Autocorrelation on Symbol Dynamics for a Chaotic Dry-Friction Oscillator

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Abstract

An autocorrelation function based on symbol dynamics is applied to a chaotic dry-friction oscillator to estimate the largest Lyapunov exponent. The friction problem is well suited for symbol dynamics since two distinct states of motion can be identified: sticking and slipping. In addition, the dynamics of the oscillator can be reduced to a non-invertible one-dimensional map, which has been studied in terms of binary symbol sequences. The study is done for an experimental oscillator and for a numerical model. The numerical result is compared to the Lyapunov exponent estimated from the continuous flow.

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Introduction

Experimentalists often wish to quantitatively characterize chaotic motion by calculating the Lyapunov spectrum. However, it is computationally difficult to estimate Lyapunov exponents directly from time series data, especially in the presence of experimental noise. Singh and Joseph [1] proposed the use of symbol dynamics to obtain an autocorrelation and an estimate of the largest Lyapunov exponent. Thus, a binary sequence of yes-no information can be used to quantitively characterize the dynamics. In this paper we report the successful use of this technique on the chaotic dynamics of a dry-friction oscillator.

The modeling of friction in dynamical systems has a long history which goes back to the ancient egyptians. In 1931, Den Hartog [2] solved the equations of a harmonic oscillator with Coulomb friction for periodic motion. Shaw [3] has used modern techniques to extend those results to include a stability analysis, and found period-two motion and beating phenomena. Grabec [4] modeled friction in cutting tools, and found self-excited chaos in a four-dimensional phase space. In diploma theses under K. Popp, of the University of Hannover, F.R.G., Ahlborn [5] and Jahnke [6] observed quasiperiodicity and chaos in a self-excited continuum and a harmonically driven self-excited friction oscillator.

In this letter, we present results for a one-degree-of-freedom oscillator with dry friction dependent on both displacement and velocity. This can occur, for example, if displacement induces elastic deformation which, in turn, causes changes in the normal load at the friction surface. The nondimensionalized equation of motion is

$$\ddot{x} + 2\zeta\dot{x} + x + n(x)f(\dot{x}) = a\cos(\Omega t), \tag{1}$$

where $f(\dot{x})$ represents the friction coefficient and $n(x)$ represents the normal load. We let the normal load vary linearly with displacement, and, for physical consistency, we restrict the normal load to be nonnegative, so that $n(x) = 1 + kx$ for $x > -1/k$, and $n(x) = 0$ for $x < -1/k$. Often, the friction is modeled by a Coulomb law, which includes a static coefficient of friction μ_s, and a dynamic coefficient of friction μ_d. If $\mu_s = \mu_d = \mu$, the friction law may be written as

$$f(\dot{x}) = \mu\,\mathrm{sign}(\dot{x}). \tag{2}$$

One might try to approximate a Coulomb law with a continuous function such as

$$f(\dot{x}) = (\mu_d + (\mu_s - \mu_d)\,\mathrm{sech}(\beta\dot{x}))\,\tanh(\alpha\dot{x}). \tag{3}$$

The *tanh* term represents the jump from positive friction to negative friction, and approaches a discontinuity as $\alpha \to \infty$. The *sech* term represents the transition from μ_s to μ_d.

The numerical solution for the continuous case is performed using a standard fifth order Runge-Kutta code with stepsize adjuster. The numerical solution for the discontinuous case (not studied in this note) requires a special algorithm which follows that of ref. [3]. A three-dimensional representation of a numerical solution of the continuous equation of motion is shown in Figure 1. In this plot we can see the stick-slip motion (described in [2] and [3]). The sticking region is plotted with small dots, and the slipping motion is plotted with large dots. Also, on the upper right portion of the portrait, we can see trajectories from the inside of the attractor stretching above the sticking region and folding back onto the outside of the attractor. This stretching and folding is typical of chaotic one-dimensional maps and two-dimensional horseshoe maps. A similarly shaped attractor was found from the experimental dry-friction oscillator described below.

