THE NONLINEAR DYNAMICS OF OSCILLATORS WITH STICK-SLIP FRICTION

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B. F. FEENY
Department of Mechanical Engineering, Michigan State University
East Lansing, Michigan 48824 USA

ABSTRACT

Stick-slip oscillators represent a special class of mechanical systems. They are modeled with a discontinuous velocity field, and they involve a collapsing phase space. In single-degree-of-freedom systems, this leads to an underlying one-dimensional map. This chapter focuses on a single-degree-of-freedom system with friction. It discusses the relationship between stick-slip, the discontinuity, and the underlying one-dimensional maps. It connects the mechanical characteristics of a chaotic stick-slip oscillator to the mathematical theory of one-dimensional maps and symbolic dynamics. Finally, the chapter points out consequences of stick-slip on system geometry, including implications regarding the analysis of experimental data. Most of the discussions take place through a running example.

1. Introduction

Stick-slip is important in mechanical systems. It is typically associated with friction. Examples of frictional systems include robot joints, braking systems, automotive squeak, micromachines, machine-tool processes, earthquake faults, and rail-wheel contacts. Ibrahim and Armstrong-Hélouvry et al. have provided thorough surveys on dynamical systems with friction.

This chapter focuses on the dynamics of oscillators with stick-slip dry friction. In this context, a "stick" refers to an event in which the relative velocity between contact surfaces is zero for an interval of time. Stick-slip may also take place in systems with lubricated contacts. However, stick-slip is not confined to friction phenomena. Many of the ideas presented here, although in the context of friction, may be applicable to other stick-slip systems.

From the point of view of nonlinear dynamics, stick-slip has important consequences. During a stick, there is a collapse in the dimension of state space. This can be visualized in state space by imagining that one of the states, velocity, is directly constrained during a stick. For a forced one-degree-of-freedom oscillator, this produces an underlying one-dimensional map. Such maps have been studied extensively by mathematicians. Thus, we have a class of mechanical systems which corresponds to a special class of mathematical systems. The theory
developed by the mathematicians for one-dimensional maps may be applied by the engineers to the mechanical systems.

A phenomena similar to sticking can occur in impacting systems also, and is sometimes referred to as dwell. For example, a model of a bouncing ball will chatter, that is undergo infinitely many impacts in finite time, before resting on the table. An oscillating table will soon throw the ball into motion again\textsuperscript{11}. Szczygielski and Schweitzer\textsuperscript{12} observed such a dimensional collapse in an impacting rotor. In terms of the collapsing phase space, dwell is more extensive than stick-slip. During a dwell, the displacement is constrained in addition to the velocity. Thus, the collapse reduces the dimension by two.

Another related phenomenon is that of sliding-mode dynamics. While a frictional discontinuity tends to occur when relative velocities are zero, a more general system can have an arbitrarily oriented discontinuity. This is typical of sliding-mode and bang-bang control systems, in which a discontinuity is formed, by design, via a control algorithm. In the limit of a continuous-time controller, the motion can be constrained within the surface of the discontinuity, leading to sliding-mode dynamics\textsuperscript{13}. This is an example of a dimensional collapse.

This chapter summarizes the author’s participation in this subject. Section 2 discusses some examples of oscillators showing stick-slip motion. We will then focus on one of these oscillators. Section 3 examines this oscillator experimentally, and section 4 numerically. In section 5, the geometry of stick-slip is discussed with respect to this oscillator. Section 6 studies the one-dimensional map dynamics through symbol sequences and bifurcation sequences. Section 7 addresses some experimental issues.

