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SECOND ORDER PERTURBATION ANALYSIS OF A FORCED NONLINEAR MATHIEU EQUATION

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ABSTRACT

The present study deals with the response of a forced nonlinear Mathieu equation. The equation considered has parametric excitation at the same frequency as direct forcing and also has cubic nonlinearity and damping. A second-order perturbation analysis using the method of multiple scales unfolds numerous resonance cases and system behavior that were not uncovered using first-order expansions. All resonance cases are analyzed. We numerically plot the frequency response of the system. The existence of a superharmonic resonance at one third the natural frequency was uncovered analytically for linear system. (This had been seen previously in numerical simulations but was not captured in the first-order expansion.) The effect of different parameters on the response of the system previously investigated are revisited.

INTRODUCTION

This work was originally motivated by our interest in studying the in-plane dynamics of wind turbine blades. The equation of motion developed include terms of a forced Mathieu equation, which caught our interest as a fundamental equation in dynamics. The detailed development of the governing equations of motion is dealt in [1]. The incorporation of nonlinearity due to large deflections in the formulation of the model gives rise to cross-coupled displacement, velocity, and acceleration terms in the equation of motion. The single-mode nonlinear equation has elements of a forced Mathieu equation,

$$\ddot{q}+2\varepsilon\mu\dot{q}+(\omega^2+\varepsilon\gamma\cos\Omega t)q+\varepsilon\alpha q^3=F\sin\Omega t, \qquad (1)$$

which itself warrants study as a fundamental equation in dynamics.

Reference [2] dealt with the super- and sub-harmonic resonances for the forced Mathieu equation. A first-order multiple scales analysis of equation (1) reveals the existence of superharmonics at a third and half the natural frequency and subharmonics at twice and thrice the natural frequency of the system. The superharmonic at order one half persists for the linear system, while that of order one third requires nonlinearity in the first-order expansion. Numerical simulations of equation (1) validated the occurrence of these harmonics. However, numerical simulations indicated that the superharmonic resonance at order $1/3$ can indeed occur in the linear system (see figure 1), as reported in [2]. In this work, we seek to explain this with a second-order perturbation expansion.

There have been extensive studies on systems with parametric excitation that fit in into a minor variation of the Mathieu equation. Shaw *et al.* [3, 4] have studied MEMS structures with parametric amplification and have demonstrated it using

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experiments as well. Other work has examined nonlinear variations of the Mathieu equation, which have included van der Pol, Rayleigh, and Duffing nonlinear terms. [5–10] have analyzed the dynamics, stability control and bifurcations of a parametrically excited systems.

Furthermore, the Mathieu equation is well known to have stability wedges in the parametric forcing amplitude-frequency space, such that the fixed point at the origin can be stable or unstable depending on these parameters. We expect that the introduction of small direct excitation will remove the existence of a fixed point at the origin, replacing it with a periodic orbit, and also perturb the stability characteristics now in reference to the periodic orbit.

The stability wedges of the Mathieu equation can be studied by applying Floquet theory with harmonic balance solutions [11], and also by a higher-order perturbation expansion [12]. The introduction of direct forcing to the Mathieu equation turns it into a system that does not directly align it with Floquet theory. Therefore, to study the perturbations of the stability characteristics, we turn to a higher-order multiple scales expansion.

In the current work, we first analyze equation (1) with two orders of expansion in order to capture superharmonic resonance at one-third for the linear system with hard forcing. Researchers have successfully employed higher order expansions to study inherent dynamic in systems [13–15] and have otherwise employed different scaling techniques to study dynamical systems [16]. We extend our second-order perturbation analysis for a weakly forced system to two orders of expansion in order to aid us to capture information regarding stability transition curves (Arnold tongue, see figure 4 [11]). Inoue et al. [17, 18] have studied the vibrations of wind turbine out-of-plane blade motion and have reported the occurrence of superharmonic resonance both in simulations and experiments.

SECOND-ORDER PERTURBATION ANALYSIS

We consider two excitation “levels” for our second order analysis. Hard excitation i.e. direct forcing F is of order 1 and soft excitation when direct forcing is of order ε. Figure 1 shows the response of the system in equation (1) without nonlinearity for a system with $O(1)$ excitation. Based on 1^{st} order analysis, the existence of super harmonic resonance at order one-half is expected for a system with direct and parametric excitation at the same frequency. Numerical simulations confirm this, but also show a peak at a third of the natural frequency. This is not uncovered in the first order analysis. When we set $\alpha = 0$ in the first-order analysis, the resonance condition is non existent. We thus proceed to conduct the second order perturbation analysis of the linear form of equation (1)

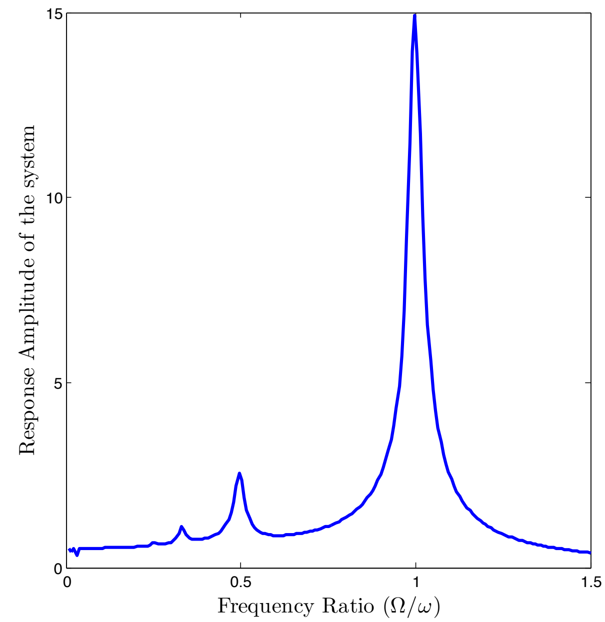

FIGURE 1. Simulated response of the linear case of equation (1) showing superharmonic resonances at orders 1/2 and 1/3; $\mu = 0.05$, $\varepsilon = 0.1, \alpha = 0, F = 0.1, \gamma = 3$

damping is mislabeled here or in Fig. 6. I think zeta = 0.025.

