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Identifying Coulomb and Viscous Friction from Free-Vibration Decrements

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Abstract. This study focuses on an algorithm for the simultaneous identification of Coulomb and viscous damping effects from free-vibration decrements in a damped linear single degree-of-freedom (DOF) mass-spring system. Analysis shows that both damping effects can indeed be separated. Numerical study of a combined-damping system demonstrates a perfect match between the simulation parameters and the estimated values. Experimental study includes two types of real systems. The method is applied to an experimental industrial bearing. Experimental results are compared with numerical simulations to illustrate the reliability of this method. An analysis provides conservative bounds on error estimates. An example of the effect of quantization error on the estimations is included.

Keywords: Friction oscillator, friction estimation, friction identification, Coulomb friction, free vibration.

1. Introduction

The idea of exploiting the exponential decay of a free vibration of a viscously damped system goes back to Helmholtz, who used it in determining frequency information in musical tones in 1863 [1]. Rayleigh [2] introduced the term 'logarithmic decrement' in applying it to the estimation of the damping factor. The linear-decay property of systems with purely Coulomb friction was found in Lorenz's work as early as 1924 [3]. Calculating the amplitude decrement for consecutive cycles gives an estimation of the dry-friction effect.

However, many machines and structures have both sources of damping. Jacobsen and Ayre [4] derived an approximate scheme for estimating both quantities from free vibration decrements by noting that the viscous friction dominates in the large-amplitude responses, and that Coulomb friction dominates in the small-amplitude oscillations. An exact estimation scheme was presented by Watari [5] in 1969. It seems to have remained unknown beyond the Japanese-speaking community until the idea was recently re-derived [6].

This paper builds on the previous work by applying it to an industrial system and including an analysis of the effects of measurement error on the estimations.

Systems with solely viscous damping or dry friction can be considered as specials cases of this study. Many real frictional systems may exhibit other frictional characteristics, such as contact compliance [7–10]. Stribeck friction or unsteady friction velocity characteristics [11, 12] and state-variable friction [13, 14]. The method is not designed to account for these dynamical friction behaviors.

The next section summarizes the derivation of the key equations from the free vibration of a single-degree-of-freedom system with viscous and dry friction. In Section 3, these ideas are applied to a linear-bearing system. Section 4 contains an analysis of the effect of measurement error on the estimates. Section 5 closes with concluding remarks.

2. Derivation of the Identification Equations

The identification equations are derived from the free-vibration solution. We consider a mechanical system modeled as

$$m\ddot{x} + c\dot{x} + kx + f(\dot{x}) = 0, \tag{1}$$

where x denotes the displacement of the mass and spring from the unstretched equilibrium position, and m, c, and k represent the mass, viscous damping coefficient, and the spring stiffness. The dry friction is modeled as $f(\dot{x}) = f_k \mathrm{sign}(\dot{x}), \dot{x} \neq 0$, and $-f_s \leq f(0) \leq f_s$. If there exists a coefficient of friction, such that $f(\dot{x}) = N\mu(\dot{x})$, where N is the normal load and μ is the coefficient of friction consisting of a static coefficient, μ_s, and a kinetic coefficient of friction, μ_k, then this friction law corresponds to Coulomb's law. Nevertheless, we loosely refer to $f(\dot{x})$ as Coulomb friction.

The equilibrium solution of this equation of motion can be obtained by letting $\ddot{x} = \dot{x} = 0$. This gives rise to a locus of equilibria, i.e. $-x_s \leq x \leq x_s$, where $x_s = f_s/k$. Equation (1) is piecewise solvable, and can be recast as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\omega_n^2 x_k, \quad \dot{x} > 0, \tag{2}$$

and

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = +\omega_n^2 x_k, \quad \dot{x} < 0, \tag{3}$$

where $\omega_n^2 = k/m, 2\zeta\omega_n = c/m$, and $x_k = f_k/k$.