2. Examples of Stick-Slip Oscillators

A standard example is a forced mass-spring system with Coulomb damping (Figure 1). The nondimensionalized equation of motion is

\[ \ddot{x} + 2\zeta \dot{x} + x + n(x)f(\dot{x}) = a \cos \Omega t, \quad (1) \]

where \( x \) is the displacement, \( \zeta \) is the damping ratio, and \( a \cos \Omega t \) represents a harmonic excitation. \( f(\dot{x}) \) represents the coefficient of friction, and is given by

\[ f(\dot{x}) = \mu \text{sign}(\dot{x}), \quad \dot{x} \neq 0, \quad -1 \leq f(\dot{x}) \leq 1, \quad \dot{x} = 0. \quad (2) \]

For a constant normal load, \( n(x) = 1 \). Stick-slip was noticed by Eckolt\textsuperscript{14} as early as in 1920. Periodic solutions with and without sticking were then formulated by Den Hartog\textsuperscript{15} in 1931. More recently, Shaw\textsuperscript{4} extended this work with a modern stability analysis, and noticed that the stick-slip dynamics reduced to that of a one-dimensional map.

Allowing the normal load to vary linearly with displacement, the normal load at the friction contact is given by

\[ n(x) = 1 + kx, \quad x > -1/k, \quad n(x) = 0, \quad x < -1/k. \quad (3) \]
Figure 1: Mechanics model for a forced oscillator with dry friction. The friction plates are fixed to the mass \( m \), and slide relative to the friction pads, which are fixed in \( z \). The friction surfaces can be arranged such that they are not parallel in the direction of displacement \( z \). In such case, the elastically loaded normal forces vary with displacement.

To prevent the existence of a negative normal load (and negative friction), the model allows for a loss of contact. This oscillator can undergo stick-slip chaos on a branched manifold, which has an underlying single-humped 1-D map$^{16,6}$. We will look at this oscillator in great detail throughout the rest of this chapter.

The model of a belt-driven, forced oscillator also undergoes stick-slip chaos$^{8,17}$. It has an underlying 1-D map which resembles a circle map. As with circle maps, the oscillator exhibits intermittency as a route to chaos.

Other types of systems may exhibit stick-slip. One fascinating example is a beam with a magnetic tip oscillating near a superconductor. Such a configuration produces a locus of fixed points, which provides an opportunity for sticking. A return map of this oscillator, when driven to chaos, uncovers a one-dimensional map$^{18}$. Another example is in a beam with a plastic member$^{19}$, which also undergoes stick-slip-like behavior, and leads to reduced-order map dynamics.

The upshot is that forced single degree-of-freedom oscillators with stick-slip represent a class of systems in which the underlying one-dimensional dynamics as a typical feature. Distributed systems with stick-slip can undergo very complicated behavior, which can include spatio-temporal chaos$^{20,21}$. However, the remainder of this chapter focuses on simple friction oscillators modeled by equations (1), (2), and (3).
3. Experimental Oscillations in a Mass/Beam System

Leonardo da Vinci (1452-1519) noticed that the force of sliding friction is roughly proportional to the normal load at the contact surface. In this section, we examine an experimental system which exploits this idea.

The experiment consisted of a mass attached to the end of a cantilevered elastic beam. The mass had titanium plates on both sides, providing surfaces for sliding friction. Spring-loaded titanium pads rested against the titanium plates. We used titanium because it is relatively light and tough, and therefore a candidate for flight and space applications\(^22\). The coefficient of static friction for titanium on titanium is in the range of 0.7 to 1.1\(^23\). The titanium plates were not parallel in the direction of sliding, and thus a displacement of the mass caused a change in the force on the spring-loaded pads. Hence a change in displacement caused a change in normal load and frictional force. The angle in the titanium plates was 2.9 degrees. The elastic beam, mass and pressure pads were fixed to a common frame which was excited harmonically by an electromagnetic shaker. Strain gages attached to the elastic beam were used to sense the displacement of the mass relative to the oscillating frame. The beam and mass had a fundamental natural frequency of 2.4 Hz with the friction removed. The frequency of the second mode was 37 Hz. The excitation frequency was typically in the range of 2.5 Hz to 6 Hz, with an amplitude typically in the range of 8-12 mm. The first-mode damping ratio was \(\zeta \approx 0.015\).

