) = 0$, and
2. $u_j g_j(x^*) = 0$ for all $j = 1, 2, 3, \ldots\ldots, J$
3. $u_j \geq 0$, $\lambda_j \geq 0$, for all j and $\lambda_j > 0$ for at least one j

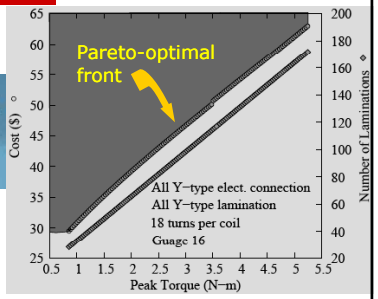
- To use above conditions requires differentiable objectives and constraints
- Yet, it lurks existence of some properties among Pareto-optimal solutions

Brushless DC Permanent Magnet Motor Design for Cost and Peak Torque

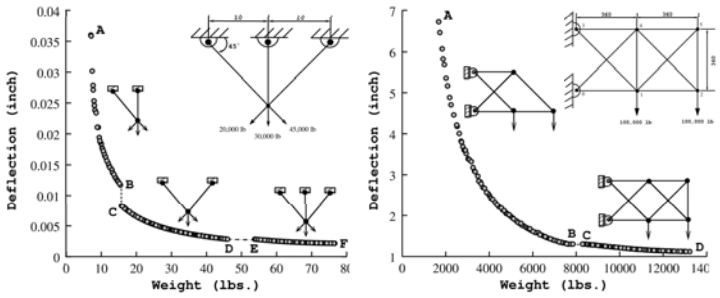
- Six variables (all discrete), three constraints



- *Innovizations*:
  - Connection: Y (betn. Y & Δ)
  - Lamination Type: Y (X, Y, Z)
  - 1 out of 16 wire guages
  - 18 turns per coil (10,80)
- *More peak torque by adding linearly more laminations* ⟵ Recipe for design



Revealing Salient Insights: Truss Structure Design

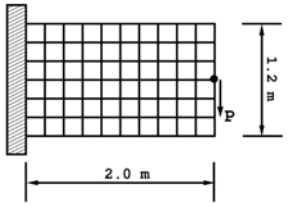


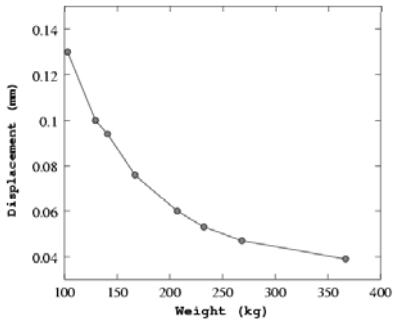
(Deb, Khan and Jindal, 2000)

Revealing Salient Insights: A Cantilever Plate Design

Eight trade-off solutions are chosen

Base Plate

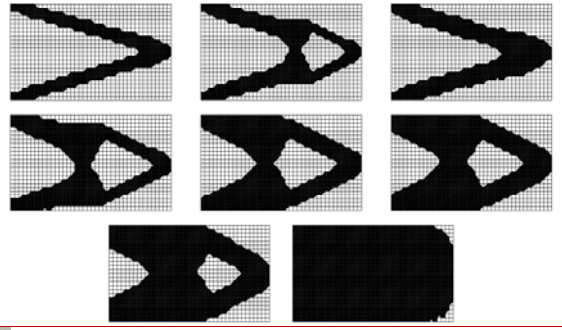




(Deb and Chaudhuri, 2003)

Trade-Off Solutions

- Symmetry in solutions about mid-plane, discovery of stiffener

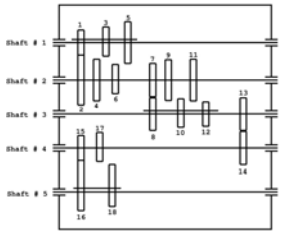


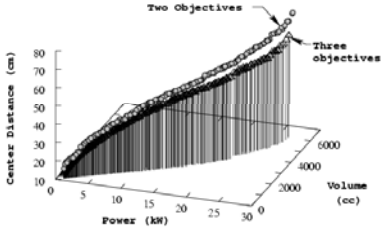
Innovized Principles

- Mid-line symmetry
- Straight arms to reach load is minimum-weight strategy
- Two ways to increase stiffness
  - Thickening of arms
  - Use of a stiffener
  - Additional stiffening by a combination
- Chamfering of corners helpful

Gear-box Design

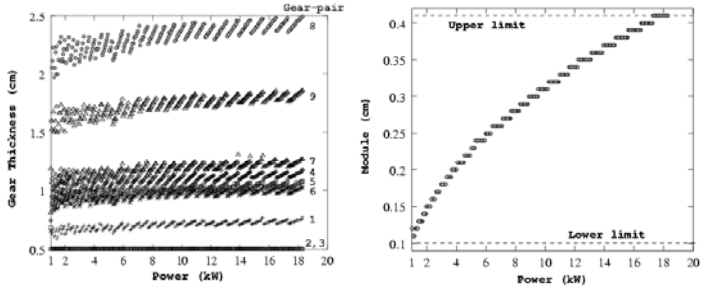
- A multi-spindle gear-box design (Deb and Jain, 2003)
- 28 variables (integer, discrete, real-valued)
- 101 non-linear constraints
- Important insights obtained
  (larger module for more power)