Linear Forced Mathieu Equation with Hard Excitation

Consider the system, given by equation (1) with no nonlinearity. The equation is of the form:

$$\ddot{q} + 2\varepsilon\mu\dot{q} + (\omega^2 + \varepsilon\gamma\cos\Omega t)q = F\sin\Omega t. \tag{2}$$

Employing MMS, we incorporate three time scales (T_0, T_1, T_2), and allow for a dominant solution q_0 and slow variations of that solution q_1, q_2, such that

$$q = q_0(T_0,T_1,T_2) + \varepsilon q_1(T_0,T_1,T_2) + \varepsilon^2 q_2(T_0,T_1,T_2) + \ldots \tag{3}$$

where $T_i = \varepsilon^i T_0$. Then $\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2$ and $D_i = \frac{\partial}{\partial T_i}$.

We substitute this into our ODE and then simplify and extract the expressions for coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2$:

$$\begin{aligned}
O(1):\ & {D_0}^2 q_0 + \omega^2 q_0 = F\sin\Omega T_0 \\
O(\varepsilon):\ & {D_0}^2 q_1 + \omega^2 q_1 = -2\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos\Omega T_0 \\
O(\varepsilon^2):\ & {D_0}^2 q_2 + \omega^2 q_2 = -2D_0 D_1 q_1 - ({D_1}^2 + 2D_0 D_2) q_0 \\
& \qquad -2\mu(D_0 q_1 + D_1 q_0) - \gamma q_1 \cos\Omega T_0
\end{aligned} \tag{4}$$

Solving O(1) equation, we arrive at the solution of q_0 as

$$q_0 = Ae^{i\omega T_0} - i\Lambda e^{i\Omega T_0} + c.c. \tag{5}$$

where $\Lambda = \dfrac{F}{2(\omega^2 - \Omega^2)}$, and $A = \dfrac{1}{2}ae^{i\beta}$.
The coefficient A, and hence a and β are functions of T_1 and T_2.

Substituting this in $O(\varepsilon)$, we arrive at the expression

$$\begin{aligned} {D_0}^2 q_1 + \omega^2 q_1 \; = & -2\mu(Ai\omega e^{i\omega T_0} + \Lambda\Omega e^{i\Omega T_0}) - 2D_1 Ai\omega e^{i\omega T_0} \\ & -\frac{\gamma}{2}(Ae^{i(\omega+\Omega)T_0} + \bar{A}e^{i(\Omega-\omega)T_0} - i\Lambda e^{2i\Omega T_0}) + c.c \end{aligned}$$

Here, $\Omega \approx \omega/3$ is not a combination that would lead to secular terms. As we are seeking that specific resonance condition, we find the solution of q_1 for a general case.

The solvability condition at $O(\varepsilon)$ is $-2\mu Ai\omega - 2i\omega D_1 A = 0$ and hence, the particular solution for q_1 is

$$\begin{aligned} q_1 \; = & -\frac{2\mu\Lambda\Omega}{(\omega^2 - \Omega^2)}e^{i\Omega T_0} + \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)}e^{i(\Omega+\omega)T_0} \\ & + \frac{\gamma\bar{A}}{2(\Omega^2 - 2\omega\Omega)}e^{i(\Omega-\omega)T_0} + \frac{i\gamma\Lambda}{2(\omega^2 - 4\Omega^2)}e^{2i\Omega T_0} + c.c \end{aligned} \tag{6}$$

We substitute the solutions for q_0, q_1 into the $O(\varepsilon^2)$ expression in equation (4). We recognize that the terms in the ODE for q_2 have either q_0 or q_1 differentiated over different time scales. Only the $\gamma q_1 \cos\Omega T_0$ term i.e. the last term in the expression has cross coupled terms.

Expanding that term we notice that by multiplying the $\dfrac{i\gamma\Lambda}{2(\omega^2 - 4\Omega^2)}e^{2i\Omega T_0}$ from the q_1 solution by the $\dfrac{e^{i\Omega T_0}}{2}$ term from the $\cos\Omega T_0$ component that appears in the O(ε^2) terms in equation (4) produces the exponential term $e^{3i\Omega T_0}$.

This would give rise to the $1/3$ superharmonic. To capture this we needed to go one level deeper in our analysis and correspondingly the numerical simulation of the system shows that the peak at this frequency is an order smaller than the superharmonic obtained at half the frequency.

The solvability conditions up to $O(\varepsilon^2)$ for $3\Omega \approx \omega$ are

$$\begin{aligned} O(\varepsilon): & \;\; -2\mu\omega Ai - 2D_1 Ai\omega = 0 \\ O(\varepsilon^2): & \; -{D_1}^2 A - 2i\omega D_2 A - 2\mu D_1 A - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)} \\ & \; -\frac{\gamma^2 A}{4(\Omega^2 - 2\omega\Omega)} - \frac{i\gamma^2\Lambda}{4(\omega^2 - 4\Omega^2)}e^{i\sigma T_1} = 0 \end{aligned} \tag{7}$$

where σ is the detuning parameter defined as $3\Omega = \omega + \varepsilon\sigma$.

From this it is clear that the forced linear Mathieu equation has a one-third superharmonic. Furthermore, as evidenced from figure 1 the response is ε order lower than the peak at one-half. In order to get an expression for A we need to solve the solvability conditions at $O(\varepsilon)$ and $O(\varepsilon^2)$ together. Eliminating $D_1 A$ and $D_1^2 A$ from the $O(\varepsilon^2)$ equation we can show that the resulting solvability conditions come from a multiple scale expansion of

$$\begin{aligned} & -2i\omega\frac{dA}{dt} - 2\varepsilon i\omega\mu A + \varepsilon^2\Bigg(\mu^2 A - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)} \\ & -\frac{\gamma^2 A}{4(\Omega^2 - 2\omega\Omega)} - \frac{i\gamma^2\Lambda}{4(\omega^2 - 4\Omega^2)}e^{i\sigma\varepsilon t}\Bigg) = 0 \end{aligned} \tag{8}$$