Further details regarding the experimental setup can be found in the literature\(^16\).

3.1. Dynamical Behavior

Both periodic and chaotic dynamics were observed in this system. The principal periodic motions observed were of periods one and two. The experiment seemed to be inherently noisy (i.e. small high-frequency signals on the low-frequency oscillators), so period-four and higher subharmonics were not observed for a sufficiently long time to make good measurements. Speculatively, this noise may be intrinsic to friction. Contact surfaces have a random irregularity at some small scale. During the sliding process, the temperature of the contacts may vary, inducing a variation in the friction properties. Also, wear may effect the experiment by continually changing the number and distribution of contact points between the surfaces. Strain hardening and oxidation may change the amount of plastic deformation at the asperities. Some of these phenomena may contribute to hidden, unseen state variables in the modeling of friction. The "noise" may in fact consist of deterministic chaos involving such hidden variables.

A phase portrait of the oscillator, driven at 3.7 Hz, is shown in Figure 2. The motion seems to go through a funnel structure, something like with the Rössler attractor\(^24\). Unlike the Rössler attractor, sticking motion tends to take place within the funnel structure at positive displacements, where the normal load (and hence the friction force) is high. As the system is excited, the mass tends to step out of this region of high friction, toward a region of lower friction, producing the funnel...
shape. Eventually, the friction is low enough for the mass to enter into a larger orbit, which may take it back to the high-friction sticking funnel.

Sticking motion in this oscillator may be associated with heavy dissipation. In the familiar case of a block sliding freely on a surface, energy is lost during sliding, until motion stops, at which time the kinetic energy is zero. However, in the forced oscillator, the simulation model is in some sense not dissipative (by Coulomb effects) during sliding (see section 5.2). However, when the oscillator is stuck, all energy associated with the input excitation is absorbed without producing motion. Hence, there is heavy dissipation during a stick.

Quantitative statistical measurements can be found in Feeny and Moon\textsuperscript{16}.

3.1.1. Three-dimensional flow and one-dimensional map

Since the oscillator is driven periodically, we can look at the motion in $R^2 \times S^1$, where $R^2$ consists of displacement and velocity, and $S^1$ represents periodic time $t(\text{mod}\,2\pi/\Omega)$, with $\Omega$ denoting the driving frequency. A projection of the three-dimensional phase portrait is shown in Figure 3. The toroidal image is similar to a deformed, flattened donut. In a cylindrical coordinate system, the time variable moves clockwise around the donut, the displacement variable goes radially away from the center, and the velocity points up.
Figure 3: 3-D representation of motion for the experimental oscillator driven at 4.2 Hz. The radial, circumferential, and longitudinal axes are displacement, time, and velocity.
Geometrically, the motion can be approximately described by a sheet which, as time evolves, stretches and folds (Figure 4(a)), forming a branched manifold. After a driving period has evolved, the end of the sheet is folded and identified with the beginning, and it is wrapped around and attached, so that it looks like a flattened, deformed torus (Figure 4(b)). Trajectories on the sheet orbit the hole as time evolves. Motions on the outside edge of the object wind around in time, and return slightly inside the outside edge. Motions on the inside edge wind in time, push up into positive velocity, and return on the outside edge (overshooting it slightly in the experimental motion). Trajectories on the region between undergo stretching and folding.

A slice through the $S^1$ component of phase space at a fixed time defines a Poincaré section. It provides a cross-sectional view of the dynamical object and aids in visualizing the attractor. A Poincaré section for the motion driven at 3.7 Hz is shown in Figure 5(a). The cross section of the attractor seems to be confined to a one-dimensional object bent in two-dimensional space. It is possible to define a coordinate $s$ along the one-dimensional Poincaré plot and to construct a delay map from the resulting sequence of points. This delay map is shown in Figure 5(b). It is a single-humped map, like the logistic map (defined by $s_{n+1} = as_n(1 - s_n)$) and the tent map (defined by $s_{n+1} = as_n$ for $0 \leq s_n \leq \frac{1}{2}$ and $s_{n+1} = a(1 - s_n)$ for $\frac{1}{2} < s_n \leq 1$). The implication is that the dynamics of a mass-beam system with displacement-dependent dry friction may be approximately reduced to a non-invertible one-dimensional map! The tent map manifests the stretching and folding of the attractor. We will return to maps in sections 5 and 6.2.

