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FRICTION-INDUCED VIBRATION IN PERIODIC LINEAR ELASTIC MEDIA

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Abstract

For a one-dimensional finite elastic continuum with distributed contacts and periodic boundary conditions, the presence of unstable waves is investigated. The stability of the waves is evaluated and explanations for instabilities under a constant coefficient of friction are provided. A negative slope in the coefficient of friction as a function of sliding-speed is not a necessary condition for the occurrence of dynamic instability. Dynamic instability occurs in the form of self-excited, unstable, traveling waves. The stabilizing effects of external and internal damping were studied. Low and high frequency terms of the traveling waves are stabilized by adding external and internal damping, respectively. Responses corresponding to unstable eigenvalues can dominate the system response. It is presumed that this can lead to squeaking or squealing noise.

1. INTRODUCTION

When there is frictional contact between two materials, noise and vibrations are often generated. They are associated with chattering, squeaking, squealing noise, and stick-slip oscillations of elastic materials.

In classical studies of the causes for the noise and vibrations, analyses dealing with discretized mathematical models have prevailed (Brockley *et al.*[1], Sakamoto [2, 3]). Moreover, most of the causes cited for steady sliding instabilities have been based on friction-speed relations: a coefficient of friction that decreases with sliding speed has played a primary role in instability of the system based on linear stability criteria

(Simpson [4], Krauter [5], Nakai and Yokoi [6]).

However, according to research by Adams [7, 8] and Martins *et al.* [9] dynamic instability, which can generate noise and vibrations, can occur even without a negative-slope in the friction-speed relations. Experimental studies by De Togni *et al.* [10], Rorrer *et al.* [11], and Vallette and Gollub [12] also showed such dynamic instabilities can occur in the absence of negative-slope characteristics in friction-speed relations. Because of couplings in spatial coordinates caused by friction stresses, steady sliding motions are dynamically unstable even when the coefficient of friction is constant.

In this paper investigations on the stability of waves of an elastic medium under steady frictional sliding are presented in order to understand the mechanisms which cause vibrations and noise. The focus is on the mathematical model of a one-dimensional elastic system with distributed contacts and periodic boundary conditions. The periodic boundary conditions are motivated by elastic bushing systems, in which the elastic media are approximately annular. The annulus is simplified as a one-dimensional rectilinear medium with periodic boundary conditions. Through evaluations of characteristic solutions of waves, we show that finite elastic systems with periodic boundary conditions and a constant coefficient of friction can have unstable steady sliding. In addition, the effects of external and internal damping on overall system stability are analyzed.

2. EQUATION OF MOTION

We consider the system shown in Figure 1. A linear elastic medium, placed between a moving belt and a frictionless linear bearing, represents a one-dimensional, undamped, continuous system in distributed sliding contact. The friction coefficient is constant. In addition, any parameters having random properties, such as roughness of contact surface, are not included in this development in order to emphasize the dynamic stability effects of uniform properties of materials and structures. Moreover, any non-uniform motions, such as stick-slip motion or loss of contacts are not included.

The equation of axial motion for a homogeneous, undamped linear elastic medium is

$$A\frac{\partial \sigma_x(x,t)}{\partial x} + f(x,t) = \rho \frac{\partial^2 \hat{u}}{\partial t^2}, \tag{1}$$

where $A = wh$ is the cross sectional area of an elastic medium with constant width w (into the page) and height h, ρ is the mass per unit length of the elastic material, $\sigma_x(x,t)$ is a stress over the cross section, $\hat{u}(x,t)$ is an axial displacement, and $f(x,t)$