Also substituing, $\Lambda = \dfrac{F}{2(\omega^2 - \Omega^2)}$ and $\Omega = \dfrac{\omega + \varepsilon\sigma}{3}$ and using expansion rules and retaining only up to two powers of ε we get

$$-2i\omega\frac{dA}{dt} - 2\varepsilon i\omega\mu A + \varepsilon^2\left(\mu^2 A + \frac{9\gamma^2 A}{70\omega^2} - \frac{81 i\gamma^2 F}{320\omega^4}e^{i\sigma\varepsilon t}\right) = 0 \tag{9}$$

We seek a solution in the form $A = (B_r + iB_i)e^{i\sigma\varepsilon t}$, with real B_r and B_i. We enforce this solution in equation (9), separate real and imaginary parts and cancel the common exponential term to obtain,

$$2\omega\frac{dB_i}{dt} + 2\varepsilon\sigma\omega B_r - 2\varepsilon\mu\omega B_i + \varepsilon^2\mu^2 B_r + \varepsilon^2\frac{9\gamma^2 B_r}{70\omega^2} = 0 \tag{10}$$

$$2\omega\frac{dB_r}{dt} - 2\varepsilon\sigma\omega B_i - 2\varepsilon\mu\omega B_r - \varepsilon^2\mu^2 B_i - \varepsilon^2\frac{9\gamma^2 B_i}{70\omega^2} + \varepsilon^2\frac{81\gamma^2 F}{320\omega^4} = 0 \tag{11}$$

We have to solve for B_r and B_i to get relations between $\gamma, F, \mu, \varepsilon, \sigma$ and ω. In a standard Mathieu analysis, we would not encounter the direct forcing term as in our present study; which prevents us from admitting a solution of the form $(B_r, B_i) = (b_r, b_i)e^{kt}$, as the origin is not a solution for the above equation. The analysis will proceed to seek solutions for the set of equations given by (10) and (11). Naturally the solutions will be time varying and we can capture the trend to comment on the variations in amplitude A of the system with time and as a function of system parameters.

Alternatively, we can transform A in equation (9) to polar coordinates i.e. we substitute $A = \dfrac{1}{2}ae^{i\beta}$, separate real and imaginary parts to get,

$$a\omega\dot{\beta}+\varepsilon^2\frac{\mu^2 a}{2}+\varepsilon^2\frac{9\gamma^2 a}{140\omega^2}+\varepsilon^2\frac{81\gamma^2 F}{320\omega^4}\sin(\sigma\varepsilon t-\beta)=0 \quad (12)$$

$$-\omega\dot{a}-\varepsilon\omega\mu a-\varepsilon^2\frac{81\gamma^2 F}{320\omega^4}\cos(\sigma\varepsilon t-\beta)=0 \quad (13)$$

We make the system of equation (12) and (13) autonomous by substituting $\sigma\varepsilon t-\beta=\phi$ and correspondingly $\sigma\varepsilon-\dot{\beta}=\dot{\phi}$ to get to expressions for $\dot{a}$ and $\dot{\phi}$.

For steady state solutions we substitute $\dot{a}=\dot{\phi}=0$, combine the two equations by squaring and adding to get,

$$\left(a\sigma\varepsilon+\frac{\varepsilon^2\mu^2 a}{2\omega}+\frac{\varepsilon^2 9\gamma^2 a}{140\omega^3}\right)^2+(\varepsilon\mu a)^2=\left(\frac{\varepsilon^2 81\gamma^2 F}{320\omega^5}\right)^2 \quad (14)$$

We solve equation (14) as a quadratic a. The equation is of the form $pa^2-q=0$. The solution of which (after canceling a common ε term)is

$$a=\sqrt{\frac{q}{p}}$$

where

$$p=\left(\sigma+\frac{\varepsilon\mu^2}{2\omega}+\frac{\varepsilon 9\gamma^2}{140\omega^3}\right)^2+\mu^2$$
$$q=\left(\frac{\varepsilon 81\gamma^2 F}{320\omega^5}\right)^2 \quad (15)$$

Hence, we have an expression for the amplitude of the system as a function of all its parameters. Since, p and q are positive solution exist over the entire parameter space. To find a_{max} we differentiate the expression for a^2 with respect to σ i.e. compute $\frac{d(14)}{d\sigma}$ and find the value of σ_{max} where $\frac{da^2}{d\sigma}=0$. This yields,

$$2\left(\sigma_{max}+\frac{\varepsilon\mu^2}{2\omega}+\frac{\varepsilon 9\gamma^2}{140\omega^3}\right)a^2=0$$

From this we get the value of σ_{max} and the corresponding value of $a_m ax$ as

$$\sigma_{max}=-\left(\frac{\varepsilon\mu^2}{2\omega}+\frac{\varepsilon 9\gamma^2}{140\omega^3}\right)$$
$$a_{max}=\left(\frac{\varepsilon 81\gamma^2 F}{320\mu\omega^5}\right) \quad (16)$$

Figure 2 shows a numerical plot of the variation of amplitude with respect to the detuning parameter. The maximum value for the parameter used in the plot give $a_{max}=0.2278$ at $\sigma=-0.07$ which is the same values as obtained by using equation (16). Alternatively, we can compute the total amplitude of oscillation by summing up the amplitudes of the free and forced oscillation components i.e. compute $|q|$ by summing $|a|$ and $|2\Lambda|$ and compare the response to the numerical simulations shown in figure 1. The value of a_{max} for the parameters shown in figure 1 is calculated from equation (16) and is found to be $a_{max}=0.455$, which is the corresponds to the rise above the primary resonance curve.