Innovized Principles

- Module varies proportional to square-root of power
- Keep other 27 variables more or less the same

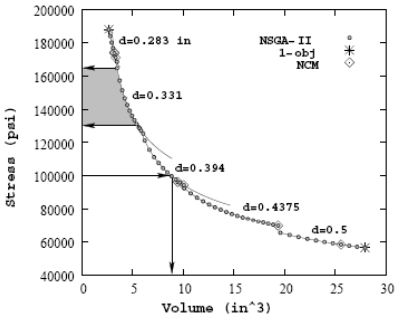


Mechanical Spring Design

- Minimize material volume
- Minimize developed stress
- Three variables: (d, D, N):
  discrete, real, integer
- Eight non-linear constraints
  - Solid length restriction
  - Maximum allowable deflection (P/k≤6in)
  - Dynamic deflection (P_m-P)/k≥1.25in
  - Volume and stress limitations

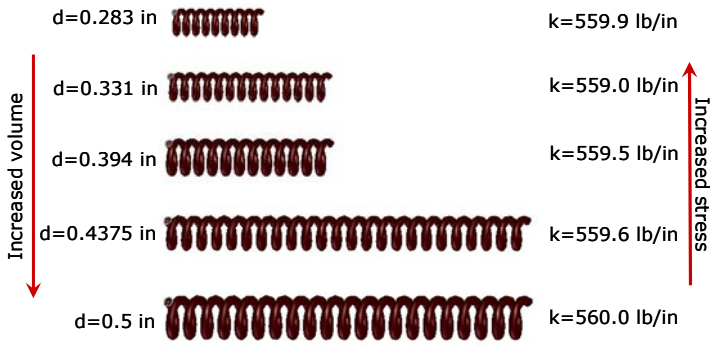
Innovized Principles

- Pareto-optimal front have niches with d
- Only 5 (out of 42) values of d (large ones) are optimal
- A blue-print for optimal design



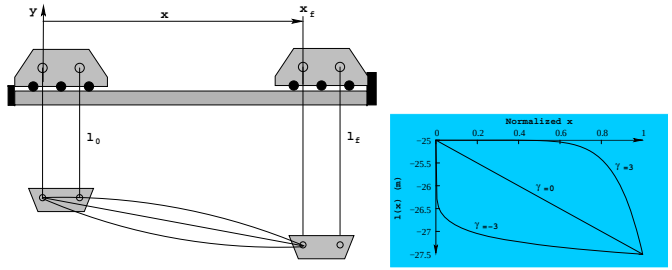
(Deb and Srinivasan, 2005)

Optimal Springs, Optimal Recipe



Overhead Crane Maneuvering

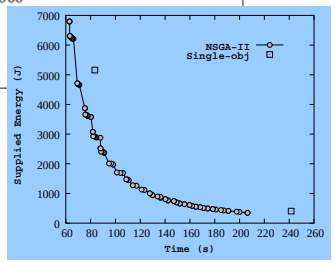
- Minimize time of operation
- Minimize operating energy



Simulation Results

γ	Force (N)	Pattern
2.98	149.26	1111111100100010000000000000000000000000000000000000000
3.00	187.76	11111111001000100000000000000000000000000000
3.00	238.56	1111111100100010000000000000000000
3.00	327.86	11111011001001100000000000
2.98	427.93	1111101100100110000000
3.00	544.94	11111111001000100
2.87	629.62	11111111010001
2.88	818.99	110111010010001

- NSGA-II finds trade-off and interesting properties



Innovative Principles

- For optimum operation:
  - Lower load suddenly at the end
  - Spend energy only at the beginning
- Fast unloading demands more energy
  - For fixed θ, E ∞ l
  - Delay lowering for saving energy

$$E = mg(1 - \cos\theta)l$$

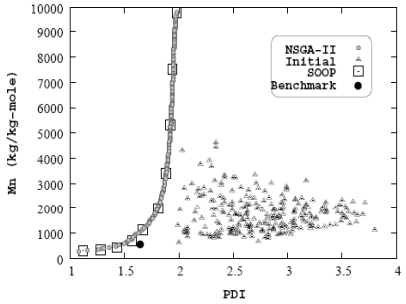
Epoxy Polymerization

- Three ingredients added hourly
- 54 ODEs solved for a 7-hour simulation
- Maximize Chain length (Mn)
- Minimize Polydispersity index (PDI)
- Total 3x7 or 21 variables

$$\begin{aligned} \text{Maximize} \quad & I_{11} = M_n, \\ \text{Minimize} \quad & I_{12} = \text{PDI}, \\ \text{subject to} \quad & \text{satisfying mass and moment balance equations,} \\ & u_i^{\min} \le u_i \le u_i^{\max} \quad i = 1, \ldots, 21. \end{aligned}$$

Pareto-Optimal Front

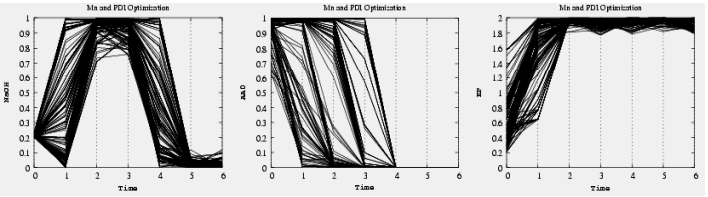
- NSGA-II results
- Verified with NCM
- Batzer and Zahir (1992)'s solution as benchmark



