A DECREMENT METHOD FOR THE SIMULTANEOUS ESTIMATION OF COULOMB AND VISCOUS FRICTION

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1. INTRODUCTION

In his *Theory of Sound*, Lord Rayleigh [1] noted that, for the free vibration of a linear damped oscillator, "the difference in the logarithms of successive extreme excursions is nearly constant, and is called the logarithmic decrement." In fact, the idea goes back to Hermann Helmhotz, who in 1863 applied the logarithmic decrement to determine frequency information in musical tones given a known damping coeffient [2]. This observation can be applied to estimate the damping factor from a free vibration of a single-degree-of-freedom linear oscillator.

By 1924, in his lectures on dynamics, Lorenz [3] had included the free-response solution of a mass and spring with Coulomb friction. He commented, "die aufeinanderfolgenden absoluten Scheitelwerte oder Umkehrmpunkte einen Unterschied von $2x_r$ erhalten," essentially meaning that the successive extreme excursions decrease at a constant rate. From this property, it is possible to extract the Coulomb friction from the constant decrement of a harmonic oscillator with constant frictional damping.

These ideas are summarized in modern texts on vibrations, such as Meirovitch [4]. However, we are not aware of a method for extracting both Coulomb and viscous damping parameters by the examination of the free-vibration decrements of a system with both forms of dissipation (although it is conceivable that multiple damping parameters be obtained from general nonlinear parametric identification schemes, such as that of Stry and Mook [5]). In this letter, we present preliminary ideas and tests on such a decrement method. The method is applicable to linear single-degree-of-freedom systems with both viscous and dry damping.

2. FREE VIBRATION WITH COULOMB AND VISCOUS DAMPING

We consider a mass-spring-dashpot with a dry contact. By Newton's second law, the equation of motion is

$$m\ddot{x} + c\dot{x} + kx + f(\dot{x}) = 0, \tag{1}$$

where x is the displacement of the mass and spring from the unstretched position, m, c, and k are the mass, viscous damping coefficient, and stiffness. The dry-friction has the form $f(\dot{x}) = f_k \operatorname{sign}(\dot{x}), \dot{x} \neq 0$, and $-f_s \leq f(0) \leq f_s$. If we were to assume the existence of a coefficient of friction, such that $f(\dot{x}) = N\mu(\dot{x})$, where N is the normal load and μ is the coefficient of friction consisting of a static coefficient of friction, μ_s, and a kinetic coefficient of friction, μ_k, then this friction law would correspond to Coulomb's law. Nontheless, we loosely refer to $f(\dot{x})$ as Coulomb friction.

The equilibrium solution of this equation of motion can be obtained by letting $\ddot{x} = \dot{x} = 0$. This leaves us with a locus of equilibria for $-x_s \leq x \leq x_s$, where $x_s = f_s/k$. Equation (1) is piecewise solvable, and can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\omega_n^2 x_k, \qquad \dot{x} > 0, \tag{2}$$

and

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = +\omega_n^2 x_k, \qquad \dot{x} < 0, \tag{3}$$

where $\omega_n^2 = k/m, 2\zeta\omega_n = c/m$, and $x_k = f_k/k$.

If we begin with $x(t_0) = X_0 > x_s$ and $\dot{x}(t_0) = 0$, then motion starts with $\dot{x} < 0$. The response to equation (3) then has the form

$$x(t) = (X_0 - x_k)e^{-\zeta\omega_n(t-t_0)}(\cos\omega_d(t-t_0) + \beta\sin\omega_d(t-t_0)) + x_k, \tag{4}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\beta = \zeta/\sqrt{1-\zeta^2}$. This equation is valid until $\dot{x} = 0$, at which time $t = t_1 = t_0 + \pi/\omega_d$, and $X_1 = x(t_1) = -e^{-\beta\pi}X_o + (e^{-\beta\pi} + 1)x_k$. If $X_1 < -x_s$, then the mass will reverse direction and continue sliding with $\dot{x} > 0$ according to equation (2). The solution for this interval of motion is

$$x(t) = (X_1 + x_k)e^{-\zeta\omega_n(t-t_1)}(\cos\omega_d(t-t_1) + \beta\sin\omega_d(t-t_1)) - x_k, \tag{5}$$

which is valid until $\dot{x} = 0$, at which time $t = t_2 = t_1 + \pi/\omega_d$, and $X_2 = x(t_2) = -e^{-\beta\pi}X_1 - (e^{-\beta\pi} + 1)x_k$. If $X_2 > x_s$, motion will continue.