Description of Experiment

The experiment consisted of a mass attached to the end of a cantilevered elastic beam. A diagram is shown in Figure 2. The mass had titanium plates on both sides, providing surfaces for sliding friction. Spring-loaded titanium pads rested against the titanium plates. The titanium plates were not parallel in the direction of sliding, thus a displacement of the mass caused a change in the force on the spring-loaded pads. Hence a change in displacement caused a change in normal load and friction. The elastic beam, mass and pressure pads were fixed to a common frame which was excited harmonically by an elecromagnetic shaker. Strain gages attached to the elastic beam were used to sense the displacement of the mass relative to the oscillating frame.

Discussion of Results

An experimental Poincaré section is shown in Figure 3a. It represents a slice of the three-dimensional phase portrait of Figure 1 parallel to the x-$\dot{x}$ axes, perpendicular to the Ωt axis [7]. The standard autocorrelation of the experimental strain signal is shown in Figure 4. It consists of a rapid decay into a small oscillation, suggesting that the signal is uncorrelated, although the influence of the harmonic driver is present. The autocorrelation of a periodic signal would be periodic, with no component of decay.

Notice that the Poincaré map appears to be confined to a one-dimensional object embedded in two-dimensional phase space. By defining a coordinate s along the Poincaré map as shown in Figure 3a, we can obtain a return map as shown in Figure 3b. The return map (Figure 3b) resembles a tent map. The tent map is well known to be chaotic, and the dynamics of similar maps have been studied in terms of binary symbol sequences[8].

The Poincaré section in Figure 3a is located near the axes labeled in Figure 1. Note that it cuts through the sticking region. The sharp, horizontal part of the Poincaré plot is in the sticking region, and the fuzzy part is in slipping motion. This suggests a natural use of symbol dynamics for the motion, namely sticking or slipping. By considering whether the motion is sticking or slipping at each pass through the Poincaré section, we can construct a binary symbol sequence.

Singh and Joseph [1] have proposed a technique of extracting quantitative information from a binary symbol sequence. First it is necessary to represent the symbol sequence $u(k)$ as a string of 1's and -1's. These values are chosen so that the expected mean of a random sequence of equally likely symbols is zero. As the trajectory passes through the Poincaré section for the kth time, if it is not sticking, we set $u(k) = 1$. If it is sticking, we set $u(k) = -1$. An autocorrelation on such a symbol sequence is defined as

$$r(n) = \frac{1}{N} \sum_{k=1}^{N} u(k+n)u(k), \quad n = 0, 1, 2, \ldots, \qquad N \gg n. \tag{4}$$

If the sequence is chaotic, the autocorrelation should have the property $r(n) \to 0$ as $n \to \infty$.

If the sequence becomes uncorrelated, an estimate of the largest Lyapunov exponent can be obtained using the binary autocorrelation function. The largest Lyapunov exponent can be defined as

$$\lambda = \frac{1}{N} \sum_{i=1}^{N} \log_2 \frac{d(n)}{d_o(n-1)}, \tag{5}$$

where $d_o(n-1)$ is the starting distance between two trajectories, and $d(n)$ is the distance between them after one iteration. Since the binary sequence is uncorrelated, we can estimate $d_o(n-1)$ as the expected value of the distance $\bar{d}_o(n-1)$ between two randomly chosen points in the same symbol region. In our example, we measure the distance using coordinate s on the Poincaré plot. Two points chosen from the sticking region have an expected distance $\bar{d}_o(n-1) = 1/3$. Two points from the nonsticking region also have an expected distance $\bar{d}_o(n-1) = 1/3$. If $u(n-1)$ and $u(n)$ are in the same region, their iterates will either stay in that region, be in different regions, or both be in the other region. One defines[1]

$$\alpha = \log_2 \frac{\bar{d}(n)}{\bar{d}_o(n-1)}, \tag{6}$$

where $\bar{d}(n)$ is the expected distance of two points, each chosen from separate regions. For our problem, $\bar{d}(n) = 1$ and $\alpha = \log_2 3$. Replacing $d_o(n-1)$ and $d(n)$ in (5) with their expected values defines the macroscopic Lyapunov exponent, λ_m, which is rewritten via a derivation in ref. [1] as

$$\lambda = \frac{\alpha}{2}(1 - r(1)^2), \tag{7}$$

Application of equations (4), (6) and (7) to a symbol sequence derived from the tent map yields a rapidly decaying autocorrelation and a Lyapunov exponent $\lambda_t = 0.787516$ for a string of 100000 symbols, and an exponent of $\lambda_t = 0.787705$ for a string of 2048 symbols, compared to its exact value, calculated using log_2, $\lambda_{te} = 1$. Application to the logistic map yields a rapidly decaying autocorrelation and a Lyapunov exponent of $\lambda_l = 0.791578$ for 100000 symbols, and $\lambda_l = 0.791116$ for 2048 symbols, compared to its exact value of $\lambda_{le} = 1$.