There is some ambiguity in assigning $s$ values to points near the cusp at $s = 1$. This ambiguity causes the horizontal step-like features at $s = 0.5$ and $s = 1.5$, since such points are iterated to the area of the cusp. The true map underlying the dynamics does not necessarily have these horizontal-step features.

3.1.2. Varying attractor dimension

The Poincaré section in Figure 5(a) consists of a fuzzy part and a crisp part. The crisp part contains sticking orbits, where the velocity is zero. Qualitatively, these two portions are strikingly different. Is it possible that they have different dimensions? In the experiment, it is likely that higher modes of the beam, noise, and complications in the friction surfaces are mechanisms for disrupting this “ideal” one-dimensional Poincaré section. This causes us to wonder if higher modes or additional nonlinearities, combined with friction, could produce and example of an attractor with nonuniform dimension or topology. (“Nonuniform dimension” refers here to the localized box count in the attractor$^{25}$.) This topic is currently under investigation.
Figure 4: A geometric illustration of the motion shows trajectories confined to a sheet. (a) A sheet stretches and folds as time evolves, forming a branched manifold. (b) The beginning and end of the sheet, at time zero and after the driving period, are identified and joined to form a flattened, deformed toroidal structure. Trajectories move around the sheet, orbiting the hole. Trajectories that start on the inside fold to the outside, while those that start on the outside return near the outside.
Figure 5: (a) A Poincaré section is defined as a slice in time. (b) A return map on the Poincaré section reveals a one-dimensional map.
4. Numerical Modeling

4.1. Modeling Friction

Modeling an oscillator with dry friction is not a straightforward task. Much progress has been made in understanding the mechanisms of friction; these have been summarized in survey papers\textsuperscript{26,1,2}. Friction descriptions range from to detailed surface-tribological descriptions, to the simple Coulomb friction law, which states that there is a kinetic coefficient of friction. The former approach, although useful, is too complicated to put into a low-order differential equation of motion. On the other hand, when simple friction laws are used, dynamical friction phenomena will not all be modeled, and their effects will not be observed in the dynamics. This can effect analysis, prediction, and control\textsuperscript{27}.

The Coulomb law has been used to model stick-slip\textsuperscript{15,4} and forced response\textsuperscript{29}. A simple velocity-dependent friction law with a negative slope can model self-excited oscillations\textsuperscript{30,31}. State-variable friction laws have successfully modeled hysteresis in sliding rocks\textsuperscript{32,33,34}. Other models may be used to describe phenomena such as rising static friction\textsuperscript{27} or normal vibrations of contact surfaces\textsuperscript{26}.

Our interest in this chapter is in stick-slip, and its effects on dynamical systems. Hence, we will focus on the Coulomb friction law, since it can model stick-slip motion (although it may not model fine details of stick-slip itself). We will also take a glance at a simple, smooth friction law, motivated by the idea of static and kinetic coefficients of friction, and a state-variable friction law which was written up based on experimental observations of the friction forces\textsuperscript{23}.

In each case, we assume that the friction force $F$ is proportional to the normal load $N$, such that $F = \mu N$. In the experiment, the normal load varies with displacement, such that $N = n(x)$. In the simple friction laws the coefficient of friction is dependent on velocity, i.e. $\mu = f(v)$. Thus $F = n(x)f(v)$. If another state variable $\theta$ is involved, then the coefficient of friction might be written as $\mu = f(v, \theta)$.