This process can be iterated until $-x_s \leq X_n \leq x_s$, at which time the motion stops. This iterated process leads to a recursive relation for the successive peaks and valleys in the oscillatory response:

$$X_i = -e^{-\beta\pi}X_{i-1} + (-1)^{i-1}(e^{-\beta\pi} + 1)x_k, \qquad i = 1, 2, \ldots, n. \tag{6}$$

From this evolution of decaying peaks and valleys, we can isolate the viscous effect, and then extract the Coulomb effect. A sum of consecutive extreme displacement values cancels out the dry-friction contribution. Taking the ratio between successive sums yields

$$\frac{X_i + X_{i+1}}{X_{i-1} + X_i} = -e^{-\beta\pi}. \tag{7}$$

Thus, a logarithmic decrement reveals the viscous dependence:

$$\log(-\frac{X_i + X_{i+1}}{X_{i-1} + X_i}) = -\beta\pi. \tag{8}$$

Once the quantity β has been estimated, we can estimate ζ, and also the dry-friction parameter x_k from equation (6).

A fundamental problem in an experimental system is that, because of the locus of equilibria, it may be difficult to determine the position in which the spring is unstretched. Thus, measurements may have a constant bias with respect to our formulation. There is a simple way to deal with this. If the biased measurement is $y = x + \delta$, we remove the bias δ by subtracting two measured peaks (or valleys) Y_i. Since $Y_i - Y_j = X_i - X_j$, we can work with the difference between two recursive relations (6) such that

$$X_{i+1} - X_i = -e^{-\beta\pi}(X_i - X_{i-1}) + 2(-1)^i(e^{-\beta\pi} + 1)x_k, \qquad i = 1, 2, \ldots, n-1. \quad (9)$$

By summing the equation for $X_{i+1} - X_i$ with that of $X_i - X_{i-1}$, we eliminate dry-friction contribution. An alternative decrement equation can then be written as

$$\frac{X_{i+1} - X_{i-1}}{X_i - X_{i-2}} = -e^{-\beta\pi}, \quad (10)$$

and

$$\log\left(-\frac{X_{i+1} - X_{i-1}}{X_i - X_{i-2}}\right) = -\beta\pi. \quad (11)$$

In what follows, we will apply this decrement idea numerically and experimentally to estimate both the linear viscous-damping factor and the kinetic friction quantity $x_k = f_k/k$.

3. NUMERICAL EXAMPLE

In this example, we numerically integrate the equation (1) with the parameter values $m = 1, k = 10, c = 0.2$, and $f_k = 0.2$. Corresponding to these values, we have $\omega_n^2 = 10, \zeta = 0.0316$, and $x_k = 0.02$. The numerical integration algorithm, based on a fifth-order Runge-Kutta method, carefully accounts for the discontinuous switching in the friction, by checking whether the velocity has changed sign, and iterating to precisely find the time of switching. Similar algorithms are discussed in Shaw [6] and Feeny and Moon [7]. Switching points locate the extreme excursions during the free vibration. The initial conditions were $x(0) = 1$ and $\dot{x}(0) = 0$.

Under these conditions, the numerical solution goes through six periods of damped oscillation prior to sticking. Among the extreme excursions determined by the numerical simulation are $X_1 = -0.8673, X_{10} = 0.1164$, and $X_{11} = -0.0673$. Applying the logarithmic decrement based on equation (7), we can write

$$\frac{X_{10} + X_{11}}{X_0 + X_1} = -e^{-10\beta\pi}, \quad (12)$$

whence viscous parameter estimates are $\hat{\beta} = 0.0316$ and $\hat{\zeta} = 0.0316$. Inserting $\hat{\beta}$ in equation (6) leads to the estimate $\hat{x}_k = 0.02$, and hence $\hat{f}_k = 0.2000$. Thus the estimates are identical to the known values of these parameters.