If we begin with initial conditions $x(t_0) = X_0 > x_s$, and $\dot{x}(t_0) = 0$, then motion starts with $\dot{x} < 0$. The response to Equation (3) has the form

$$\begin{aligned} x(t) &= (X_0 - x_k)e^{-\zeta\omega_n(t-t_0)}(\cos\omega_d(t-t_0) \\ &\quad + \beta\sin\omega_d(t-t_0)) + x_k, \end{aligned} \tag{4}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\beta = \zeta/\sqrt{1-\zeta^2}$. This equation is valid until $\dot{x} = 0$ at which time $t = t_1 = t_0 + \pi/\omega_d$, and $X_1 = x(t_1) = -e^{-\beta\pi}X_0 + (e^{-\beta\pi}+1)x_k$. If $X_1 < -x_s$, then the mass will reverse direction and continue sliding with $\dot{x} > 0$ according to Equation (2). The solution for this interval of motion is

$$\begin{aligned} x(t) &= (X_1 + x_k)e^{-\zeta\omega_n(t-t_1)}(\cos\omega_d(t-t_1) \\ &\quad + \beta\sin\omega_d(t-t_1)) - x_k, \end{aligned} \tag{5}$$

which is valid until $\dot{x} = 0$, at which time $t = t_2 = t_1 + \pi/\omega_d$, and $X_2 = x(t_2) = -e^{-\beta\pi}X_1 - (e^{-\beta\pi}+1)x_k$. If $X_2 > x_s$, motion will continue.

This process can be iterated until $-x_s \leq X_n \leq x_s$, at which time the motion stops. This iterated process leads to a recursive relation for the successive peaks and valleys in the oscillatory response:

$$X_i = -e^{-\beta\pi}X_{i-1} + (-1)^{i-1}(e^{-\beta\pi}+1)x_k, \quad i = 1, 2, \ldots, n. \tag{6}$$

From this evolution of decaying peaks and valleys, we can isolate the viscous effect, and then extract the Coulomb effect. A sum of consecutive extreme displacement values cancels out the dry-friction contribution. Taking the ratio between successive sums yields

$$\frac{X_i + X_{i+1}}{X_{i-1} + X_i} = -e^{-\beta\pi}. \tag{7}$$

Thus, a logarithmic decrement reveals the viscous dependence:

$$\log(-\frac{X_i + X_{i+1}}{X_{i-1} + X_i}) = -\beta\pi. \tag{8}$$

Once the quantity β has been estimated, we can estimate ζ, and also the dry-friction parameter x_k from Equation (6).

A fundamental problem in an experimental system is that, because of the locus of equilibria, it may be difficult to determine the position in which the spring is unstretched. Thus, measurement may have a constant bias with respect to our formulation. There is a simple way to deal with this. If the biased measurement is $y = x + \epsilon$, we remove the bias ϵ by subtracting two measured peaks (or valleys) Y_i. Since $Y_i - Y_j = X_i - X_j$, we can work with the difference between two recursive relations (6) such that

$$Y_{i+1} - Y_i = -e^{-\beta\pi}(Y_i - Y_{i-1}) + 2(-1)^i(e^{-\beta\pi} + 1)x_k, \quad i = 1, 2, \ldots, n-1 \tag{9}$$

By summing the equation for $Y_{i+1} - Y_i$ with that of $Y_i - Y_{i-1}$, we eliminate dry-friction contribution. An alternate decrement equation is thus

$$\frac{Y_{i+1} - Y_{i-1}}{Y_i - Y_{i-2}} = -e^{-\beta\pi}, \tag{10}$$

or

$$\log(-\frac{Y_{i+1} - Y_{i-1}}{Y_i - Y_{i-2}}) = -\beta\pi. \tag{11}$$

This idea has been tested numerically and in controlled experiments [6].

3. Experimental Linear Bearing System

Linear bearings are widely used in high-speed position control systems. It is important to quantify the friction contribution to provide modeling information for control engineers.

This system has inherent damping with no modeling information known *a priori*. Its presence in the system cannot be controlled. The displacement will be referred to as $y(t)$ instead of $x(t)$ due to the unknown unstretched spring position.

The system consists of two linear bearings with very low viscous and dry friction effects. The linear bearings were made by Thomson Industries, Inc. (Model, 1CC-08-HAA). In order to compare the estimated results to the data provided by the linear bearing company, the seals at both ends of the linear bearings were removed. A sliding table mounted on the top of linear bearings was connected by three helical springs. The schematic diagram is shown in Figure 1.

The displacement response was sensed by a linear variable differential transformer (LVDT) which had a resolution of 0.01 mm after quantization in the data-acquisition process.