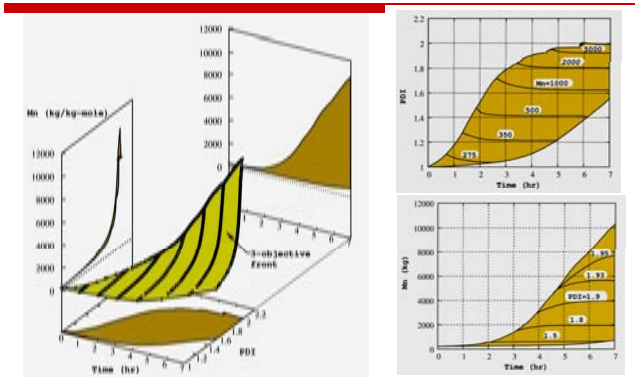
A non-convex frontier

Epoxy Polymerization (cont.)

- Some patterns emerge among obtained solutions
- Chemical significance unveiled

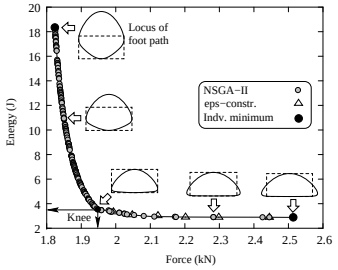


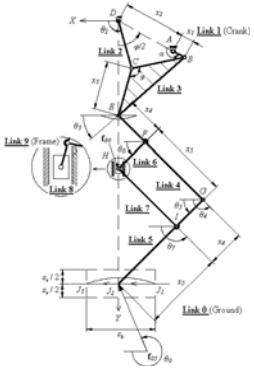
Higher-Level Innovization:
More Objectives and Operating Chart



Robotic Leg Mechanism Design

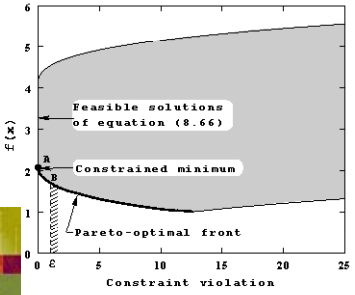
- Minimize energy
- Minimize force





Multiobjectivization for Other Optimization Tasks

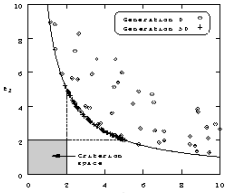
- Constrained handling
  - Constraint violations as additional objectives
- Find partial front near zero-CV
- May provide a flexible search
- A new book ->





Goal Programming and Others

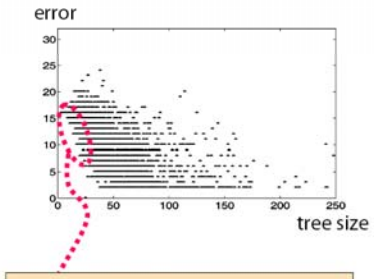
- Goal programming to find multiple solutions
- Avoids fixing a weight vector (Deb, 2001)



- Reducing the chance of getting trapped in local optima (Knowles et al., 2001)
- Use secondary objectives for maintaining diversity (Abbass and Deb, 2003, Jensen, 2003)

Reducing Bloating in GP

- Bleuler et al., (2001)
- Find small-sized programs with small error
- Minimization of Size of Program as second objective



Keep and optimize small trees (potential building blocks)

Optimization Problems with Uncertainty

- Decision variables: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$
- Constraints restrict some solutions to be feasible

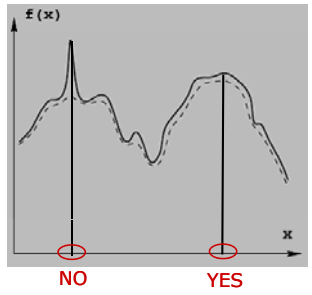
$$\begin{aligned} &\text{Min. } f(\boldsymbol{x},\boldsymbol{a}) \\ &\text{s.t. } g_j(\boldsymbol{x},\boldsymbol{b}) \geq 0 \quad j = 1,2,\ldots,J \\ &\quad\; h_k(\boldsymbol{x},\boldsymbol{c}) = 0 \quad k = 1,2,\ldots,K \\ &\quad\; x_i^L \leq x_i \leq x_i^U \; i = 1,2,\ldots,n \end{aligned}$$

- Multiple conflicting objectives can exist
- Other problems: permutations, mixed types
- What sources of uncertainties?

Robust Optimization:

Handling uncertainties in variables/parameters

- Not noise in obj/constraints
- DV/Parameters uncertain and sensitive to implementation
  - Tolerances in manufacturing
  - Material properties are uncertain
  - Loading is uncertain
- Who wants a sensitive optimum solution?



- Real Issue: How to perform EAs sub-optimally?