With this in mind, the general nondimensional equation of motion for the mechanics model in Figure 1 is given in equation (1). The normal load is assumed to vary linearly with displacement, with the ability for the contact surfaces to separate rather than undergo tension. Thus $n(x)$ has the form of equation (3). Anderson and Ferri\textsuperscript{38} have studied the case of bilateral contacts, with $n(x) = |1 + kx|$. For our case, normal loads in the direction of sliding have been ignored.

4.2. The Coulomb Model

People often use coefficients of static and kinetic friction $\mu_s$ and $\mu_k$. The friction relation is given in equation (2). For simplicity, we let $\mu_s = \mu_k = \mu = 1$. Based on dynamical friction measurements in an experimental oscillator with titanium contacts, this simplification is not too unreasonable\textsuperscript{23}. Instabilities that might occur when $\mu_s > \mu_k$ are eliminated. It turns out that such a material property is not
necessary to generate chaotic dynamics. We also set the viscous damping term to zero for simplicity, and since its contribution in the experiment was small.

Before presenting numerical results, we discuss \textit{sticking regions}, and how they effect the integration algorithm.

4.2.1. Sticking regions

At the beginning of the discussion, we will relax the condition that \( n(x) = 0, x < -1/k \), and impose it again later. We first write equation (1) as a first order system. Letting \( x_1 = x \) and \( x_2 = \dot{x} \) yields

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - (1 + kx)f(x_2) + a \cos(\Omega t). \tag{4}
\]

We can determine the sticking regions by looking for fixed points. Usually, in such a non-autonomous system, there are no fixed points because \( \cos(\Omega t) \neq \text{constant} \). But here, the multi-valued friction force can balance, at least temporarily, the driving force. Looking for fixed points, we set the right-hand sides of equations (4) to zero, and obtain

\[
\dot{x}_2 = 0, \quad -x_1 - (1 + kx)f(x_2) + a \cos(\Omega t) = 0, \tag{5}
\]

whence, for \( x_1 \neq -1/k \), we have

\[
[x_1 - a \cos(\Omega t)]/(1 + kx) = -f(0).
\]

However, \(-1 \leq f(0) \leq 1\). Therefore, if

\[-1 \leq [x_1 - a \cos(\Omega t)]/(1 + kx) \leq 1,
\]

the time-dependent excitation force can be instantaneously balanced by the multi-valued friction force.

For the case in which \( 0 \leq k < 1 \), and \( 1 + kx > 0 \) (positive normal force), we find that \( x_1 \) is temporarily fixed, i.e., \( x_1 \) is in the sticking region, when

\[
x_1 \leq [1 + a \cos(\Omega t)]/(1 - k), \quad x_1 \geq [-1 + a \cos(\Omega t)]/(1 + k).
\]

For \( 0 \leq k < 1 \), and \( 1 + kx < 0 \) (negative normal force), a similar analysis can be carried out for the bounds on \( x_1 \).

For \( k > 1 \), and \( 1 + kx > 0 \), \( x_1 \) is temporarily fixed (\( x_1 \) is in the sticking region) when

\[
x_1 \geq [1 + a \cos(\Omega t)]/(1 - k), \quad x_1 \leq [-1 + a \cos(\Omega t)]/(1 + k).
\]

For \( k > 1 \), and \( 1 + kx < 0 \), a similar analysis can be conducted.

In each case, the boundaries of the sticking regions are given by

\[
C_1 : \quad x_1 = [-1 + a \cos(\Omega t)]/(1 + k) \tag{6}
\]
Figure 6: Sticking regions $R$ in the $(x_1, t)$ plane (at $x_2 = 0$) for various values of $a$ and $k$: (a) $k < 1$ and $ak < 1$; (b) $k < 1$ and $ak > 1$; (c) $k > 1$ and $ak < 1$; (d) $k > 1$ and $ak > 1$. Plus and minus signs indicate where the flow is upward and downward through the page. Part (a) includes the sketch of a sticking solution, where sticking starts at point $p$ and slip resumes at point $q$. 
and

\[ C_2 : \quad x_1 = \frac{[1 + a \cos(\Omega t)]}{(1 - k)}, \]

(7)

\[ k \neq 1, \] which intersect at \( x_1 = -1/k \) when \( ak \geq 1. \)

The sticking regions \( R \) for various parameters \( a \) and \( k \) are displayed in Figure 6.