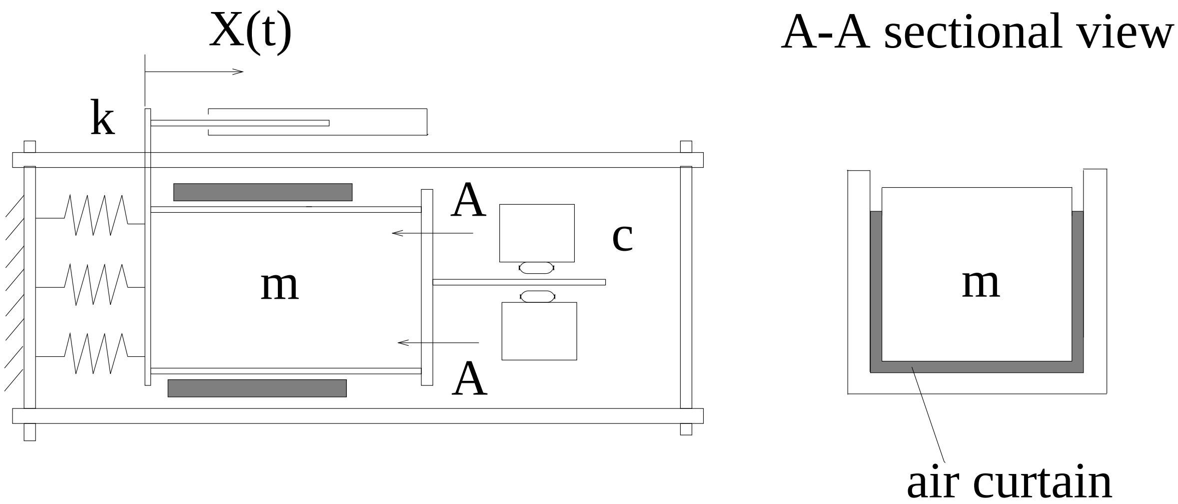

Figure 1: Schematic diagram of the experimental mass-spring-damper with dry friction. The parameters k, m, and c represent the springs, mass, and eddy-current damping mechanism, respectively. The mass is guided by an air track depicted in the A-A cross section.

4. EXPERIMENTAL EXAMPLE

We tested these ideas on an experimental mass-spring-damper with a dry contact. A schematic diagram of the experiment is shown in Figure 1. The experiment consists of a mass m of 1.51 kg attached to a rigid frame through three springs, k, with a total stiffness of 54 N/m. The mass glides in an air track. An eddy-current damper is used to provide a nearly linear damping characteristic. For this damper, coiled wires are used to produce a magnetic field, which interacts with a steel flange attached to the mass to provide the damping. A dry contact was created by packing paper between the electromagnets and the steel flange. An impulse was given by the finger in such a way that the impulse ended prior to the initial peak in the free response.

An LVDT (linear voltage differential transformer) is used to sense the displacement of the mass. The damping factor with the eddy-current damper switched off was 0.003.

We present results for two test cases. Case 1 has lighter damping than Case 2. For some basis of evaluating the results, we estimated the isolated effects of each damping type. The eddy-current damping ratio ζ could be determined by removing the dry contact and performing a logarithmic decrement. The dry-friction damping x_k was estimated by switching off the eddy-current damper. Figure 2 shows example free responses with eddy-current damping alone (dry friction removed) and with dry friction alone (eddy-current switched off). The exponential envelope $Ce^{-\zeta\omega_n t}$ is added to visualize the linearity of the response with eddy-current damping. Close examination of the decrements under dry friction suggests a slightly higher friction during oscillations of larger amplitudes. Estimates from 18 sets of consecutive peaks and valleys in this time trace reveal an average dry-friction parameter of $x_k = 0.116$ with a standard deviation of 7.7%.