To investigate damping characteristics of this system, initial conditions were applied to conduct a free-vibration test. Figures 2 and 3 show the displacement response and the decay

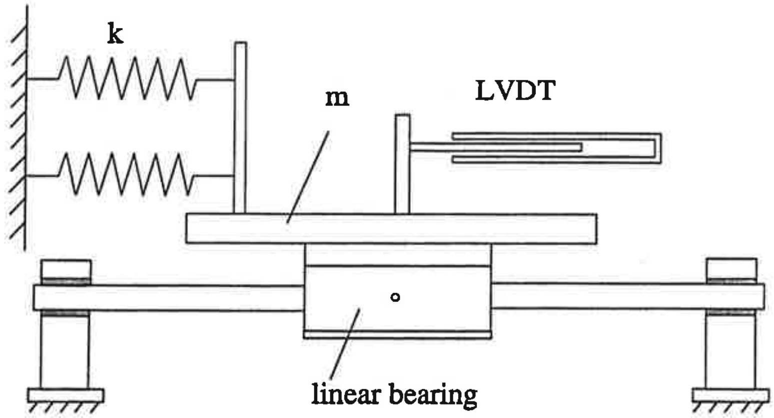

Figure 1. Schematic diagram of the experimental linear bearing system.

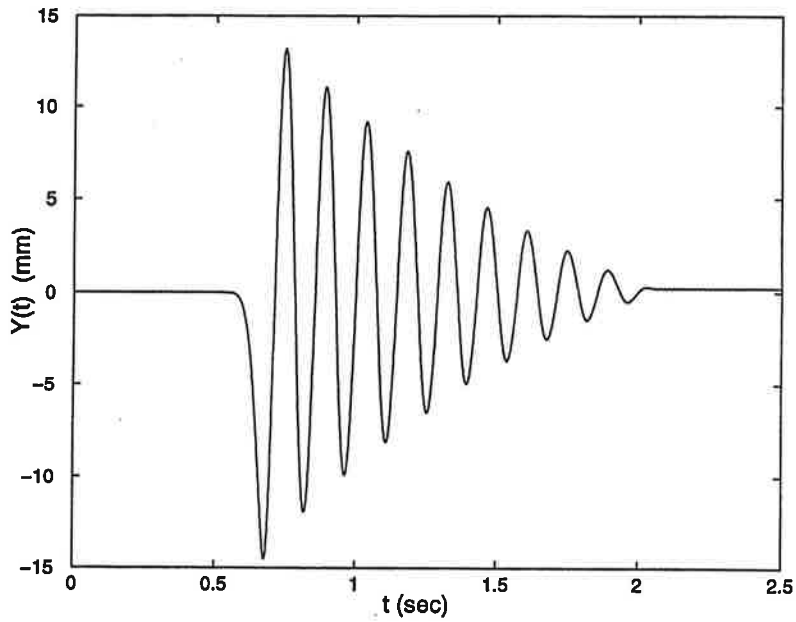

Figure 2. Comparison between experimental and identified displacement responses for the linear-bearing system. The solid line represents the experimental time trace, and the dashed line indicates the numerical result based on the identified parameters.

of amplitude differences respectively. The estimations of system parameters are $\hat{\beta}_3 = 0.0177$ (with a standard deviation of $\sigma = 0.0164$), $\hat{\zeta}_3 = 0.0177$ ($\sigma = 0.0164$), and $\hat{f}_k = 0.413$ N ($\sigma = 0.0249$ N). The subscript k refers to the word 'kinetic'.

Figure 3 suggests that the system has low viscous damping because the envelopes bend slightly and the deviations of amplitude differences from two envelopes are quite small. To check the validity of the Coulomb and viscous friction model, we numerically simulated the system response by applying the estimated parameters. The mass and stiffness were determined: $m = 1.92$ kg and $k = 2310$ N/m. These parameters were incorporated with the estimated damping information for accomplishing the numerical simulation. The simulation