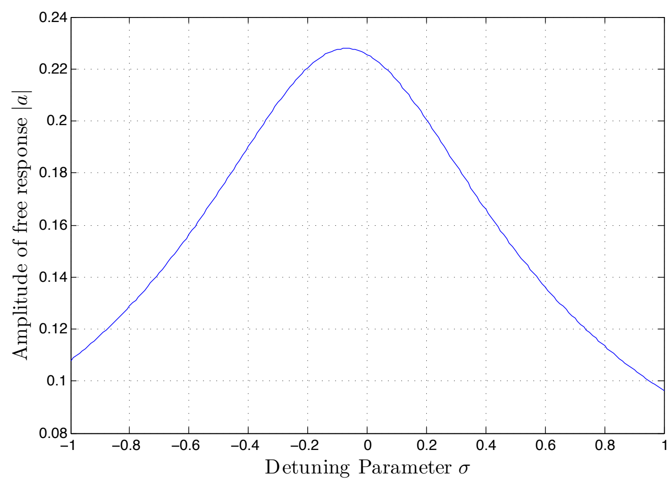

FIGURE 2. Numerical simulation to generate amplitude vs detuning curve from equation(14). Graph shown in generated using $\mu=0.5,\varepsilon=0.1,\alpha=0,F=0.5,\gamma=3,\omega=1$. $a_{max}=0.2278$ occurs at $\sigma=-0.07$

From these equations we can also plot the parameter space in which solutions exist. However, one of the primary objectives to perform second-order expansions was to uncover *hidden* resonances in the system. Preliminary numerical simulation with numerous sets of parameters showed the presence of the one-third superharmonic for the linear system as addressed in this section. We thus revert our attention back to the forced nonlinear Mathieu to study its dynamics which is our primary focus.

Further numerical simulation were carried out varying the system parameters that would make the system go unstable at primary resonance. However, if a system could operate below $\Omega\approx\omega$ excitation, such as with wind turbines, this instability would not be encountered. Figure 3 shows one such scenario. We notice the existence of multiple peaks. These can be correlated to either a harmonic frequency or the seeds of instability wedges of the unforced Mathieu equation, which occur at frequency ratios of $\sqrt{4/n^2}$ in the frequency response curve.

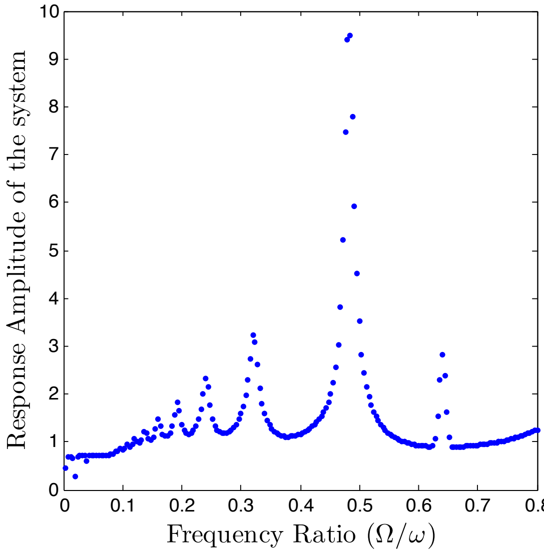

FIGURE 3. Simulated response of the linear case of equation (1) showing multiple superharmonic resonances $\mu = 0.05, \varepsilon = 0.1, \alpha = 0, F = 0.5, \omega = 0.5, \gamma = 7$. $a_{max} = 2.477$ by (16)

As we know, the instability wedges become slender and weaker as we go to the right in a typical Mathieu plot especially in the presence of damping (see figure 4). The stable and unstable characteristics of the system become local to the origin in the case of a nonlinear Mathieu equation. Global stability is determined by the other fixed points that arise in the system.

Higher δ in figure 4 translates to lower frequency ratios in our analysis. As we can see from figure 3, the response near a ratio of $2/3$ is distinct in the sense that the amplitude increases abruptly. This increase could be attributed to the presence of an instability wedge. The responses at other ratios are due to resonance phenomena. In some cases, there exists both a resonance curve and an instability wedge. Future work will explore to parameters that will make the system operate in such critical zones.

Figure 5 shows the response of the system when excited at higher frequencies than the natural frequency. We can clearly see the subharmonic occurring at twice the frequency. We also suspect that the system goes unstable if the parametric pump is sufficiently high as there is also an instability wedge at $\Omega = 2\omega$. The presence of instability wedges and harmonics of excitation need to be distinguished analytically. to do this we carry out a second order perturbation analysis of a weakly forced nonlinear Mathieu equation. Strong forcing would dominate the system response and we may lose information regarding the effect of nonlinearity, damping and parametric excitation. We could also extend the response curve higher frequency ratios and numerically plot the other subharmonics in the system.

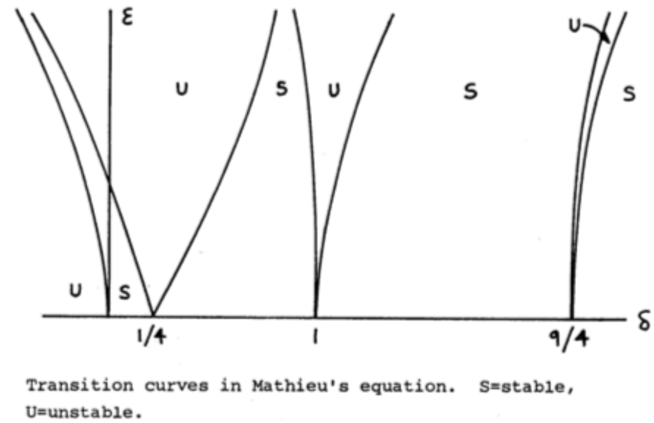

FIGURE 4. Transition curves in Mathieu equation (Figure taken from [11]). S - Stable region; U - Unstable region

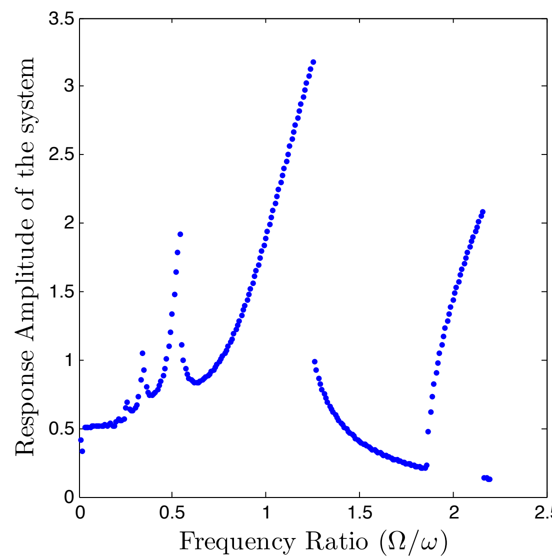

FIGURE 5. Simulated response of the linear case of equation (1) showing superharmonic resonances at orders 1/2 and 1/3 and subharmonic resonance at 2; $\mu = 0.05, \varepsilon = 0.1, \alpha = 1, F = 0.5, \omega = 0.5, \gamma = 3$