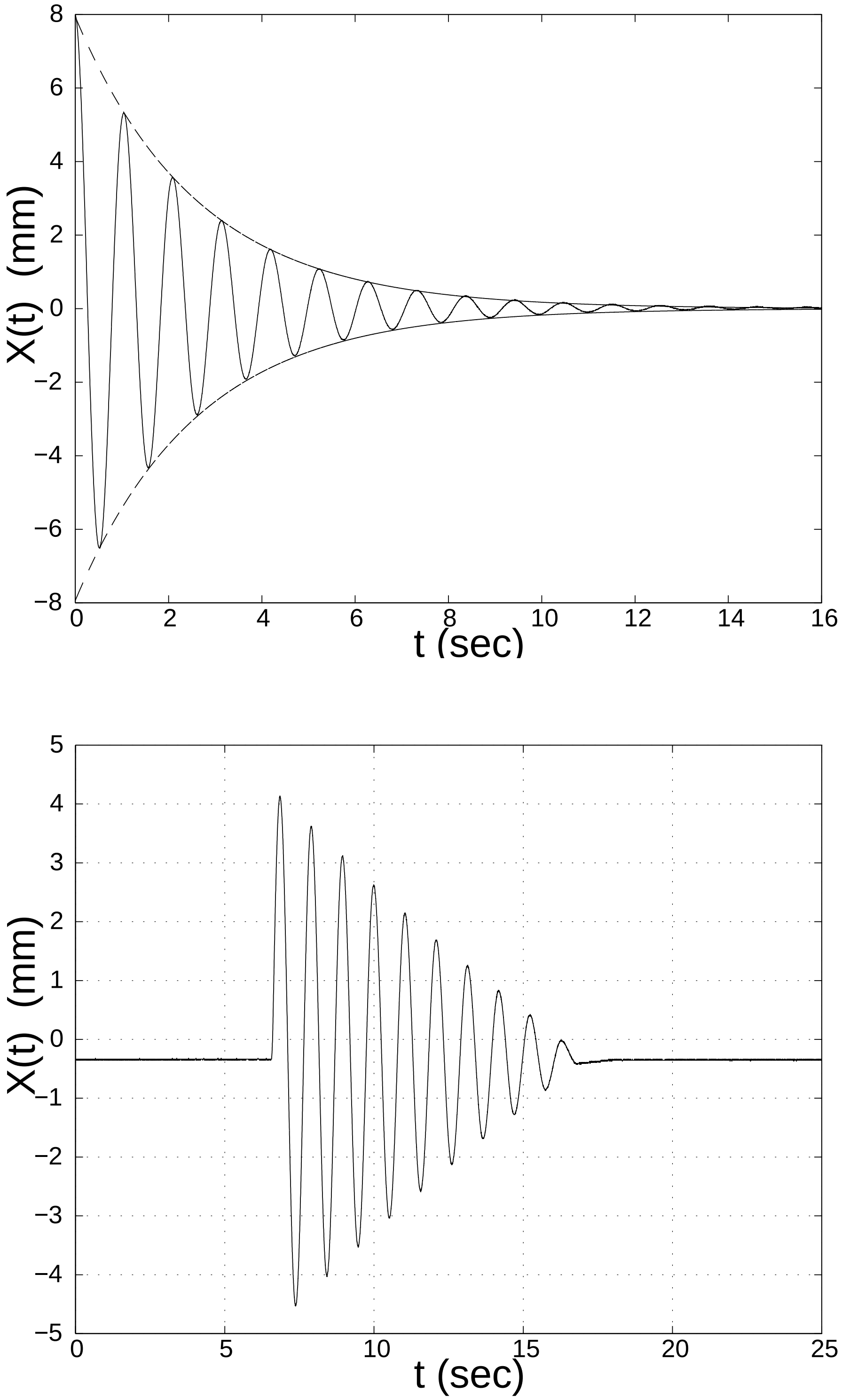

Figure 2: (a) The free response with eddy-current damping alone (dry friction removed). The solid line indicates the computed exponential envelope based on ζ estimated by the traditional logarithmic decrement. (b) The free response with dry friction alone (eddy current switched off) indicates a nearly constant decrement x_k per half period.

Table 1: Experimental estimates of damping parameters. Hatted quantities correspond to estimates under combined damping. The reference quantities correspond to estimates under a single damping source.

Parameters	$\hat{\beta}$	$\hat{\zeta}$	ζ	Error	$\hat{x}_k$	x_k	Error
Case 1	0.0645	0.0644	0.0638	0.93%	0.106 mm	0.115 mm	7.8%
Case 2	0.0281	0.0281	0.0275	2.1%	0.124 mm	0.130 mm	4.6%

The results are shown in Table 1. The values $\hat{\beta}$, $\hat{\zeta}$, and $\hat{x}_k$ are estimated from the combined decrement method, applied to all consecutive peaks and consecutive valleys and then averaged. During oscillations with combined damping, the estimated viscous damping factors are within 2% of the values estimated without dry friction. The estimated dry damping terms are less consistent, with the estimates under combined damping differing by 5% and 8% of the estimates under dry damping alone. However, if we consider the dry damping in this case to be "small", then the magnitudes of the deviations between the two estimates is smaller.