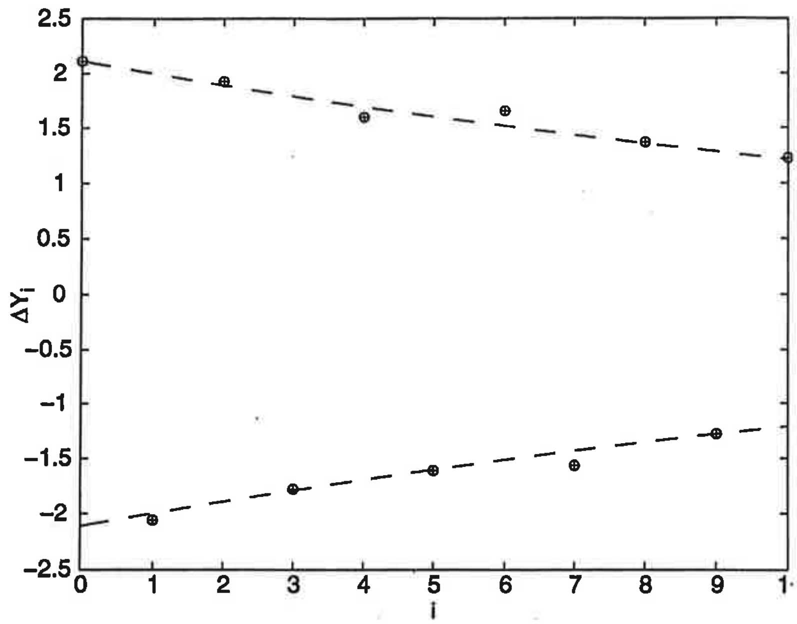

Figure 3. Experiment results showing the exponential decay of amplitude differences in the linear-bearing system.

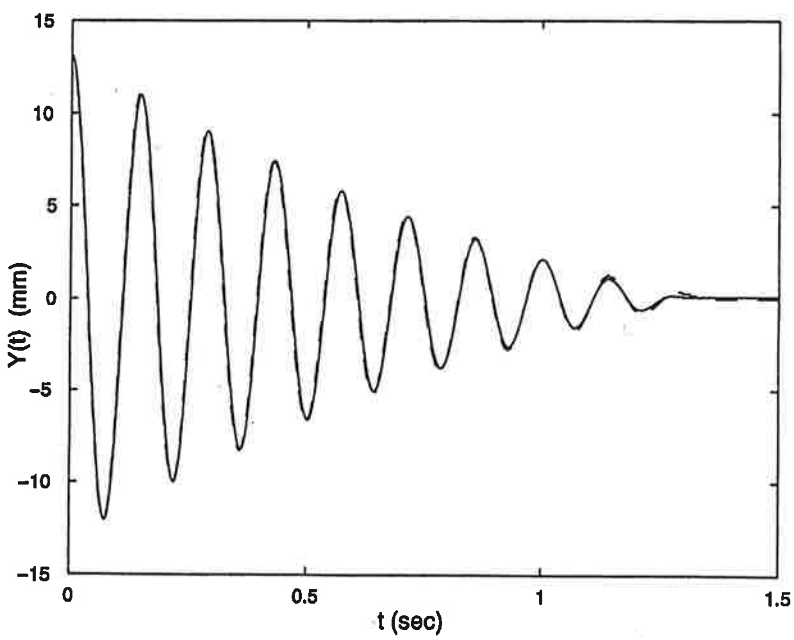

Figure 4. Comparison between experimental and identified displacement responses of the linear-bearing system (solid line: experimental result; dashed line: identified result).

initial conditions were similar to those of experimental study. The comparison between numerical and experimental displacement responses is presented in Figure 4. Figure 4 shows that the numerical result catches most of the features of the experimental data. Dividing the estimated friction force by the weight of sliding table, the coefficient of sliding friction was found to be 0.022, which is about ten times greater than the value provided by the manufacturer. The reason for this discrepancy calls for further investigation. (The manufacturer measured the dry friction coefficient by applying a normal load N, measuring the tangential load P when the bearing is set in motion, and taking $\mu = P/N$. It did not distinguish whether the coefficient was for static or kinetic friction.)

4. Error Analysis

The experimental implementation of this identification method involves measurements of the oscillation amplitudes. The reliability of the estimation depends on the accuracy of the measurements and the coherence of the model to the physical system. We address the measurement error, which is inevitable, and may result from digitization in time and displacement magnitude or other sources. It is of interest to understand how measurement error affects the estimation accuracy.

Consider the oscillation amplitudes and denote the physical or real values of amplitudes by $Y_i, i = 1, 2, \ldots, n$. If the measured values of amplitudes are represented by $\bar{Y}_i, i = 1, 2, \ldots, n$, then $Y_i = \bar{Y}_i + \delta_i$, $i = 1, 2, \ldots n$, where δ_i is the error associated with digitization, random error, transducer error, etc. Equations (10) and (11) are the crux of the proposed decrement method for experimental systems. In order to study the effects of quantization error, we extend the idea of Equation (10) into

$$\frac{Y_{i+m+1} - Y_{i+m-1}}{Y_{i+1} - Y_{i-1}} = (-1)^m e^{-\beta m \pi} \quad i = 1, 2, \ldots, n, \quad \text{and} \quad m = 1, 2, \ldots, n-i-1. \tag{12}$$

where m is the number of extreme excursions accumulated for the estimation process.