The binary autocorrelation function for an experimental sequence of length 2048 was obtained using (4) as shown in Figure 5a. Applying equations (6) and (7), the resulting Lyapunov exponent is $\lambda_{exp} = 0.79055$. Using equations (4), (6) and (7) on numerical smooth-law data (2048 symbols) yields the autocorrelation in Figure 5b, and a Lyapunov exponent of $\lambda_{s_1} = 0.79219$. The largest Lyapunov exponent of the flow can be estimated numerically[7], and can be related to that of the Poincaré map via $\lambda_{flow} = \frac{\lambda}{T}$, where T is the driving period. This calculation of exponent for the Poincaré map from the equations of motion gave $\lambda_{s_2} = 0.77$.

Conclusions

Binary symbol dynamics were used to describe the stick-slip motion of a dry-friction oscillator. Sticking and slipping motion were used as the states in the binary sequence. It was shown that the dynamics of the dry-friction oscillator is reducible to a one-dimensional map, well suited for symbol dynamics. The proposed method in ref. [1] produced a binary autocorrelation function which was used to estimate the order of magnitude of the largest Lyapunov exponent. This estimate was done for experimental and numerical data. The implication is that for limited information, i.e. a series of 'yes' and 'no' information, quantitative information of the dynamics can be easily obtained.

References

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Figure Captions

Figure 1. Three-dimensional phase portrait of the numerical solution using a continuous friction law with parameters $\mu_s = 1$, $\mu_d = 0.7$, $\zeta = 0.015$, $\alpha = 50$, $\beta = 5$, $\Omega = 1.3$, and $a = 1.45$.

Figure 2. Schematic of experiment.

Figure 3. (a) Experimental Poincaré section, (b) Return map of Poincaré section.

Figure 4. Standard autocorrelation function for experimental time series.

Figure 5. Binary autocorrelation function for (a) symbol sequence from experimental data, (b) symbol sequence from numerical simulation.