We can speculate on the sources error. There might be error inherent to the assumption that the sliding friction is constant. In fact, many friction models incorporate velocity dependences. There may also be some variation in the normal load. Our activities regarding forced oscillations on a very similar experimental apparatus (with a frictional load cell in place of the eddy-current apparatus) indicate very small variations in the friction and normal load during the motion. Furthermore, errors in the estimate of $\hat{\beta}$ are propagated into the estimate of $\hat{x}_k$. The numerical sensitivity of this aspect of the scheme is under study.

To estimate the overall quality of the identification of our damping parameters, we simulated the identified mass-spring system. Figure 3 shows a comparison between the experimental free vibration with combined damping and a simulated response with damping parameters estimated simultaneously from the peaks of the solid curve. The frequency of free oscillation was estimated from the positions of peaks in the response. The comparison is visually quite good. Thus, an interpretation might be that, in this case, the damping parameters estimated for a model including viscous and Coulomb friction fit the experimental system well with regard to its physical damping mechanisms.

5. CONCLUSION

In this letter, we have presented a method for estimating both viscous and Coulomb damping parameters simultaneously from the decrement of a single free-vibration response. We tested the idea numerically and experimentally. The results are promising. This method can be easily applied whenever the free vibration in a regime where the undamped oscillator is nearly linear, and when the damping is light enough that the free response undergoes a few oscillations.

In our continuing work on this idea, we are analyzing the effects of certain sources

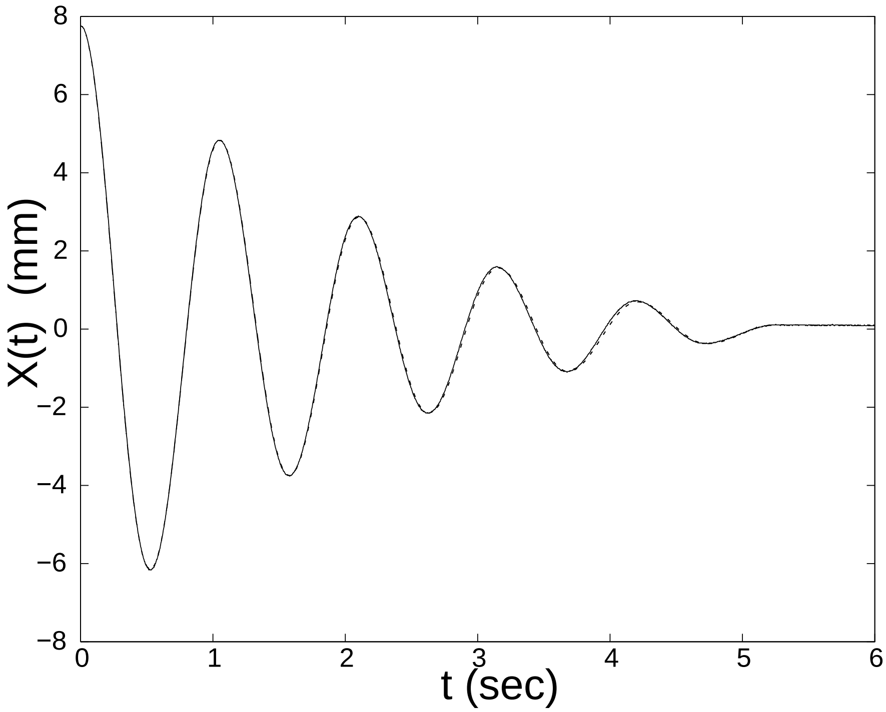

Figure 3: The solid line shows a free vibration of the experimental system with both eddy-current and dry damping. The experimental parameters correspond to case 2. Superimposed is a dashed line representing a numerical simulation which incorporated the damping parameters $\hat{\zeta}$ and $\hat{x}_k$ estimated from the solid curve. The experiment and the simulation are nearly indistinguishable.

of error. These might include estimating the effect that an error in $\hat{\beta}$ has on the estimation of x_k. We may also consider how deviations of the physical damping from either the linear viscous model or the Coulomb model affect the estimates. We are applying the method to characterize the dry and viscous damping in an industrial system.

6. ACKNOWLEDGEMENTS

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