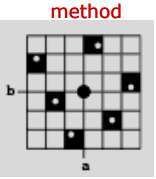
Robustness in Evolutionary Optimization

- Relate to solution quality or performance
  - Every $\boldsymbol{x}$ represents a set of neighboring solutions $\boldsymbol{y}^j \in B(\boldsymbol{x})$, $j=1,\ldots,H$
  - Good expected performance
    - Minimize μ_f instead of f
  - Good worst-case performance
    - Minimize (max $\boldsymbol{y}^j$)
  - Low variability in performance
    - Minimize σ_f <- *not a good idea alone*
    - Minimize $\mu_f + k\sigma_f$ is a better idea

Expected Performance, μ_f

- Most popular approach (Ex., Jin and Branke, 2005)
- Assume a distribution of variation
- Compute objective variation
- Reduce to a single performance measure
- Cannot be computed analytically
- Estimation procedures
  - EA studies vary on different estimation schemes
  - Archieving, samples, meta-models

Latin Hypercube method

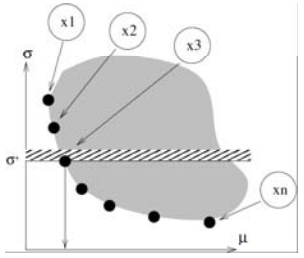


Worst Case Performance, max $\boldsymbol{y}^j$

- Complete risk aversion
- Many research groups (Ong, Haftka, Tan)
- Difficult to find worst neighbor analytically
- Mostly simulation based
  - Nested optimization -> Minimize (maximize $\boldsymbol{y}^j$)
- Meta-modeling for a quick computation
- *Co-evolutionary method may be a good idea*
  - One population handles $\boldsymbol{x}$, other population relative neighboring points ($\boldsymbol{y}^j - x$)

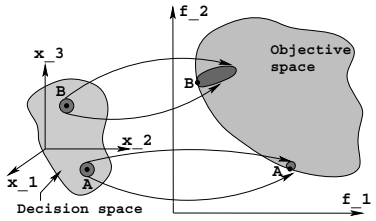
Other Approaches

- Bi-objective approach (Avigad, Jin+Sendhoff)
  - Optimize performance and minimize its variation
    - Provide valuable information through a Pareto-optimal frontier
  - Involves a decision maker
- Good survey (Beyer and Sendhoff, 2007)



Robust EAs for Multi-Objective Optimization

- Not all Pareto-optimal points may be robust
- A is robust, but B is not
- Decision-makers will be interested in knowing robust part of the front
- Generic decision-making idea



Multi-Objective Robust Solutions of Type I and II (Deb and Gupta, ECJ 2004)

- Similar to single-objective robust solution of type I

$$\begin{aligned} &\textit{Minimize} && (f_1^{\text{eff}}(\mathbf{x}), f_2^{\text{eff}}(\mathbf{x}), \ldots, f_M^{\text{eff}}(\mathbf{x})), \\ &\textit{subject to} && \mathbf{x} \in \mathcal{S}, \end{aligned}$$

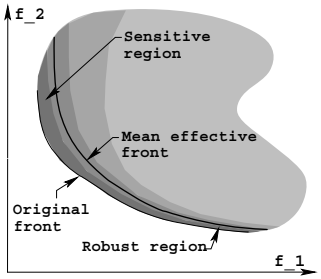
- Type II

$$\begin{aligned} &\textit{Minimize} && \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x})), \\ &\textit{subject to} && \frac{\|\mathbf{f}^p(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \leq \eta, \\ & && \mathbf{x} \in \mathcal{S}. \end{aligned}$$

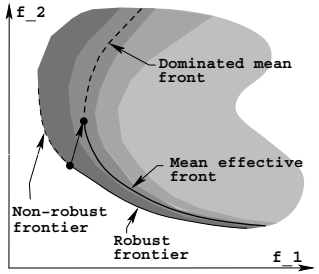
- Other ideas can be used as well

Different Scenarios: Cases 1 and 2 (No Local Front)

Case 1: Complete Pareto-optimal front is robust

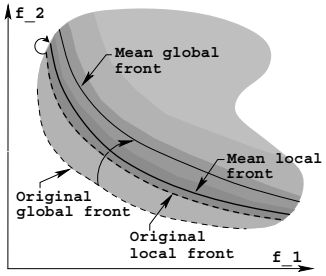


Case 2: Part of original PO front is robust

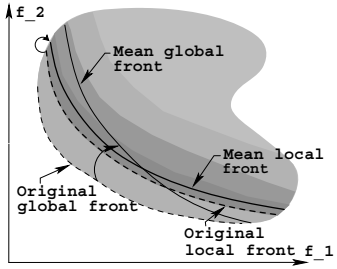


Cases 3 and 4 (Multimodal Fronts)

Case 3: Original local PO front is robust

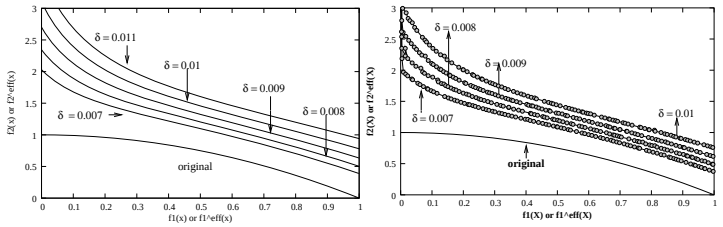


Case 4: Parts of original global and local PO fronts make robust frontier



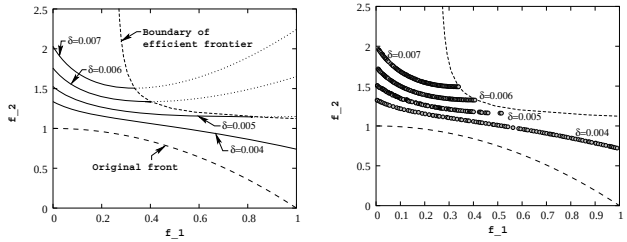
Type I Robustness: Effect of δ-neighborhood Size