Second-Order Perturbation Analysis: Weak Forcing

As discussed in the prvious section, typically while analyzing the Mathieu equation we look for stability transition curve in the $\varepsilon - \delta$ (i.e. magnitude of parametric forcing - square of frequency) space. The transition curves are the Arnold tongues/Mathieu wedges. In our analysis of the weakly forced Mathieu, we seek to reconstruct the Arnold tongue in a similar space and study the effect of direct forcing and nonlinearity on the system. To do this we perform second order analysis to uncover resonance conditions and instability wedges. In order to do this we need to consider a system with soft excitation (O(ε) forcing). We focus our attention back to the original equation (1) restated below.

$$\ddot{q}+\varepsilon\mu\dot{q}+(\omega^2+\varepsilon\gamma\cos\Omega t)q+\varepsilon\alpha q^3=\varepsilon F\sin\Omega t$$

We follow the second-order expansion analysis done in the previous section for the case of a linear forced Mathieu equation to arrive at expressions for coefficients of $\varepsilon^0,\varepsilon^1,\varepsilon^2$ as

$$\begin{aligned}
O(1):\ & {D_0}^2q_0+\omega^2q_0 = 0\\
O(\varepsilon):\ & {D_0}^2q_1+\omega^2q_1 = -2\mu D_0q_0-2D_0D_1q_0-\gamma q_0\cos\Omega T_0\\
& \qquad -\alpha {q_0}^3+F\sin\Omega T_0\\
O(\varepsilon^2):\ & {D_0}^2q_2+\omega^2q_2 = -2D_0D_1q_1-({D_1}^2+2D_0D_2)q_0-2\mu\\
& \qquad (D_0q_1+D_1q_0)-\gamma q_1\cos\Omega T_0-3\alpha {q_0}^2q_1
\end{aligned} \tag{17}$$

The solution of $O(1)$ equation is

$$q_0=A(T_1,T_2)e^{i\omega T_0}+c.c \tag{18}$$

We substitute this into the $O(\varepsilon)$ equation and identify other resonance conditions to eliminate secular terms and seek solution of q_1.

The equation for q_1 obtained by substituting the solution for q_0 into $O(\varepsilon)$ equation is

$$\begin{aligned}
{D_0}^2q_1+\omega^2q_1 = & -2i\omega D_1Ae^{i\omega T_0}-2\mu i\omega Ae^{i\omega T_0}\\
& -\alpha(A^3e^{3i\omega T_0}+3A^2\bar{A}e^{i\omega T_0})\\
& -\gamma(\frac{A}{2}e^{i(\omega+\Omega)T_0}+\frac{\bar{A}}{2}e^{i(\Omega-\omega)T_0})-\frac{iF}{2}e^{i\Omega T_0}+c.c
\end{aligned} \tag{19}$$

We have three cases for equation (19) which can contribute towards resonance condition.

1. No specific relation between Ω and ω
2. $\Omega\approx\omega$
3. $\Omega\approx 2\omega$

Case 1: When there is no specific relationship between the forcing frequency Ω and the natural frequency ω we equate the secular terms to zero, such that

$$-2i\omega D_1A-2\mu i\omega A-3\alpha A^2\bar{A}=0$$

and solve the remaining ODE in equation (19) to get the particular solution, by treating A as constant with respect to the independent variable T_0, as

$$\begin{aligned}
q_1 = & \frac{\alpha A^3}{8\omega^2}e^{3i\omega T_0}+\frac{\gamma A}{2(\Omega^2+2\omega\Omega)}e^{i(\Omega+\omega)T_0}\\
& +\frac{\gamma\bar{A}}{2(\Omega^2-2\omega\Omega)}e^{i(\Omega-\omega)T_0}+\frac{iF}{2(\Omega^2-\omega^2)}e^{i\Omega T_0}+c.c
\end{aligned} \tag{20}$$

Case 2: For the second condition of when $\Omega\approx\omega$ i.e. $\Omega=\omega+\varepsilon\sigma_1$ (σ_1 being the detuning parameter for this case) from equation (19),

$$-2i\omega D_1A-2\mu i\omega A-3\alpha A^2\bar{A}-\frac{iF}{2}e^{i\sigma_1T_1}=0$$

forms the solvability condition. The particular solution for q_1 then becomes

$$\begin{aligned}
q_1 = & \frac{\alpha A^3}{8\omega^2}e^{3i\omega T_0}+\frac{\gamma A}{2(\Omega^2+2\omega\Omega)}e^{i(\Omega+\omega)T_0}\\
& +\frac{\gamma\bar{A}}{2(\Omega^2-2\omega\Omega)}e^{i(\Omega-\omega)T_0}+c.c
\end{aligned} \tag{21}$$

Case 3: And finally, when $\Omega\approx 2\omega$ i.e. $\Omega=2\omega+\varepsilon\sigma_2$ (σ_2 being the detuning parameter for this case)

$$-2i\omega D_1A-2\mu i\omega A-3\alpha A^2\bar{A}-\frac{\gamma\bar{A}}{2}e^{i\sigma_2T_1}=0$$

forms the solvability condition. The particular solution for q_1 then becomes

$$\begin{aligned}
q_1 = & \frac{\alpha A^3}{8\omega^2}e^{3i\omega T_0}+\frac{\gamma A}{2(\Omega^2+2\omega\Omega)}e^{i(\Omega+\omega)T_0}\\
& +\frac{iF}{2(\Omega^2-\omega^2)}e^{i\Omega T_0}+c.c
\end{aligned} \tag{22}$$

We now substitute the solutions for q_0 and q_1 from equations (5) and (6) into the expression (17) at $O(\varepsilon^2)$. After expanding the terms in the expression for the q_2 ODE, we seek to identify terms that can contribute to a resonance condition.