If the estimated nondimensional viscous-damping parameter $\bar{\beta}$ (note in previous sections this magnitude was called $\hat{\beta}$) were calculated from Equation (12) using the measured extreme excursions $\bar{Y}_i$, then

$$\frac{\bar{Y}_{i+m+1} - \bar{Y}_{i+m-1}}{\bar{Y}_{i+1} - \bar{Y}_{i-1}} = (-1)^m e^{-\bar{\beta} m \pi}. \tag{13}$$

We relate the estimated and real damping parameters such that $\beta = \bar{\beta} + \delta\beta$, where $\delta\beta$ represents the estimation error.

We would like to understand how the bound of the estimation error in viscous-damping parameter, $|\delta\beta|$, depends on the measurement error. Therefore, Equations (12) and (13), and expressions $Y_i = \bar{Y}_i + \delta_i$, and $\beta = \bar{\beta} + \delta\beta$ must be considered. If we expand Equation (12) into a Taylor series and consider only the first-order terms, the following equation can be achieved:

$$\begin{aligned} \delta_{i+m+1} - \delta_{i+m-1} &= m\pi(-1)^{m+1} e^{-\bar{\beta} m \pi} (\bar{Y}_{i+1} - \bar{Y}_{i-1}) \delta\beta \\ &\quad + (-1)^m e^{-\bar{\beta} m \pi} (\delta_{i+1} - \delta_{i-1}), \end{aligned} \tag{14}$$

where $\delta_{i+m+1}, \delta_{i+m-1}, \delta_{i-1}, \delta_{i+1}$ are errors of different displacement measurements. We next assume that there exists an upper bound δ on the deviations in displacement measurements, namely $|\delta_j| \leq \delta\ \forall j = 1, 2, \ldots, n$. Based on this assumption and taking the bounds of both sides of Equation (14), an expression which relates the estimation error and the upper bound of the measurement error can be obtained as follows:

$$|\delta\beta| \leq \frac{2(1+e^{-\bar{\beta}m\pi})\delta}{m\pi e^{-\bar{\beta}m\pi}|\bar{Y}_{i+1}-\bar{Y}_{i-1}|} = \alpha. \tag{15}$$

This expression suggests that the bound α of the estimation error is directly proportional to the bound δ of the measurement error, and that α can be minimized by a clever choice of m. If $\bar{\beta}$ is small, for example, the optimal choice of m will be large. Additionally, to increase $|\bar{Y}_{i+1} - \bar{Y}_{i-1}|$ so as to reduce α, use of the initial cycles of a free-vibration response is recommended so that a larger amplitude difference can be achieved.

We proceed to investigate the estimation of the dry-friction effect by applying Equation (9). As with the study of the viscous-damping estimation, the real value of the Coulomb friction parameter can be written as $x_k = \bar{x}_k + \delta x_k$. Moreover, if Equation (9) is expressed in a Taylor series, the following equation can be obtained after neglecting the high-order terms, and using the jth extreme excursion as a reference:

$$\begin{aligned}\delta_{j+1} - \delta_j &= \pi e^{-\bar{\beta}\pi}(\bar{Y}_j - \bar{Y}_{j-1})\delta\beta - e^{-\bar{\beta}\pi}(\delta_j - \delta_{j-1}) \\ &\quad + 2(-1)^{j+1}(\pi e^{-\bar{\beta}\pi})\bar{x}_k\delta\beta + 2(-1)^j(1+e^{-\bar{\beta}\pi})\delta x_k.\end{aligned} \tag{16}$$

Rearranging and taking bounds in the above equation yields

$$|\delta x_k| \leq \frac{\pi e^{-\bar{\beta}\pi}(|\bar{Y}_j - \bar{Y}_{j-1}| + 2\bar{x}_k)}{2(1+e^{-\bar{\beta}\pi})}|\delta\beta| + \delta = \psi. \tag{17}$$

Applying the bound of the error on β derived for general value of m (inequality (15)) to the above equation, and taking $m = 1$, leads to

$$|\delta x_k| \leq \{1 + \frac{|\bar{Y}_j - \bar{Y}_{j-1}| + 2\bar{x}_k}{|\bar{Y}_{i+1} - \bar{Y}_{i-1}|}\}\delta = \gamma. \tag{18}$$

This equation gives the bound on the error in the estimated dry friction for the case in which the viscous friction is estimated based on Equation (13) using $m = 1$ and the extreme excusions centered at the i, and then imported into Equation (9) to estimate the dry friction based on extreme excusions centered at j. The equation is not applicable if $m > 1$ is used, although if the effort increases the accuracy of $\bar{\beta}$, it will increase the accuracy of $\bar{x}_k$.