- Theory and NSGA-II simulation on Case 1
- Larger δ, more shift from original front
- Some part is more sensitive than others



Case 2: Partial Frontier is Robust

- Theory and NSGA-II simulation on Case 2
- Larger δ, more shift from original front
- Some part is no more robust



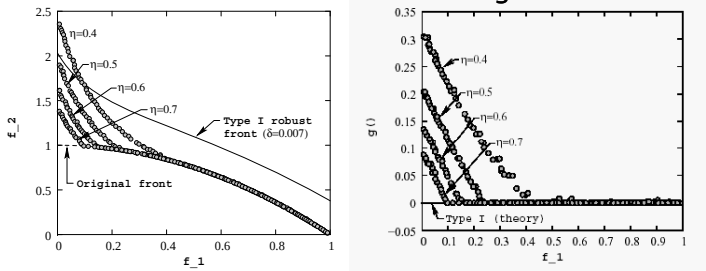
Type I Robustness may not be Practical

- Entirely depend on averages
- Decision-maker may like to have control on extent of robustness
- Type II is more pragmatic
- Define extent of relative perturbation (η) allowed

$$\begin{aligned} \textit{Minimize} \quad & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x})), \\ \textit{subject to} \quad & \frac{\|\mathbf{f}^p(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|} \leq \eta, \\ & \mathbf{x} \in \mathcal{S}. \end{aligned}$$

Type II Robustness on Case 1

- As η reduces, more stringent requirement for robustness
- Solutions deviate more from original front



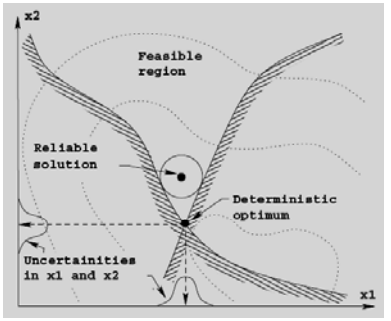
Uncertainties in Presence of Constraints

- Two kinds of uncertainty
  - Aleatory (random and irreducible), inherent to the problem
    - Probability model (RBDO)
    - Bayesian model (BADO)
  - Epsitemic (subjective and reducible), ignorance or lack of info
    - Possibility model (PBDO)
    - Evidence model (EBDO)

Reliability-Based Optimization:
Making designs safe against failures

- Deterministic optimum is not usually reliable
- Reliable solution is an interior point
- *Chance* constraints with a given reliability

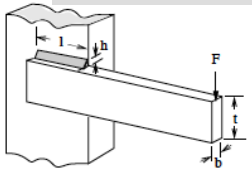
Minimize $\mu_f + k\sigma_f$
Subject to $Pr(g_j(x) \geq 0) \geq \beta_j$
β_j is user-supplied



An Illustrative Example

Minimize Cost
Subject to:
stress ≤ strength
deflection ≤ d_allow

- Decision variables:
  - Dimensions: *h, l, t, b*
- Parameters:
  - Material strength
  - Load *F*, weld quality, etc.



- Each (or some) vary with known distribution
- Causes objective (cost) and constraints (failure) to vary
- Optimal solution is not desired
- Specify an *overall* reliability (*R*) against failure and identify which solution has best cost w.r.t. *R*

RBDO: Basics

- *x*: uncertain variables
- *d*: fixed variables
- *p*: uncertain parameters

Original Problem

$$\begin{aligned} \underset{(\mathbf{x},\mathbf{d})}{\text{Minimize}} \quad & f(\mathbf{x},\mathbf{d},\mathbf{p}), \\ \text{Subject to} \quad & g_j(\mathbf{x},\mathbf{d},\mathbf{p}) \geq 0, \quad j = 1,2,\ldots,J, \\ & h_k(\mathbf{d}) \geq 0, \quad k = 1,2,\ldots,K, \\ & \mathbf{x}^{(L)} \leq \mathbf{x} \leq \mathbf{x}^{(U)}, \\ & \mathbf{d}^{(L)} \leq \mathbf{d} \leq \mathbf{d}^{(U)}. \end{aligned}$$

- *x* and *p* are assumed to vary with (μ, σ)
- Functions *f* and g_j are uncertain and have distributions of variation
- Makes it difficult to use an optimization method
  - Dealing with function distributions

RBDO Basics (cont.)