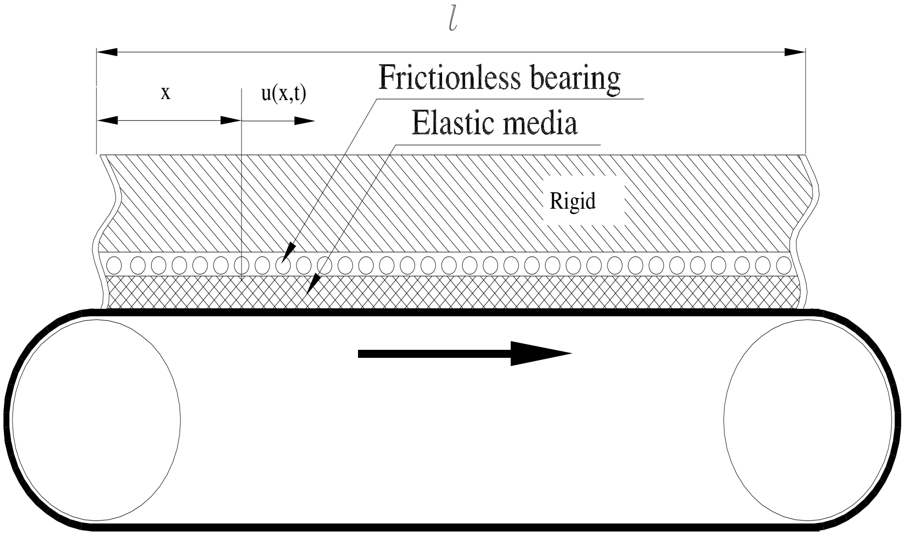

Figure 1: A schematic diagram for a one-dimensional elastic medium subjected to distributed friction. Friction between a moving belt and an infinite elastic medium induces vibrations and noise. A frictionless linear bearing is installed on top of a medium so as to allow axial motions of an elastic medium.

is a friction force per unit length. Applying a linear stress-strain relationship, stress is expressed as $\sigma_x(x,t) = E\ \epsilon_x(x,t)$, where E is Young's modulus of the material.

Axial stress accompanies a change in cross sectional area in an open elastic medium, due to the Poisson effect. Since this system is constrained, there is instead a change in contact normal stress. For plane stress, the friction stress including Poisson's ratio effect per unit length is given by

$$f(x,t) = -\mu w \sigma_y(x,t) = -\mu w \{\sigma_0 + \nu \sigma_x(x,t)\}, \tag{2}$$

where μ is a friction coefficient, $\sigma_y(x,t)$ is a contact normal stress, and σ_0 is a pre-loaded normal stress per unit length, which should be always less than zero (compression) to generate friction stress and maintain contact to the sliding rigid body. This distributed friction force contributes to the axial stresses in the medium through equation (1). Thus the friction stress is modeled as proportional to the normal stress. The normal stress consists of a static load plus a variation due to the axial stress, which through the Poisson effect, is accompanied by variations in pressure on the medium constrained between a sliding surface and a bearing.

By considering the linear strain-displacement relation, $\epsilon_x(x,t) = \frac{\partial \hat{u}(x,t)}{\partial x}$, a non-dimensional equation of motion is obtained as

$$\frac{\partial^2 \hat{u}^*}{\partial {x^*}^2} - \alpha \frac{\partial \hat{u}^*}{\partial x^*} + \alpha\beta = \frac{\partial^2 \hat{u}^*}{\partial {t^*}^2}. \tag{3}$$

The dimensionless parameters used in equation (3) are $\alpha = \frac{\mu w \nu l}{A} = \frac{\mu \nu l}{h}, \beta = -\frac{\sigma_0 l}{\nu E}, u^* = \frac{u}{l}, x^* = \frac{x}{l}$, and $t^* = \frac{t}{\sqrt{\frac{\rho l^2}{AE}}}$, where l denotes contact length, and u^*, x^* and t^* are the dimensionless displacement, coordinate and time, respectively. For the sake of simplicity, the notation $*$ will be neglected in the continuing development.

The periodic boundary conditions

$$\hat{u}(0,t) = \hat{u}(1,t), \tag{4a}$$

$$\frac{d\hat{u}(0,t)}{dx} = \frac{d\hat{u}(1,t)}{dx}, \tag{4b}$$

are applied.

3. STABILITY ANALYSIS OF ELASTIC WAVES

3.1. UNDAMPED MODEL

For the study of elastic waves we seek the dynamic equation of motion with respect to a rigid-body solution of equation (3). As such we let $\hat{u}(x,t) = u_r(t) + u(x,t)$. The rigid body solution is $u_r(t) = u_0 + v_o t + \alpha\beta t^2/2$, which in reality would be physically limited by damping or restraints. We could ground the medium with a distributed spring, but have not done so for direct comparison with the problem of fixed boundary conditions [13]. (Grounding the system with springs will effect the details in the stability analysis, but we expect the general phenomena to be similar.)

Continuing, the dynamic equation of motion can be written in self-adjoint form [13]:

$$\frac{\partial}{\partial x}\{e^{-\alpha x}\frac{\partial u}{\partial x}\} = e^{-\alpha x}\frac{\partial^2 u}{\partial t^2}. \tag{5}$$

Note that the system parameter α in equation (5) is constant, which represents a fixed coefficient of friction and Poisson's ratio.