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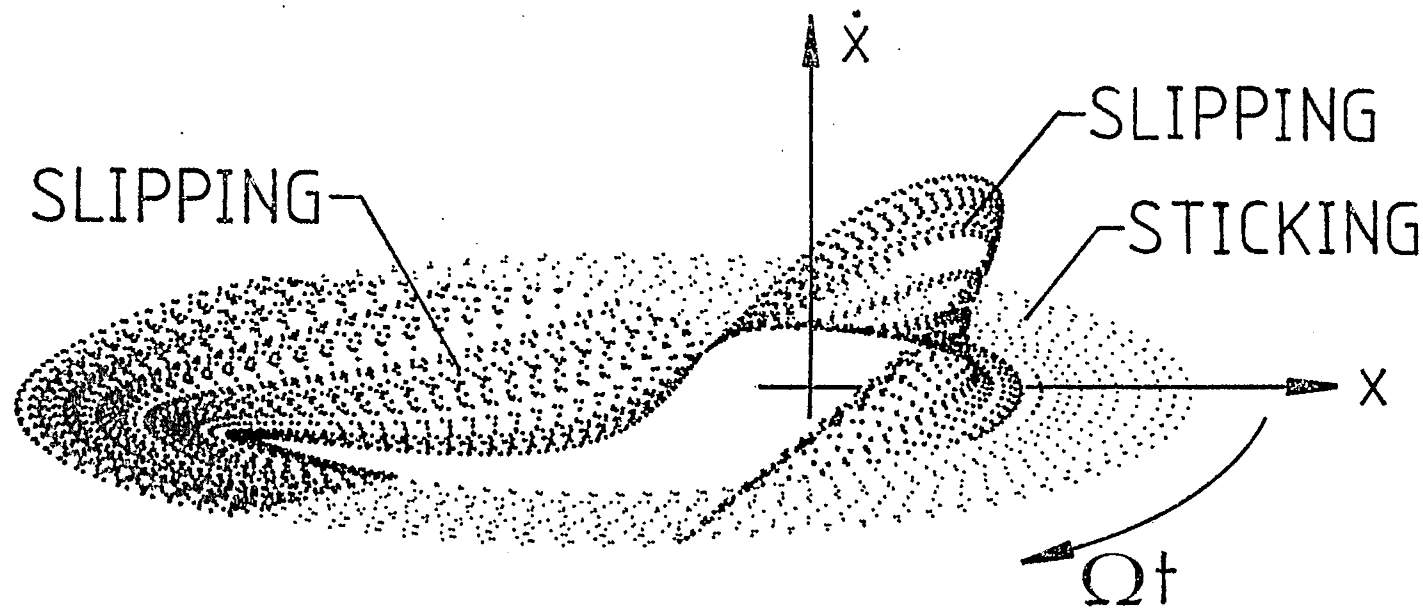
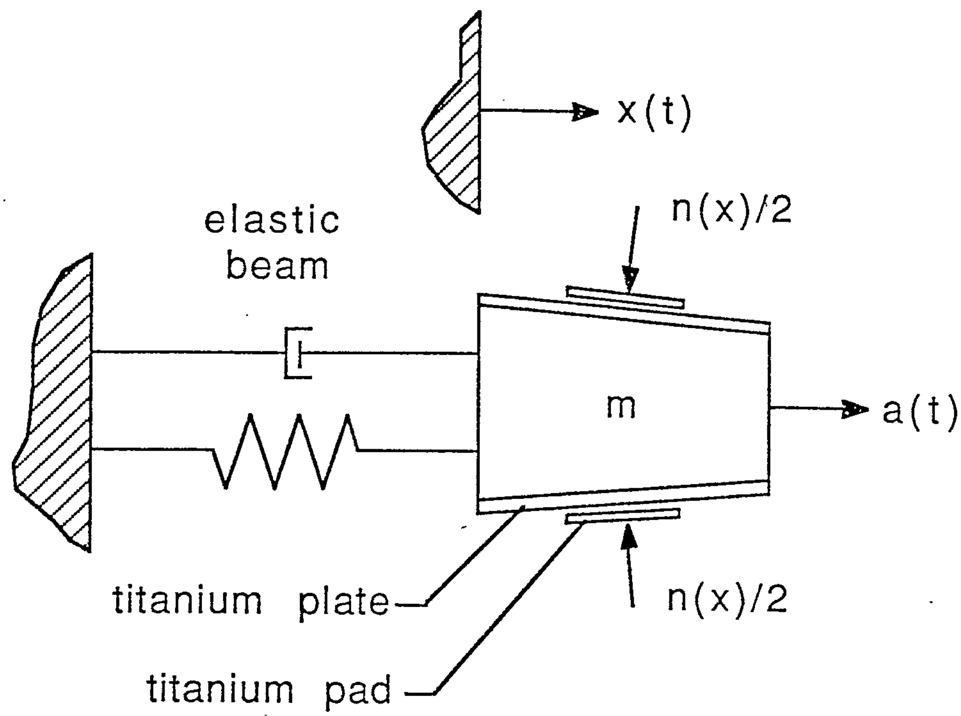
Fig. 1