- Reliability *R*
- Revised problem
- Joint prob. *P* difficult to compute

$$\begin{aligned} \underset{(\mu_{\mathbf{x}},\mathbf{d})}{\text{Minimize}} \quad & f(\mu_{\mathbf{x}},\mathbf{d},\mu_{\mathbf{p}}), \\ \text{Subject to} \quad & P(\bigwedge_{j=1}^{J}(g_j(\mathbf{x},\mathbf{d},\mathbf{p})\geq 0))\geq R \\ & h_k(\mathbf{d})\geq 0, \qquad k=1,2,\ldots,K, \\ & \mathbf{x}^{(L)}\leq \mu_{\mathbf{x}}\leq \mathbf{x}^{(U)}, \\ & \mathbf{d}^{(L)}\leq \mathbf{d}\leq \mathbf{d}^{(U)}. \end{aligned}$$

- Most studies use $P(g_j(\mathbf{x},\mathbf{d},\mathbf{p})\geq 0)\geq R_j, \quad j=1,2,\ldots,J.$ R_j=R
- System reliability versus component reliability
- Thus, the worst constraint govern the reliability constraint
- First, we discuss single-constraint strategies

Determining a Solution's Reliability

- Probability of success against constraint satisfaction

$$P_j(\mu_{\mathbf{x}},\mathbf{d},\mu_{\mathbf{p}})=\int_{g_j(\mathbf{x},\mathbf{d},\mathbf{p})<0}\varphi_{\mu_{\mathbf{x}},\mu_{\mathbf{p}}}(\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p},$$

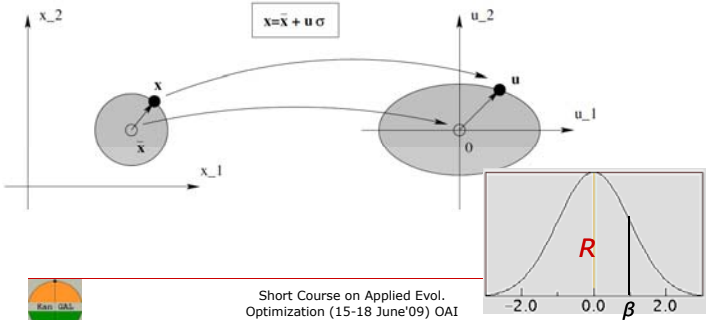
- Difficult to compute exactly
- Monte-Carlo simulation: r_j success of *N* scenarios: $1-\frac{r_j}{N}\geq R_j$ (Loughlin and Ranjithan, 1999)
- If *R*=0.9999, *N*>10,000 (Expensive!)
- Importance sampling: May still be large

Optimization Based Procedure: *Classical Approaches*

- Given a solution, determine the closest point on the constraint boundary
  - Most probable point (MPP)
- To generalize, computation is performed on std. normal coordinate system *U* for every *x*
- $g_j(x,d,p)=0 \Rightarrow G_j(U)=0$
- Two optimization approaches:
  - Performance Measure Approach (PMA)
  - Reliability Index Approach (RIA)

Standard Normal Coordinate System

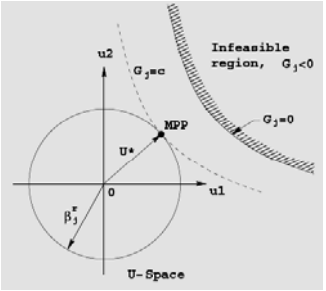
- Variance is supplied
- Any vector *x* is mapped to *u* as follows:



PMA Approach (for every *x,d,p*)

- β_j comes from R_j
- Find U^* using an optimization method
- MPP on reliability circle
- Check if MPP is feasible
- Replace $P(g_j(x,d,p) \geq 0) \geq R_j$ with $G_j(U^*) \geq 0$

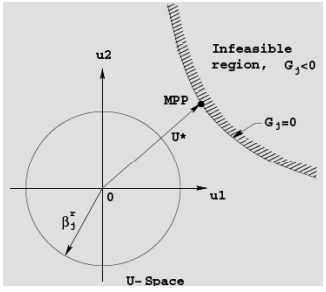
$$\begin{aligned} &\text{Minimize} && G_j(\mathbf{U}), \\ &\text{Subject to} && \|U\| = \beta_j^r, \end{aligned}$$



RIA Approach (for every *x,d,p*)

- Find U^* using an optimization method
- MPP on constraint
- Check if it equals or exceeds given R
- Replace $P(g_j(x,d,p) \geq 0) \geq R_j$ with $\|U^*\| \geq \beta_j$

$$\begin{aligned} &\text{Minimize} && \|\mathbf{U}\|, \\ &\text{Subject to} && G_j(\mathbf{U}) = 0. \end{aligned}$$

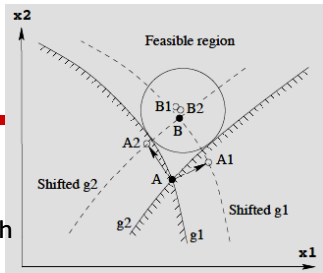


Reliability Based Optimization Procedures

- Double-loop methods (Agarwal, 2004)
  - Nested optimization with PMA or RIA
    Computationally expensive, but accurate
- Single-loop methods (Liang et al., 2004)
  - Approximate methods of PMA/RIA
  - Computationally fast, but not accurate
- Decoupled methods (Du and Chen, 2004)
  - Sequential, repetitive application of both optimization tasks
  - A good compromise

A Decoupled Method: SORA (Du and Chen, 2004)