Considering periodic boundary conditions (4), solutions are assumed to have the form

$$u(x,t) = \text{Real}\{e^{i2\pi k(x-ct)}\}, \tag{6}$$

where k is a positive integer number representing the angular frequency of solutions along the x axis, as the term $\frac{1}{k}$ shows the wave periods along the x axis. (We will use the notation in equation (6) since references from wave dynamics in continua have used such notation in their studies.) The integer values of k satisfy the periodic boundary conditions that were motivated by an annulus of unit non-dimensional circumference.

Generally, c can be a complex value and plays an important role in dynamic system stability. In the case of a real value of c, pure waves of constant shape are expected. This implies that conservative non-dispersing waves exist in the elastic medium and the system is in a neutrally stable state without damping. On the other hand, a complex value of c contains information about the characteristics of the waves. Writing $c = R + Ii$, the equation (6) can be expressed as

$$u(x,t) = \mathrm{Real}\{e^{i2\pi k(x-Rt)}e^{2\pi kIt}\}. \tag{7}$$

A positive R indicates that there is a wave propagating toward the positive direction and a positive I indicates that there is an unstable wave which increases in amplitude exponentially in time. On the other hand, a negative R indicates that there is a wave propagating toward the negative direction and a negative I indicates that there is a stable wave which decreases in amplitude exponentially in time. Thereby, the imaginary component of the characteristic solutions represents the stability of the wave.

The characteristic equation of c obtained by substituting the equation (6) into equation (5) is

$$c^2 - (1 + \frac{\alpha}{2\pi k}i) = 0. \tag{8}$$

The imaginary and real parts of the characteristic solution for c are

$$R = \pm\sqrt{\frac{1+\sqrt{1+(\alpha/2\pi k)^2}}{2}}, \tag{9a}$$

$$I = \frac{\alpha}{4\pi kR} = \pm\frac{\alpha}{4\pi k\sqrt{\frac{1+\sqrt{1+(\alpha/2\pi k)^2}}{2}}}. \tag{9b}$$

Without friction ($\alpha = 0$) the characteristic solution has pure real solutions for c and the traveling waves, which keep their wave shapes in time, are pure sinusoidal functions.

When friction is considered, however, the characteristic equation yields general, complex solutions for c. From the result in equation (9), unstable waves propagate toward the positive x axis, as indicated by a positive value R. On the other hand, any waves that propagate toward the negative x axis diminish in time since they have negative imaginary components in the characteristic solutions.

Figure 2 shows the imaginary and real parts of the characteristic solution corresponding to an unstable wave as a function of parameter α. Those solutions are presented with various undetermined frequency factors k. As α increases, i.e., as the coefficient of friction or Poisson's ratio increases (or l increases, A decreases) waves traveling toward the positive x direction (the direction of the moving rigid body) are increasingly unstable in any finite sliding velocity. Stable waves are not represented in Figure 2.