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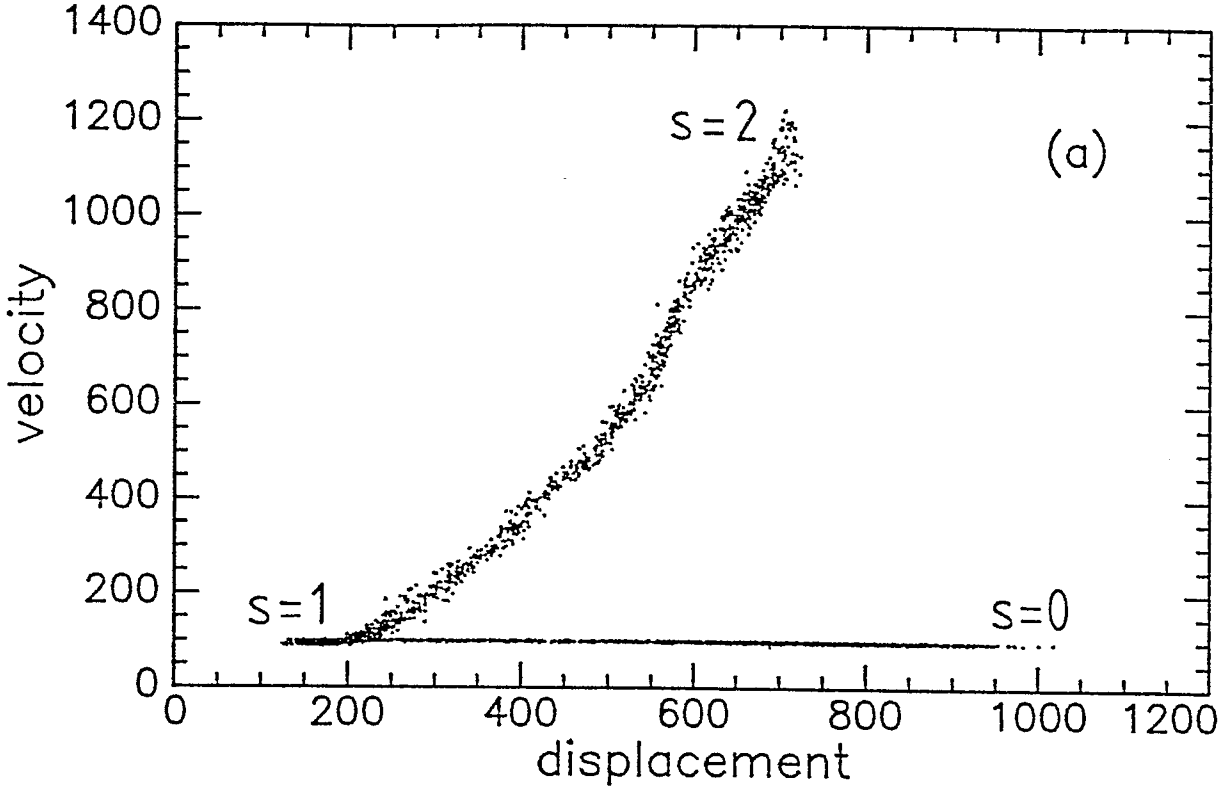