Inequality (17) relates the magnitude of the error in the dry-friction estimation, namely $|\delta x_k|$, to different quantities including the bound of measurement error, δ, the magnitude of the error in the viscous-damping estimation, $|\delta\beta|$, the magnitude of measured dry-friction effect, $\bar{x}_k$, and the span between a consecutive peak and valley, $|\bar{Y}_j - \bar{Y}_{j-1}|$. The bound on $|\delta x_k|$, ψ, is proportional to $|\delta\beta|$ and δ, and nonlinearly dependent on $\bar{\beta}$. Given $\bar{\beta}$ and $\bar{x}_k$, ψ decreases if values of δ, $|\delta\beta|$, or $|\bar{Y}_j - \bar{Y}_{j-1}|$ are small. To reduce δ, a high-precision quantization machine with a fast sampling rate and a noise-free experimental environment are required. Procedures of making $|\delta\beta|$ small have been discussed. To achieve small magnitude of $|\bar{Y}_j - \bar{Y}_{j-1}|$, the amplitudes of the last couple oscillatory cycles are recommended.

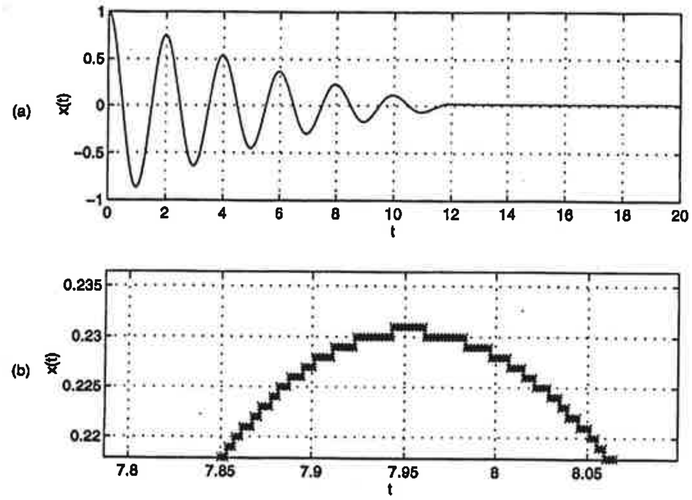

Figure 5. Simulation containing the effects of quantization error: (a) displacement response, and (b) detail showing the quantization error.

In contrast, inequality (18) suggests that given δ and $\bar{x}_k$, γ decreases when β increases. This is true because the quantity $|\bar{Y}_{i+1} - \bar{Y}_{i-1}|$, representing the amplitude difference between two consecutive peaks (or valleys) and appearing in the denominator of Equation (18), increases as β increases. Similarly, the magnitude $|\bar{Y}_j - \bar{Y}_{j-1}|$, corresponding to a span between peak and valley in one cycle, decreases as the viscous damping increases, which will reduce γ as well. The quantity $|\bar{Y}_{i+1} - \bar{Y}_{i-1}|$ in the denominator suggests using the early stage of the response for computing $\bar{\beta}$, while the term $|\bar{Y}_j - \bar{Y}_{j-1}|$ in the numerator suggests using the last few oscillations for estimating x_k.

Thus, we have shown that estimations bounds on $|\delta\beta|$ and $|\delta x_k|$ depend directly on the quantization error δ. If the viscous damping is small, a cumulative selection of amplitudes increases the accuracy of viscous-damping estimation. The accuracy of the dry-friction estimation depends on the estimation error of the viscous damping.

4.1. Numerical Example

An illustration of the effect of measurement error induced by quantization is given next. Quantization is an unavoidable source of measurement error in the digital data acquisition process.