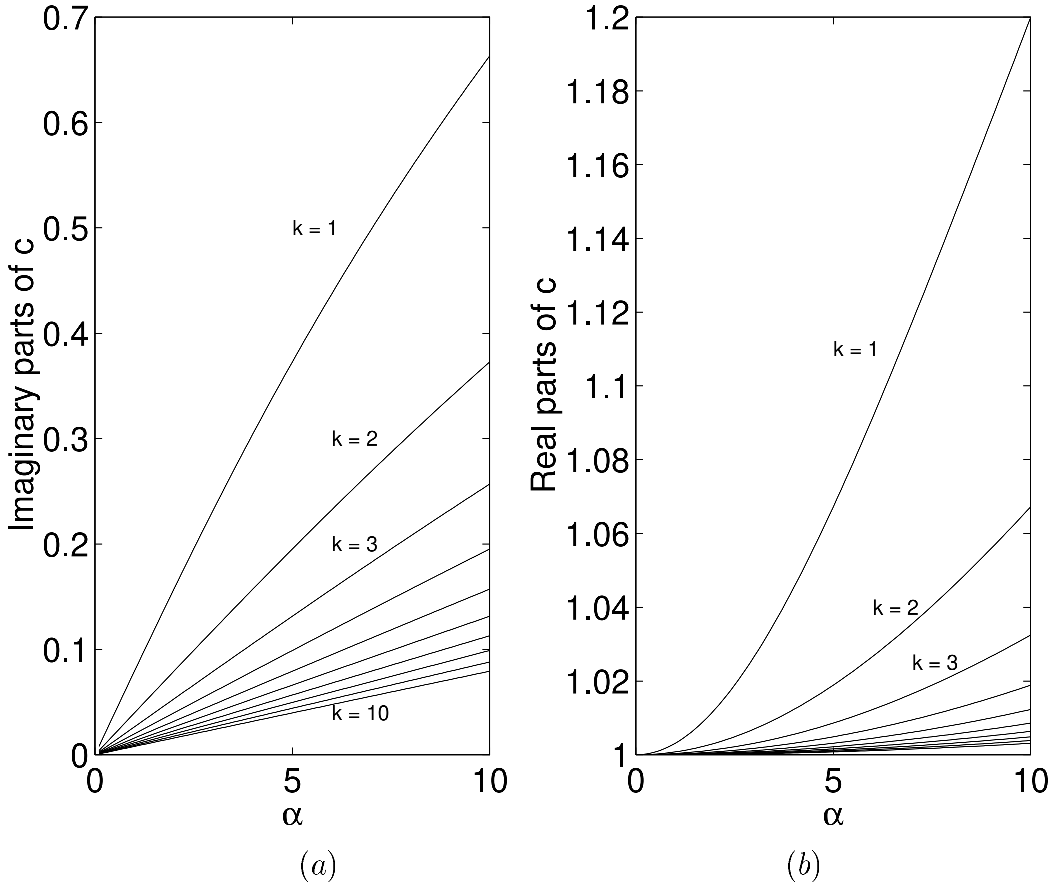

Figure 2: The unstable characteristic solutions for the undamped, periodic boundary conditioned model. (a) Imaginary and (b) real parts of the characteristic solution versus α are shown.

Similar trends associated with such unstable traveling waves were found in previous studies. Regarding the traveling direction of unstable waves, the direction of the moving rigid body indicates the direction of the unstable waves (Adams [7, 8], Martins *et al.* [9]). The instability occurs even if the coefficient of friction is constant. By considering an infinite beam subjected to distributed friction, which was modeled mathematically as a fourth order partial differential equation, Adams [8] proved that one-dimensional traveling waves caused system instability. He included properties representing the roughness of the contact surface in his modeling.

With the aid of this study, it has been analytically shown that finite elastic systems subjected to distributed friction and periodic boundary conditions can also be unstable in the presence of a constant coefficient of friction, without including any random properties. The periodic boundary conditions provide a pathway for wave propagation that is not available in the case of fixed boundary conditions. In real situations, such unstable waves are expected to lead to non-uniform motions, such as stick-slip oscillations or loss of contact in materials.

3.2. ADDITION OF EXTERNAL DAMPING

An undamped elastic continuum subject to distributed friction is considered in the previous section. In this section, effects of external damping on system stability are considered. (External damping is defined as a relative dissipation acting between an elastic material and the ground.) An equation of motion including an external damping coefficient d is

$$\frac{\partial}{\partial x}\{e^{-\alpha x}\frac{\partial u}{\partial x}\} = e^{-\alpha x}\{\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial t}\}. \tag{10}$$

Applying solutions of the form of equation (6) and periodic boundary conditions (4), the characteristic equation is

$$c^2 + \frac{di}{2\pi k}c - (1 + \frac{\alpha i}{2\pi k}) = 0. \tag{11}$$

The real and imaginary parts of the characteristic solution are

$$R = \pm r^{1/2}\cos(\phi/2), \tag{12a}$$

$$I = \pm r^{1/2}\sin(\phi/2) - \frac{d}{4\pi k}, \tag{12b}$$

where $r^{1/2} = \sqrt{\{1 - \frac{1}{4}(\frac{d}{2\pi k})^2\}^2 + (\frac{\alpha}{2\pi k})^2}, \phi = \arctan\{\frac{\frac{\alpha}{2\pi k}}{1-\frac{1}{4}(\frac{d}{2\pi k})^2}\}$. This shows that the characteristic solutions do not have complex conjugate solutions when external damping d is present.

Since instability is the primary concern of this study, we consider only the maximum value of imaginary parts in the characteristic equation (11). The maximum value of imaginary components determines the system stability, since a positive imaginary component indicates an unstable traveling wave.