For Case 1 when there is not relation between Ω and ω, the expression for q_2 is:

$$D_0{}^2 q_2 + \omega^2 q_2 = -2\Bigg(i(\Omega+\omega) D_1 K e^{i(\Omega+\omega)T_0} + i(\Omega-\omega) D_1 L e^{i(\Omega-\omega)T_0} + 3i\omega D_1 M e^{3i\omega T_0} + i\Omega D_1 N e^{i\Omega T_0} \Bigg) - \Bigg(D_1{}^2 A e^{i\omega T_0} + 2i\omega D_2 A e^{i\omega T_0} \Bigg) - 2\mu \Bigg(i(\Omega+\omega) K e^{i(\Omega+\omega)T_0} + i(\Omega-\omega) L e^{i(\Omega-\omega)T_0} + 3i\omega M e^{3i\omega T_0} + i\Omega N e^{i\Omega T_0} + D_1 A e^{i\omega T_0} \Bigg) - \frac{\gamma}{2} \Bigg(K e^{i(2\Omega+\omega)T_0} + K e^{i\omega T_0} + L e^{i(2\Omega-\omega)T_0} + L e^{-i\omega T_0} + M e^{i(\Omega+3\omega)T_0} + M e^{i(-\Omega+3\omega)T_0} + N e^{i2\Omega T_0} + N \Bigg) - 3\alpha \Bigg(A^2 K e^{i(\Omega+3\omega)T_0} + \bar{A}^2 K e^{i(\Omega-\omega)T_0} + 2A\bar{A} K e^{i(\Omega+\omega)T_0} + A^2 L e^{i(\Omega+\omega)T_0} + \bar{A}^2 L e^{i(\Omega-3\omega)T_0} + 2A\bar{A} L e^{i(\Omega-\omega)T_0} + A^2 M e^{5i\omega T_0} + \bar{A}^2 M e^{i\omega T_0} + 2A\bar{A} M e^{3i\omega T_0} + A^2 N e^{i(\Omega+2\omega)T_0} + \bar{A}^2 N e^{i(\Omega-2\omega)T_0} + 2A\bar{A} N e^{i\Omega T_0} \Bigg) + c.c. \tag{23}$$

where $K = \frac{\gamma A}{2(\Omega^2+2\omega\Omega)}$, $L = \frac{\gamma \bar{A}}{2(\Omega^2-2\omega\Omega)}$, $M = \frac{\alpha A^3}{8\omega^2}$, $N = \frac{iF}{2(\Omega^2-\omega^2)}$.

Following the analysis done in [12] for the unforced, linear case, we write the solvability conditions together for various resonance cases. Once we enlist the solvability conditions for each of the resonance case identified, we look for the fixed points and stability for each case separately.

For the case of no specific relationship between ω and Ω

$$\begin{aligned} O(\varepsilon):&\quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0 \\ O(\varepsilon^2):&\ -D_1{}^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} \\ &\quad - \frac{\gamma^2 A}{4(\Omega^2+2\omega\Omega)} - \frac{\gamma^2 A}{4(\Omega^2-2\omega\Omega)} = 0 \end{aligned} \tag{24}$$

There are various other combinations of Ω and ω appearing at second-order that could lead to resonance conditions in the system. Examining the $O(\varepsilon^2)$ equation given by equation (23) for the general case we arrive at the following combinations:

1. $\Omega \approx 3\omega$
2. $\Omega \approx \omega/2$
3. $\Omega \approx 4\omega$

The other resonance combinations identified at $O(\varepsilon)$ i.e.

4. $\Omega \approx \omega$
5. $\Omega \approx 2\omega$

lead to slightly different expressions for an ODE in q_2 based on the expression of q_1 computed for each case and given in equations (21) and (22). We examine the resulting expression for solvability conditions at $O(\varepsilon)$.

For each of these conditions, the corresponding equations at $O(\varepsilon)$ and $O(\varepsilon^2)$ are given below.

$\Omega \approx 3\omega$

$$\begin{aligned} O(\varepsilon):&\quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0 \\ O(\varepsilon^2):&\ -D_1{}^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} - \frac{\gamma^2 A}{4(\Omega^2+2\omega\Omega)} \\ &\quad - \frac{\gamma^2 A}{4(\Omega^2-2\omega\Omega)} - \frac{i3\alpha F \bar{A}^2}{2(\Omega^2-\omega^2)} e^{i\sigma T_1} = 0 \end{aligned} \tag{25}$$

$\Omega \approx \omega/2$

$$\begin{aligned} O(\varepsilon):&\quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0 \\ O(\varepsilon^2):&\ -D_1{}^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} - \frac{\gamma^2 A}{4(\Omega^2+2\omega\Omega)} \\ &\quad - \frac{\gamma^2 A}{4(\Omega^2-2\omega\Omega)} - \frac{\gamma F i}{4(\Omega^2-\omega^2)} e^{i\sigma T_1} = 0 \end{aligned} \tag{26}$$

$\Omega \approx 4\omega$

$$\begin{aligned} O(\varepsilon):&\quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0 \\ O(\varepsilon^2):&\ -D_1{}^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} - \frac{\gamma^2 A}{4(\Omega^2+2\omega\Omega)} \\ &\quad - \frac{\gamma^2 A}{4(\Omega^2-2\omega\Omega)} - \frac{3\alpha\gamma \bar{A}^3}{2(\Omega^2-2\omega\Omega)} e^{i\sigma T_1} - \frac{\gamma\alpha \bar{A}^3}{16\omega^2} e^{i\sigma T_1} = 0 \end{aligned} \tag{27}$$

$\Omega \approx \omega$

$$\begin{aligned} O(\varepsilon):&\quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} - \frac{iF}{2} e^{i\sigma T_1} = 0 \\ O(\varepsilon^2):&\ -D_1{}^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} - \frac{\gamma^2 A}{4(\Omega^2+2\omega\Omega)} \\ &\quad - \frac{\gamma^2 A}{4(\Omega^2-2\omega\Omega)} - \frac{\gamma^2 \bar{A}}{4(\Omega^2-2\omega\Omega)} e^{2i\sigma T_1} = 0 \end{aligned} \tag{28}$$

$\Omega \approx 2\omega$

$$O(\varepsilon): \quad -2i\omega D_1A - 2\mu i\omega A - 3\alpha A^2\bar{A} - \frac{\gamma\bar{A}}{2}e^{i\sigma T_1} = 0$$

$$\begin{aligned} O(\varepsilon^2): & -D_1{}^2A - 2D_2Ai\omega - 2\mu D_1A - \frac{3\alpha^2A^3\bar{A}^2}{8\omega^2} \\ & -\frac{\gamma^2A}{4(\Omega^2+2\omega\Omega)} - \frac{3\alpha\gamma A\bar{A}^2}{2(\Omega^2+2\omega\Omega)}e^{i\sigma T_1} = 0 \end{aligned} \tag{29}$$

The σ 's above are the detuning parameter in each case respectively.