We numerically simulated Equation (1) with the parameter values $m = 1.0$, $k = 10.0$, $c = 0.2$, and $f_k = 0.2$, and the initial conditions $x(0) = 1.0$ and $\dot{x}(0) = 0$. The viscous-damping parameter is $\beta = 0.0316$ and the dry-friction parameter is $x_k = 0.2$. The integration algorithm was based on a fifth-order Runge–Kutta method. The discontinuity due to Coulomb friction was handled in the same way as Shaw [15] and Feeny and Moon [16]. The quantization error was artificially imposed in the data collection.

The quantization step size in this example is 0.001, which equals the upper bound of the measurement error. The signal with this quantization resolution is shown in Figure 5. Based on this figure, different amplitude measurements were taken and listed in Table I. The values of δ_i were obtained by comparing the quantized data with the pure data. Calculations were then carried out based on $i = j = 2$ and the results are shown in Table II. These calculations

Table 1. Extreme excursions and errors in Figure ?? ($\delta = 0.001$).

extrema	$\bar{X}_0$	δ_0	$\bar{X}_1$	δ_1
	1.0	0.0000	–0.8670	–0.0003
extrema	$\bar{X}_2$	δ_2	$\bar{X}_3$	δ_3
	0.7470	0.0001	–0.6380	–0.0003
extrema	$\bar{X}_4$	δ_4	$\bar{X}_5$	δ_5
	0.5400	–0.0002	–0.4510	0.0004
extrema	$\bar{X}_6$	δ_6	$\bar{X}_7$	δ_7
	0.3700	–0.0001	–0.2970	0.0002
extrema	$\bar{X}_8$	δ_8	$\bar{X}_9$	δ_9
	0.2310	–0.0004	–0.1710	0.0003
extrema	$\bar{X}_{10}$	δ_{10}	$\bar{X}_{11}$	δ_{11}
	0.1160	0.0004	–0.0670	–0.0003

Table 2. Estimation errors and bounds for the case of quantization error ($\delta = 0.001$).

	m	$\bar{\beta}$	$\lvert\delta\beta\rvert$	$\lvert\delta\beta\rvert/\beta(\%)$	α
β	1	0.0322	0.00055	1.74	0.0059
(0.0316)	3	0.0316	0.00001	0.03	0.0022
	5	0.0318	0.00018	0.58	0.0015
	7	0.0313	0.00028	0.88	0.0012
	m	$\bar{x}_k$	$\lvert\delta x_k\rvert$	$\lvert\delta x_k\rvert/x_k(\%)$	ψ
x_k	1	0.0194	0.00058	2.91	0.0017
(0.02)	3	0.0201	0.00005	0.26	0.0010
	5	0.0198	0.00015	0.75	0.0012
	7	0.0204	0.00039	1.96	0.0013

include ψ, which is based on known values of $\delta\beta$, which would not typically be available. Some observations can be made from Table II. (1) The bound of viscous-damping error (α) decreases as m increases, which follows the results of Equation (15). (2) Although the bound of the viscous-damping error decreases as the value of m increases, the actual error $|\delta\beta|$ does not follow the same trend. This is symptomatic of the fact that Equation (15) addresses nothing about the actual estimation error. More specially, the actual estimation is affected by the actual errors in the individual extrema. (3) The estimation errors of dry friction are less accurate compared to those of viscous damping, and the trend in the bound ψ follows that of the actual estimation error of viscous damping. This agrees with Equation (17).

Other examples of the effects of error in the data were studied, such as the case of random noise and the effect of the sampling time. These examples can be found in Liang [17].

5. Conclusion

A method for simultaneously estimating the viscous and dry friction effects in a mechanical system has been applied. The method complements existing ideas for using free-vibration decrements to estimate damping parameters in linear systems with a single form of damping. Previous numerical and experimental studies focusing on a combined-damping system illustrated the validity of this decrement method.

We estimated the damping characteristics of an industrial-bearing system. Although there was no damping information provided for this system, the numerical simulation of the identified model caught most of the features of the experimental results.

An error analysis addressed the effect of measurement error on the estimations. A cumulative-based approach is recommended for high accuracy estimations in viscous damping when this damping is small. The dry-friction estimation depends on the estimation of viscous damping. Bounds of both estimation errors are proportional to measurement error. The recipe for reducing the error bounds is to reduce the measurement error δ, optimize the number m of extreme excursions, use the initial oscillations for estimating β, and use the final oscillations for estimating x_k.

A numerical example of the error analysis was performed for the case of quantization error. In this example, the bounds on the estimation errors were an order of magnitude larger than the actual estimation errors.

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