Figure 3 and Figure 4 provide the maximum imaginary part and the real part of equation (11) on the parameter domains α and d, respectively. In Figure 3, the maximum imaginary part decreases with decreasing α or increasing d. In other words, a reduction of sliding friction, or an increase in external damping is required to stabilize the system. The traveling wave speeds, which are represented as the real parts of the characteristic solution, are influenced by α and d as shown in Figure 4. The speeds of waves corresponding to the unstable ones are decreased by decreasing α or increasing d.

Figure 5 depicts the imaginary parts of c under variations in d for several frequencies k when $\alpha = 1.0$. In this example, the overall system, which was unstable

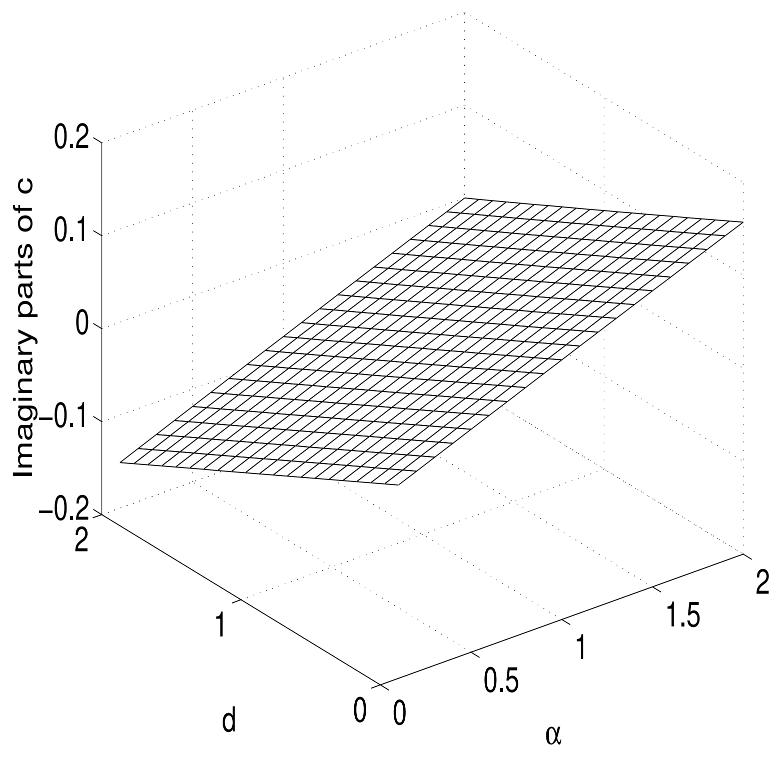

Figure 3: The imaginary parts of the characteristic solutions including the external damping coefficient d. The maximum value of the imaginary parts is presented in the parameter domains α and d. In this example $k = 1$.

by having complex conjugate pairs in the characteristic solutions when $d = 0.0$, becomes a stable system with sufficiently large external damping d. The complex solutions of c are no longer in the form of complex conjugate pairs when d is not equal to zero. Beyond the point $d = 1.0$, all characteristic solutions can have negative imaginary parts, which implies the system is fully stable. Low frequency terms, such as $k = 1$, are more easily stabilized than high frequency terms by increasing d, as indicated by the steep negative slopes in Figure 5. On the other hand, for low d, lower frequency terms are more unstable. All terms became stable at $d = \alpha$.

Based on this interpretation, the system can be unstable with both stable and unstable eigenvalues in its parameters. (For example, the range $0 < d < 1.0$ in Figure 5). Under such conditions, responses corresponding to the stable eigenvalues are damped out in time, but responses corresponding to unstable eigenvalues can dominate the system responses, presumably generating squeaking or squealing noises and vibrations in experiments.