In this analysis, using weak forcing, we do not capture the linear superharmonic at order 3. This happens because of the way the terms and the bookkeeping parameter ε appear in the equations. The terms are multiplied at varying orders, thus burying the effect of that superharmonic "one-order lower". If we were to continue our analysis further we will encounter that resonance term.

The solutions for the set of equations given for each resonance case above would give us the slow time scale variations of the amplitude A of our fast time scale solution q. In order to arrive at a solution we re-combine the terms at $O(\varepsilon), O(\varepsilon^2)$ into a single ODE and seek solutions. To arrive at this juncture would require elimination of D_1A terms from $O(\varepsilon^2)$ equation.

We sketch the treatment of one of the resonant cases listed. We consider the terms for the primary resonance case $\Omega \approx \omega$ given in equation (28). From the $O(\varepsilon)$ equation we get,

$$D_1A = -\mu A + \frac{3\alpha iA^2\bar{A}}{2\omega} - \frac{Fe^{i\sigma T_1}}{4\omega}$$

and

$$D_1\bar{A} = -\mu\bar{A} - \frac{3\alpha i\bar{A}^2A}{2\omega} - \frac{Fe^{-i\sigma T_1}}{4\omega}.$$

We compute $D_1{}^2A$ from the above expressions as

$$\begin{aligned} D_1{}^2A = \mu^2A - \frac{6\alpha i\mu A^2\bar{A}}{\omega} - \frac{9\alpha^2A^3\bar{A}^2}{4\omega^2} + \frac{F\mu}{4\omega}e^{i\sigma T_1} - \frac{Fi\sigma}{4\omega}e^{i\sigma T_1} - \\ \frac{3\alpha FiA\bar{A}}{4\omega^2}e^{i\sigma T_1} - \frac{3\alpha FiA^2}{8\omega^2}e^{-i\sigma T_1}. \end{aligned}$$

We substitute these in the $O(\varepsilon^2)$ expressions in equation (28) to arrive at

$$\begin{aligned} & -2D_2Ai\omega + \Bigg(-\mu^2A + \frac{6\alpha i\mu A^2\bar{A}}{\omega} + \frac{9\alpha^2A^3\bar{A}^2}{4\omega^2} - \frac{F\mu}{4\omega}e^{i\sigma T_1} \\ & + \frac{Fi\sigma}{4\omega}e^{i\sigma T_1} + \frac{3\alpha FiA\bar{A}}{4\omega^2}e^{i\sigma T_1} + \frac{3\alpha FiA^2}{8\omega^2}e^{-i\sigma T_1}\Bigg) - 2\mu\Bigg(-\mu A \\ & + \frac{3\alpha iA^2\bar{A}}{2\omega} - \frac{Fe^{i\sigma T_1}}{4\omega}\Bigg) - \frac{3\alpha^2A^3\bar{A}^2}{8\omega^2} - \frac{\gamma^2A}{4(\Omega^2+2\omega\Omega)} \\ & - \frac{\gamma^2A}{4(\Omega^2-2\omega\Omega)} - \frac{\gamma^2\bar{A}}{4(\Omega^2-2\omega\Omega)}e^{2i\sigma T_1} = 0 \end{aligned} \tag{30}$$

It can easily be shown that the $O(\varepsilon)$ solvability condition from equation (28) and the $O(\varepsilon^2)$ solvability condition from equation (30) result from a multiple-scales expansion of

$$\begin{aligned} & -2i\omega\frac{dA}{dt} + \varepsilon\left(-2\mu i\omega A - 3\alpha A^2\bar{A} - \frac{iF}{2}e^{i\sigma\varepsilon t}\right) + \varepsilon^2\Bigg(\mu^2A \\ & + \frac{3\alpha FiA^2}{8\omega^2}e^{-i\sigma\varepsilon t} + \frac{\gamma^2\bar{A}}{4\omega^2}e^{2i\sigma\varepsilon t} - \frac{9\alpha i\mu A^2\bar{A}}{\omega} + \frac{\gamma^2A}{6\omega^2} \\ & + \frac{15\alpha^2A^3\bar{A}^2}{32\omega^2} + \frac{3\alpha FiA\bar{A}}{4\omega^2}e^{i\sigma\varepsilon t} + \frac{Fi\sigma}{4\omega}e^{i\sigma\varepsilon t} + \frac{F\mu}{4\omega}e^{i\sigma\varepsilon t}\Bigg) = 0 \end{aligned} \tag{31}$$

Transforming the system to polar coordinates as we did in the previous section, leads to steady state equations in a and $\phi = \sigma T_1 - \beta$. The steady state equations include $\sin(\phi), \cos(\phi)$ and $\sin(2\phi), \cos(2\phi)$ terms, which make it more difficult to solve than other familiar examples.