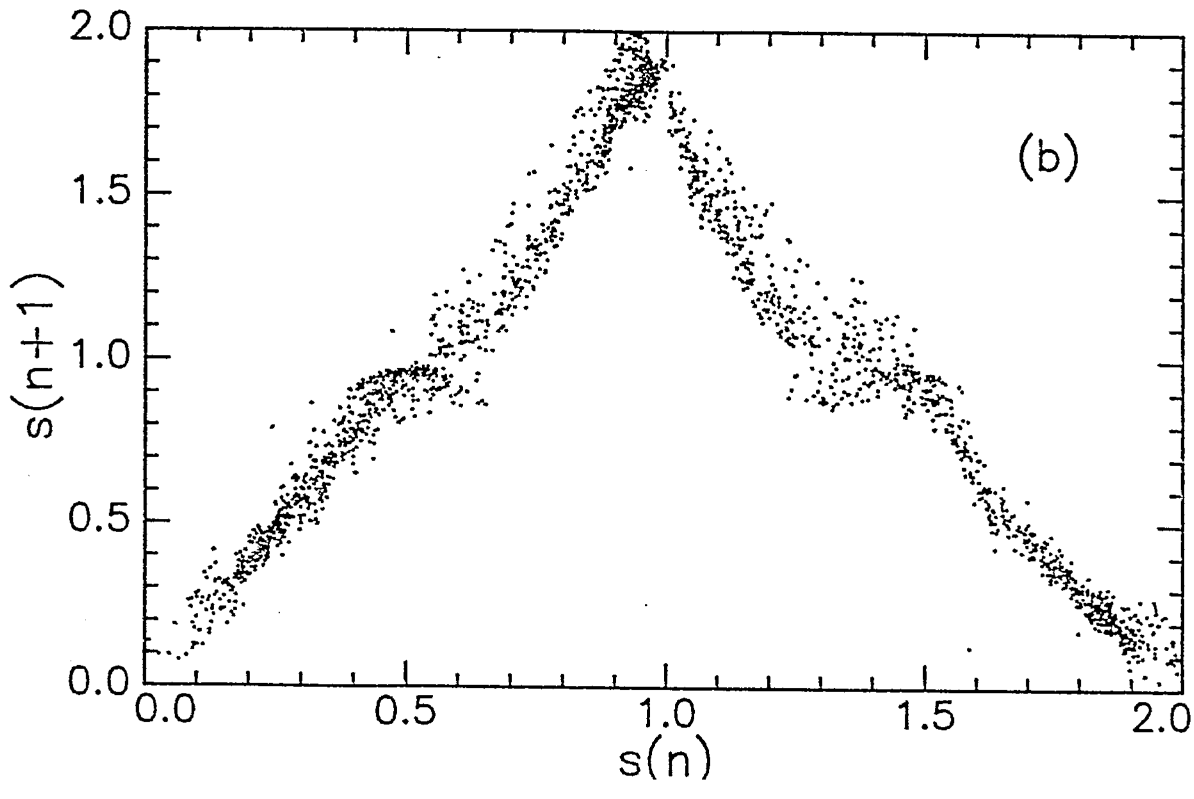
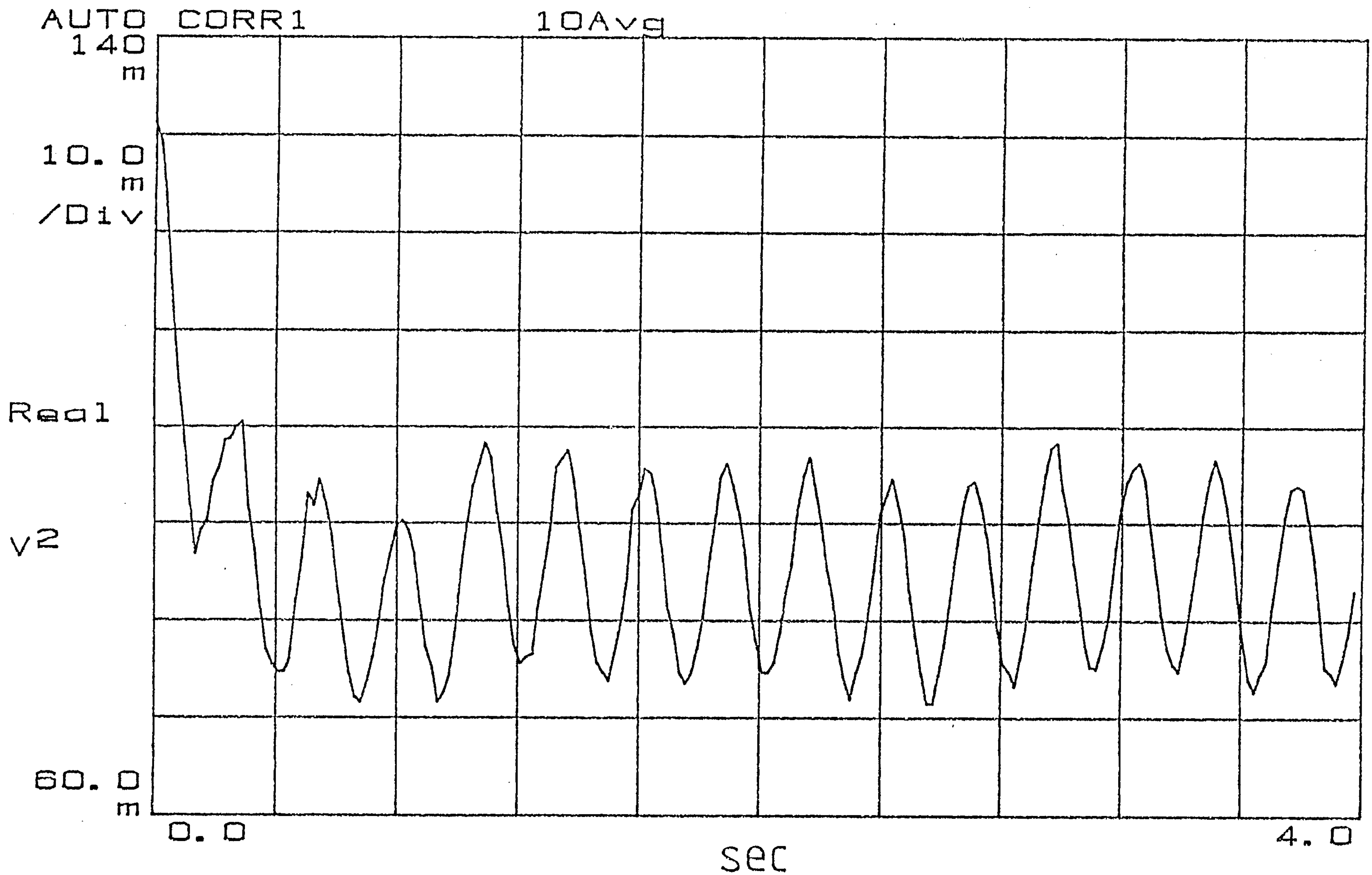