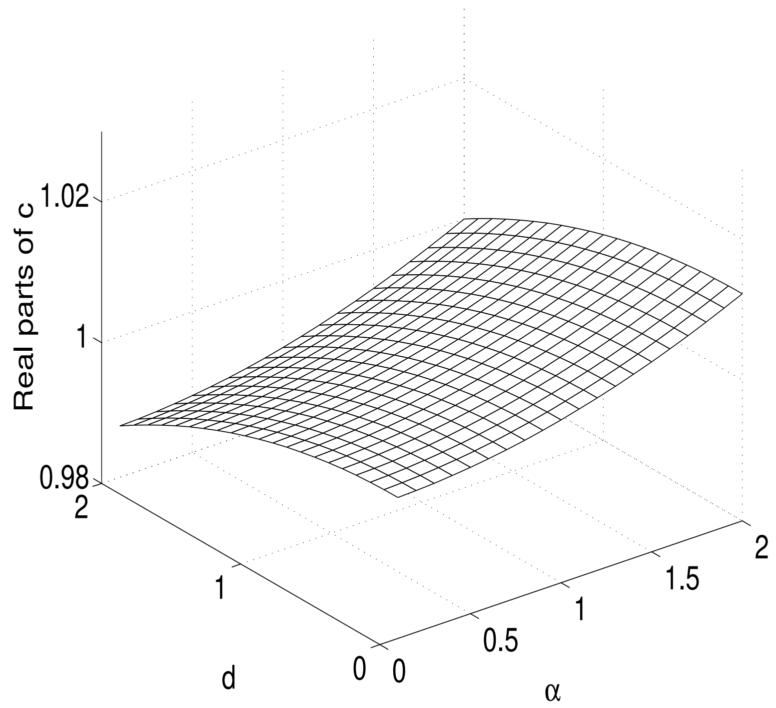

Figure 4: The real parts of the characteristic solutions including the external damping coefficient d. The real parts of the characteristic solution corresponding to the maximum imaginary value is presented in the parameter domains α and d. In this example $k = 1$.

3.3. ADDITION OF INTERNAL DAMPING

Elastic materials such as rubber contain considerable internal damping. (Internal damping is defined as a relative dissipation of strain energy in the materials.) Usually internal damping is stabilizing, but under some conditions, especially when there are non-conservative forces, internal damping can be destabilizing (Bolotin [14], Hendricks [15], Shaw and Shaw [16], and Higuchi and Dowell [20]). For the instability by modal interactions, i.e., the instability accompanied by the collision of frequencies with changing parameters, it is reported that small internal damping can be destabilizing. Effects of internal damping on the system being studied are not clear. In this section, the effects of internal damping, referred to as structural damping, on system stability are considered.

A stress-strain relation including internal damping is given by

$$\sigma_x(x,t) = E\epsilon + V\dot{\epsilon} = E\frac{\partial u}{\partial x} + V\frac{\partial \dot{u}}{\partial x}, \tag{13}$$

where E is the modulus of elasticity and V is the modulus of viscosity of the

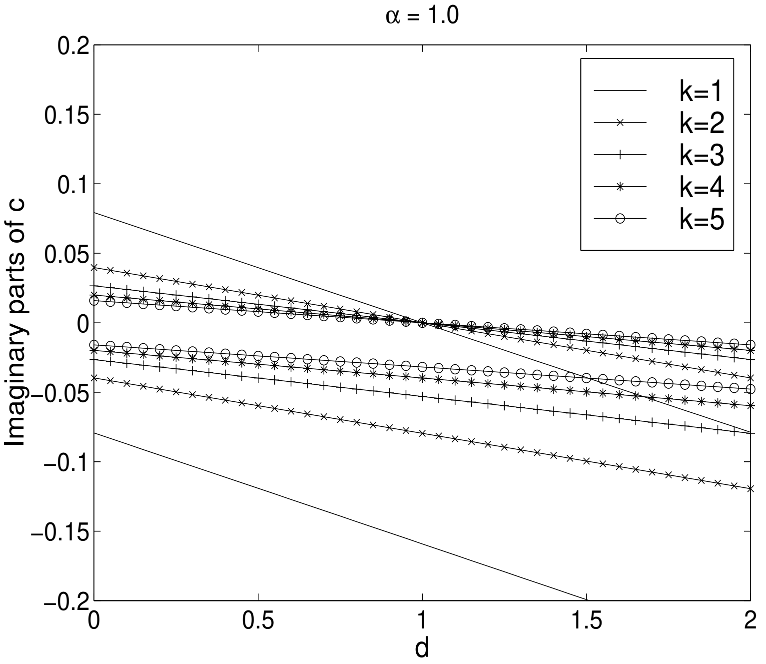

Figure 5: The imaginary parts of the characteristic solutions including the external damping coefficient d. In this example $\alpha = 1.0$.