We follow the solution procedure given in [12]. We seek a solution in the form $A = (B_r + iB_i)e^{i\sigma\varepsilon t}$, with real B_r and B_i. We enforce this solution in equation (31), separate real and imaginary parts and cancel the common exponential term to obtain,

$$\begin{aligned} & 2\omega\frac{dB_i}{dt} + \varepsilon(2\mu\omega B_i + 2\omega\sigma B_r - 3\alpha(B_r{}^3 + B_rB_i{}^2)) \\ & + \varepsilon^2\Bigg(\mu^2B_r - \frac{3\alpha F}{4\omega^2}B_rB_i + \frac{\gamma^2}{4\omega^2}B_r + \frac{\gamma^2}{6\omega^2}B_r + \frac{9\alpha\mu}{\omega}(B_i{}^3 + \\ & B_r{}^2B_i) + \frac{15\alpha^2}{32\omega^2}(B_r{}^5 + B_i{}^4B_r + 2B_i{}^2B_r{}^3) + \frac{F\mu}{4\omega}\Bigg) = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} & -2\omega\frac{dB_r}{dt} + \varepsilon(-2\mu\omega B_r + 2\omega\sigma B_i - 3\alpha(B_i{}^3 + B_iB_r{}^2)) + \frac{\varepsilon F}{2} \\ & \left(1 + \varepsilon\left(\frac{3\alpha}{4\omega^2}(B_r{}^2 - B_i{}^2) + \frac{\sigma}{2\omega}\right)\right) + \varepsilon^2\Bigg(-\frac{\gamma^2B_i}{4\omega^2} + \mu^2B_i + \frac{\gamma}{6\omega^2}B_i \\ & -\frac{9\alpha\mu}{\omega}(B_r{}^3 + B_i{}^2B_r) + \frac{15\alpha^2}{32\omega^2}(B_i{}^5 + B_r{}^4B_i + 2B_r{}^2B_i{}^3)\Bigg) = 0. \end{aligned} \tag{33}$$

As stated in the previous section, we have to solve for B_r and B_i to get relations between $\gamma, F, \mu, \varepsilon, \sigma$ and ω. The fifth degree terms still pose a challenge. If we proceed with enforcing polar coordinate forms to the B_i, B_r terms in the equation then we are faced with $\sin(\phi), \cos(\phi)$ and $\sin(2\phi), \cos(2\phi)$ terms.

Similar analysis has been done for all the resonant conditions that exist in the system. Here, as was in the case with the linear system, we are left differential equation with higher order polynomial coefficients. Origin is not a solution for this set of equation as we have direct forcing.

Having obtained these equations, we look for stability characteristics based on system parameters to may a boundary. This will require a lot more analysis before we can draw conclusions on stabilities and the parameters that dictate behavior.

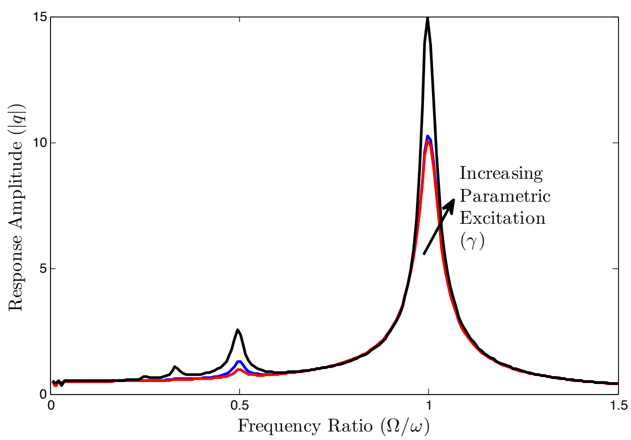

FIGURE 6. Amplitudes of simulated responses of equation (1) showing the effect of the parametric forcing amplitude; $\varepsilon = 0.1, \mu = 0.1, \alpha = 0, F = 0.5$. Different curves depict γ = 0.5, 1 and 3.

Figure 6 shows the influence of the parametric term γ on the response of the system. At primary resonance we can clearly see that beyond a certain value the curve stretches out. The first order analysis for this system once again does not capture this behavior. The expressions for the primary resonance case presented before in equation (28) are taken. For first order expansions of a linear system (the case shown in figure 6) we consider only the $O(\varepsilon)$ equation and arrive at an expression for maximum amplitude (a_{max}) as a function of system parameters.

$$a_{max} = \frac{F^2}{4\omega^2\mu^2}$$

This is consistent with linear theory. To study the influence of parametric excitation we solve a linear version of equations (32) and (33). This leads us to amplitude expressions that are dependent on $\gamma, F, \mu, \omega, \sigma$ and ε. The completed form of the expressions will show the explicit dependence of the amplitude on parametric excitation, and conclusions will be drawn in future work. The second order analysis will also provide expressions for the stability of the system.

TABLE 1. Stability Wedge and Resonance Chart. R_1: Resonance identified at 1st order of MMS expansion. R_2: Resonance identified at 2nd order of MMS expansion. W_2: Instability wedge (Arnold tongue) expression can be found at second order of MMS expansion. $-$: Known resonance case/ Instability not uncovered up to two orders of expansion

Forcing (Ω)	O(1) Forcing	O(ε) Forcing	No Forcing
2ω	R_1, W_2	R_2	R_1, W_2
ω	R_1, W_2	R_1, W_2	R_2, W_2
$2\omega/3$	-	-	-
$\omega/2$	R_1, W_2	R_1, W_2	-
$2\omega/5$	-	-	-
$\omega/3$	R_2, W_2	-	-

The boundaries that separate the solution based on stabilities are the Arnold tongues for our system. Since our system is nonlinear we expect multiple fixed points for some parameter ranges. Reference [11] has transition curves for a nonlinear Mathieu equation where the stable and unstable region have multiple fixed points. Harmonic balance was used to arrive at the power series expansions for the transition curves. Our aim is to construct the stability transition curves and analyze parameter zone where in the solution would either destabilize or be resonant. Both would lead to sustained oscillation in the system thereby causing increased loading in our wind turbine system which have been modeled using these equations. Table 1 lists the frequency ratios at which resonances or wedges have been identified.

CONCLUSION

The Mathieu equation we are dealing with is nonlinear and also has direct forcing. The analysis of the linear case with hard forcing revealed superharmonics that was previously identified using numerical simulations. The technique was extended for the nonlinear system in an effort to determine stability transition curves. We introduced forcing at two orders to identify different resonance conditions. Since this is primarily a parameter study to uncover underlying dynamics, we liberally choose the relative magnitude of parameters. Mathieu stability wedges are typically constructed for a unforced system and the forcing component poses some considerable challenge for analysis of these equations. We listed the resonance conditions. Future work will focus on the solutions of the equations obtained at various harmonics and instability boundaries and aim to get a complete picture of the inherent dynamics of a nonlinear forced Mathieu equation.

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