- Start with deterministic optimum
- Find MPPs (J lower level opt.)
- Next, original opt. problem with revised constraints



$$\begin{aligned} &\underset{(\mu_{\mathbf{X}},\mathbf{d})}{\text{Minimize}} && f(\mu_{\mathbf{X}},\mathbf{d},\mu_{\mathbf{P}}), \\ &\text{Subject to} && g_j(\mu_{\mathbf{X}} - \mathbf{s}_j^t, \mathbf{d}, \mathbf{p}_{MPP,j}^t) \geq 0, \quad j = 1,2,\ldots,J, \\ &&& h_k(\mathbf{d}) \geq 0, \quad k = 1,2,\ldots,K, \\ &&& \mathbf{x}^{(L)} \leq \mu_{\mathbf{X}} \leq \mathbf{x}^{(U)}, \\ &&& \mathbf{d}^{(L)} \leq \mathbf{d} \leq \mathbf{d}^{(U)}. \end{aligned}$$

$(\mu_{\mathbf{X}}^t, \mathbf{d}^t)$ ↓

↑ $t = t+1$, $\mathbf{s}_j^t = \mu_{\mathbf{X}}^t - \mathbf{x}_{MPP,j}^t$, $\forall j$

For $j = 1,2,\ldots,J$
PMA Approach to find MPP for j-th constraint → $(\mathbf{x}_{MPP,j}^t, \mathbf{p}_{MPP,j}^t)$

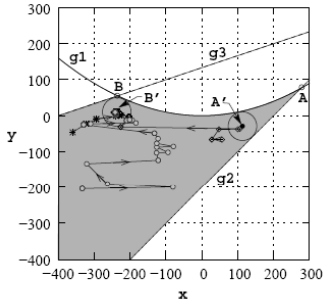
Three Problem Niches for EAs

1. Finding reliable optimum in multi-modal problems
2. Enhancing knowledge of reliable solutions with increased reliability
3. Locating reliable frontier in multi-objective optimization problems

1. Finding Global Reliable Solution: *Single-objective reliable solution*

- Multiple deterministic optima
- Reliable solution for local is better
- Classical methods starts from deterministic global optimum
- EAs solve the overall problem and are better

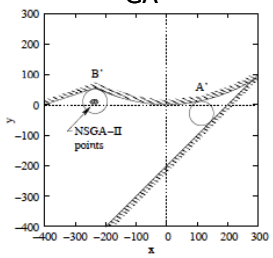
PMA approach is used



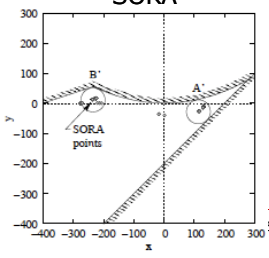
Comparison with SORA

Method	# success	Proportion success	Solution Evaluations Best	Average	Worst
GA ($\epsilon_{PMA} = 0.001$)	2,500	100%	9,483	9,756.2	9,768
GA ($\eta_{PMA} = 2$)	2,500	100%	3,663	3,663.0	3,663
SORA	1,630	∼65%	491	1,394.0	15,679
Double-loop	1,219	∼49%	2,910	147,994.7	413,040

GA



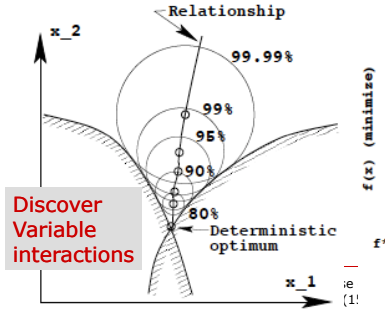
SORA



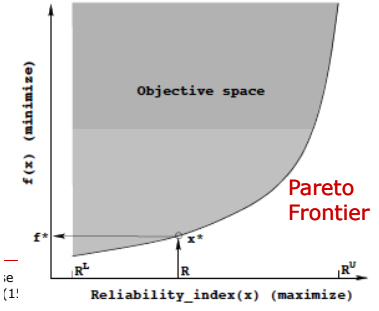
2. Solutions with Increased Reliability

- Bi-objective problem
- Use an EMO

$$\begin{aligned} &\text{Minimize}_{(\mu_{\mathbf{X}},\mathbf{d})} && f(\mu_{\mathbf{X}},\mathbf{d},\mu_{\mathbf{p}}),\\ &\text{Maximize}_{(\mu_{\mathbf{X}},\mathbf{d})} && R(\mu_{\mathbf{X}},\mathbf{d},\mu_{\mathbf{p}})\\ &\text{Subject to} && h_k(\mathbf{d}) \geq 0,\\ & && \mathbf{x}^{(L)} \leq \mu_{\mathbf{X}} \leq \mathbf{x}^{(U)},\\ & && \mathbf{d}^{(L)} \leq \mathbf{d} \leq \mathbf{d}^{(U)}. \end{aligned}$$