material. Applying equation (13) to equations (1) and (2), an equation of motion including internal damping γ is given by

$$\frac{\partial}{\partial x}\{e^{-\alpha x}\frac{\partial u}{\partial x}\} + \gamma\frac{\partial}{\partial x}\{e^{-\alpha x}\frac{\partial^2 u}{\partial x \partial t}\} = e^{-\alpha x}\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

where γ is an internal damping coefficient defined as V/AE. The characteristic equation obtained by substituting a solution of the form of equation(6) into equation(14), and applying the periodic boundary conditions of equation(4), is

$$c^2 + \gamma(2\pi ki - \alpha)c - (1 + \frac{\alpha i}{2\pi k}) = 0. \tag{15}$$

The real and imaginary parts of the characteristic solution are

$$R = \pm r^{1/2}\cos(\phi/2) + \frac{\gamma\alpha}{2}, \tag{16a}$$

$$I = \pm r^{1/2}\sin(\phi/2) - \pi\gamma k, \tag{16b}$$

where $r^{1/2} = \sqrt{\{1 + \gamma^2(\frac{1}{4}\alpha^2 - \pi^2 k^2)\}^2 + \{\alpha(\frac{1}{2\pi k} - \pi\gamma^2 k)\}^2}, \phi = \arctan\{\frac{\alpha(\frac{1}{2\pi k} - \pi\gamma^2 k)}{1+\gamma^2(\frac{1}{4}\alpha^2 - \pi^2 k^2)}\}$. This shows that the characteristic solutions do not have complex conjugate solutions when the internal damping γ exist.

Figures 6 and 7 show the imaginary and real parts of the characteristic solution corresponding to the maximum imaginary value in the parameter domains α and γ, respectively. The imaginary parts attain negative values when increasing γ or decreasing α, as shown in Figure 6. The speed of waves versus γ and α is shown in Figure 7. From these results, it is concluded that the system is stabilized by increasing internal damping.

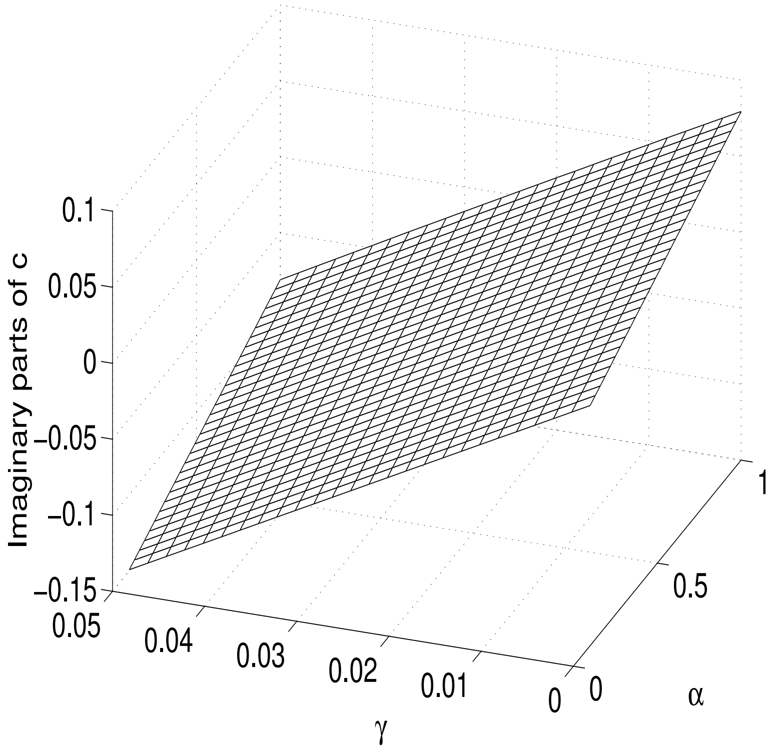

Figure 6: The imaginary parts of the characteristic solution including the internal damping coefficient γ. The maximum value of the imaginary parts is presented in the parameter domains α and γ. In this example $k = 1$.

Figure 8 shows the imaginary parts of c under variations in γ for several frequencies k. The system is stabilized beyond the point $\gamma = 0.025$. Note that the high frequency terms, such as $k = 5$ in Figure 8, are more easily influenced and stabilized with increased internal damping γ than low frequency terms.