Feeny and Moon, Fig. 3b

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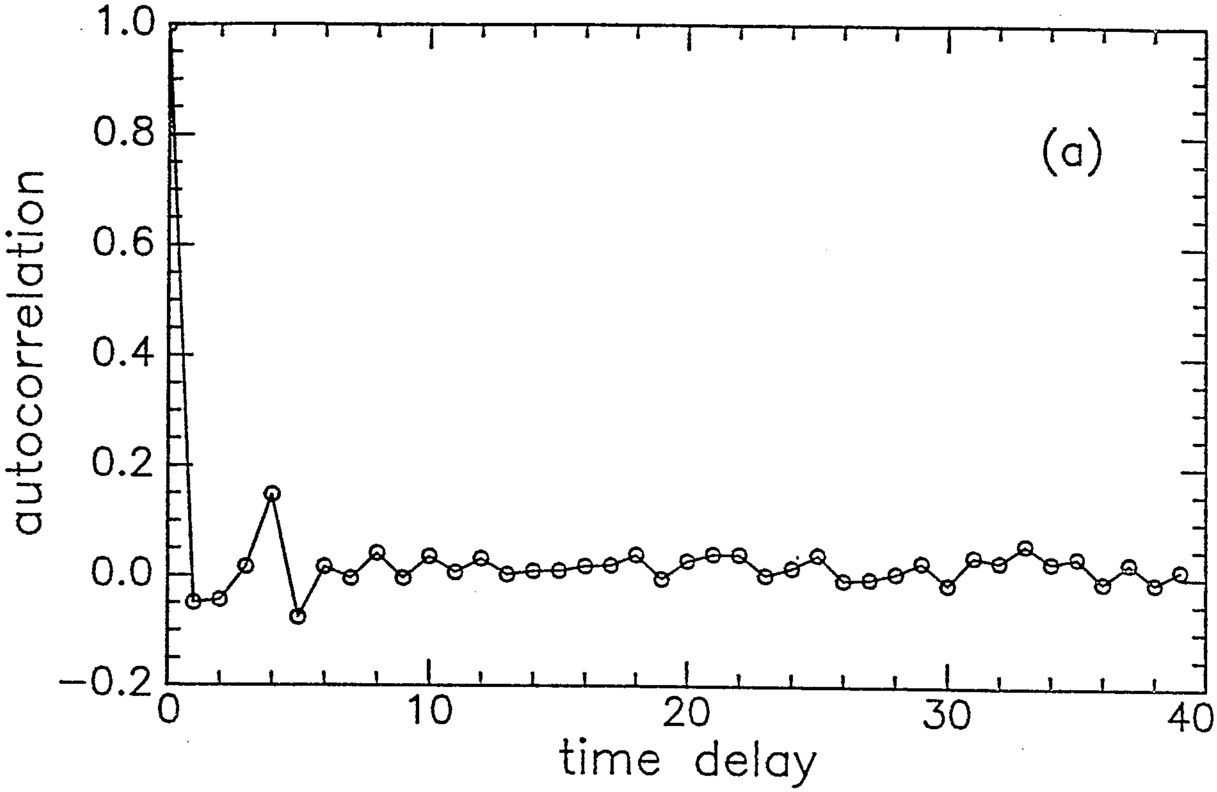

Feeny and Moon, Fig 5b