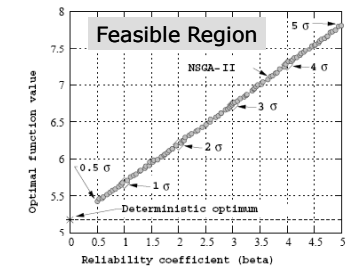


Discover Variable interactions

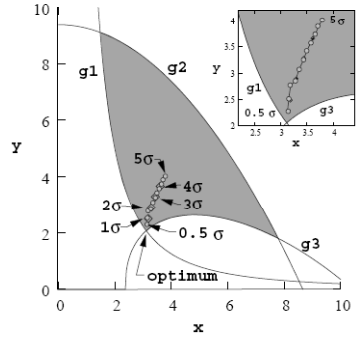


Pareto Frontier

An Example Problem: *Get a better insight*



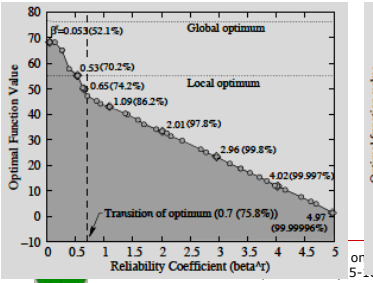
RIA approach is used

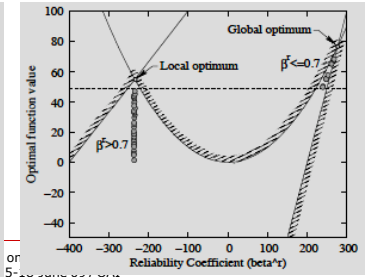


Maximize $x + y$,
Subject to $g_1(x,y) \equiv \frac{1}{20}x^2y - 1 \geq 0$,
$g_2(x,y) \equiv \frac{1}{30}(x+y-5)^2 + \frac{1}{120}(x-y-12)^2 - 1 \geq 0$,
$g_3(x,y) \equiv \frac{80}{x^2+8y+5} - 1 \geq 0$,
$0 \leq x, y \leq 10$.

Bi-Modal Problem Revisited

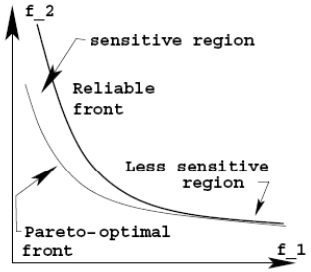
- Trade-off is clear (maximize *(f,R)*)
- After 76% reliability, local optimum is robust





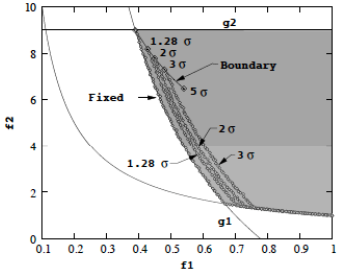
3. Multi-Objective Reliable Frontier

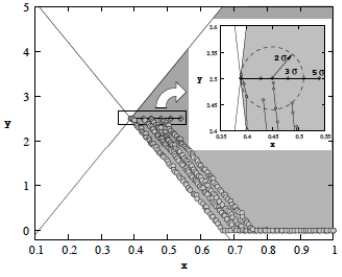
- Instead of finding deterministic Pareto-optimal front, find the reliable front
- Chance constraints
- Objectives as they are
- PMA approach is used



A Test Problem

- PO front moves inside with more *R*
- Provides information in decision space
- Can be used a decision-making tool





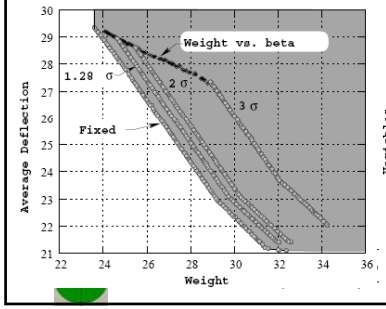
Innovization: Innovative design principles through optimization

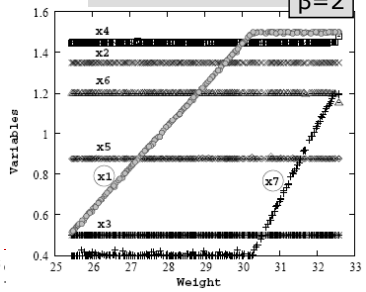
A Car Side Impact Design

- Reliable fronts show rate of movement
- What remains unchanged and what gets changed!
  - Better resource management

x1: thk. of B-Pillar inner
x7: thk. of roof rail

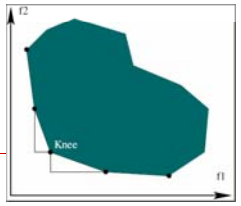
β=2





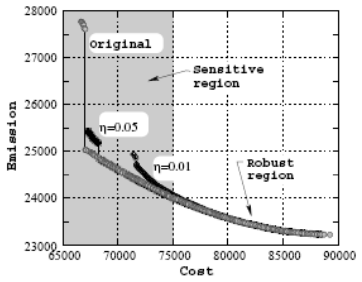
Robustness/Reliability in Decision Making

- EMO and decision making
  - An important remaining task
- Decision making usually subjective
- Collaborations with MCDM in progress
- Objective decision making
  - Select robust/reliable front, instead of PO front
  - Select knee-like regions

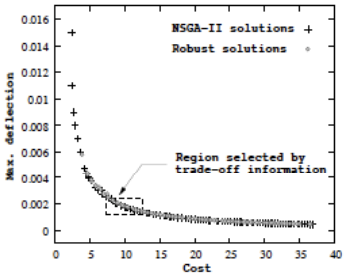


Practical Design Problems

Hydro-electric power dispatch



Welded Beam Design



Summary

- Practice demands handling constraints and uncertainties
  - Optimal solution is not reliable
- Monte Carlo simulation not sufficient
- Use RBDO using EAs:
  - Get parameter/variable distributions
  - Find reliable solution(s) for one or more objectives
  - Learn how reliability affects solution quality
- A pragmatic application and a definite niche for EAs