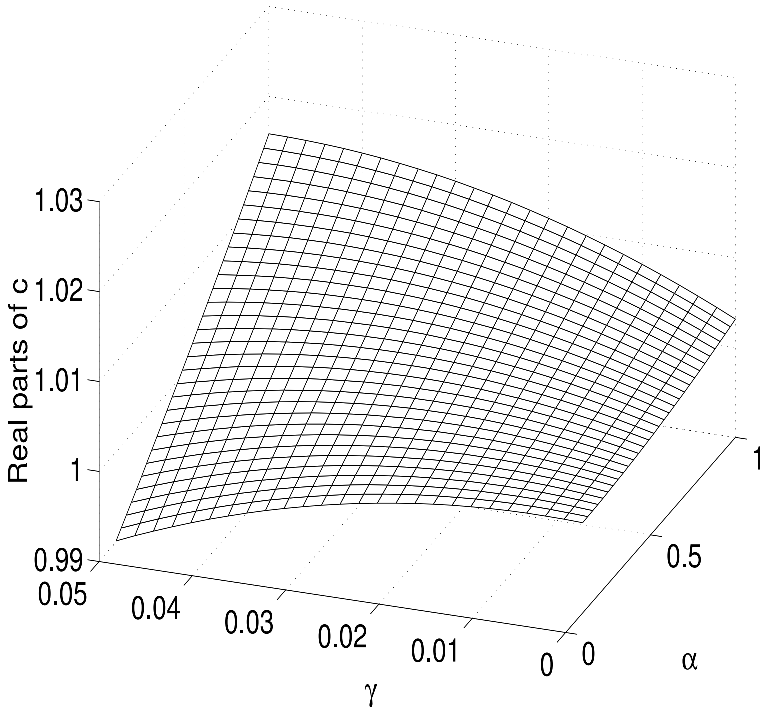

Figure 7: The real parts of the characteristic solution including the internal damping coefficient γ. The real parts of the characteristic solution corresponding to the maximum imaginary value is presented in the parameter domains α and γ. In this example $k = 1$.

4. CONCLUSION

In this study, the dynamic stability of steady frictional sliding in a finite one-dimensional system was investigated. Under periodic boundary conditions unstable traveling waves in the one-dimensional elastic system were found to be dependent upon a constant coefficient of friction and Poisson's ratio. The effect of Poisson's ratio enabled the destabilization of the elastic continuum.

A negative slope in the friction as a function of relative velocity is not a necessary condition for the occurrence of dynamic instability. This complements previous research (Adams [7, 8], Martins *et al.* [9]). In addition, the characteristic analysis showed that dynamic instability occurs in the form of self-excited, unstable, traveling waves. The stabilizing influences of external and internal damping were studied. Low and high frequency terms of the traveling waves are stabilized by adding external and internal damping, respectively.

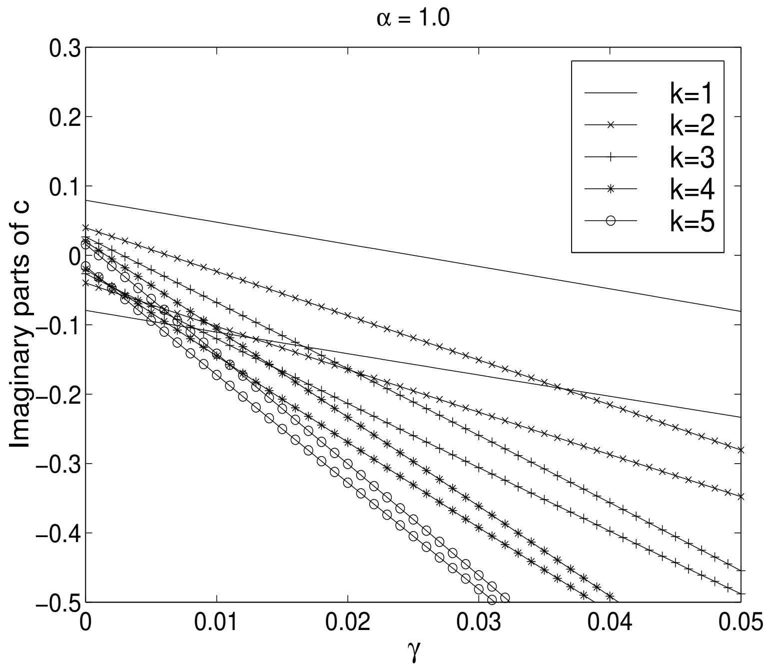

Figure 8: The imaginary parts of the characteristic solutions including the internal damping coefficient γ. In this example $\alpha = 1.0$.

5. ACKNOWLEDGMENTS

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