We now enforce \( n(x) = 0, x < -1/k \) so that the surfaces can lose contact. To visualize the sticking regions in this case, we can erase all of the sticking regions plotted for \( x_1 < -1/k \). For \( k > 0 \) there is a permanent-sticking region defined by

\[ x_1 > \max\left(\frac{1 - a}{1 - k}, \frac{-1 + a}{1 + k}\right). \]

As \( k \to 1^- \), the upper boundary \( C_1 \) of the sticking region in Figure 6(a) goes to \(+\infty\). As \( k \) passes through the value \( k = 1 \), the sticking region boundary reappears from \(-\infty\), and the orientation of the sticking region is inverted. The case of \( k = 1 \) can be specially analyzed to show that the sticking boundaries are vertical lines.

Consider an orbit which crosses the \( x_2 = 0 \) plane either in a sticking region or in a non-sticking zone. If it is in the sticking region, it will remain stuck, i.e., \( x_1 \) remains constant, until time evolves such that it is on the boundary of the sticking region. When motion resumes, will trajectories flow into \( x_2 < 0 \) or \( x_2 > 0 \)? The vector field indicates the direction of flow normal to the \((x, t)\) plane. By restricting the right side of the second of equations (4) to be greater than zero,

\[-x_1 - (1 + k x)f(x_2) + a \cos(\Omega t) > 0,\]

and we obtain

\[-x_1 + a \cos(\Omega t) > (1 + k z_1)f(0),\]

(8)

which describes a region in which flow may go from \( x_2 < 0 \) to \( x_2 > 0 \). We must satisfy this condition for both extremes, \( f(0) = 1 \) and \( f(0) = -1 \). As a result, for "upward" flow, and for \( f(0) = 1 \), we must satisfy

\[ x_1 < \frac{-1 + a \cos(\Omega t)}{(1 + k)}, \quad k \neq 1, \]

and, for \( f(0) = -1 \), depending on the value of \( k \), either

\[ x_1 < \frac{1 + a \cos(\Omega t)}{(1 - k)}, \quad k < 1, \]

or

\[ x_1 > \frac{1 + a \cos(\Omega t)}{(1 - k)}, \quad k > 1. \]

To find the region of "downward" flow, i.e., flow from \( x_2 > 0 \) to \( x_2 < 0 \), we reverse the inequality in equation (8). From \( f(0) = 1 \), we must satisfy

\[ x_1 > \frac{-1 + a \cos(\Omega t)}{(1 + k)}, \quad k \neq 1, \]
and, for \( f(0) = -1 \), depending on the value of \( k \), either

\[
x_1 = \frac{1 + a \cos(\Omega t)}{(1 - k)}, \quad k < 1,
\]

or

\[
x_1 < \frac{1 + a \cos(\Omega t)}{(1 - k)}, \quad k > 1.
\]

Thus, the curves \( C_1 \) and \( C_2 \) in the \((x_1, t)\) plane (Figure 6) represent the boundaries of both the sticking region and the regions of upward and downward flow.

This analysis will be revisited from a geometric point of view in section 5.

4.2.2. Integration algorithm

We now return to the task of describing an algorithm for numerical integration of the discontinuous equations of motion. (See Shaw\(^4\) for a similar algorithm for friction without displacement dependence). While we discuss this algorithm particularly for this system, researchers have also been developing more general algorithms for dealing with discontinuities or unsteady topologies\(^38,36\).

Starting with initial conditions outside of the sticking region, say \( x_2 > 0 \), numerical integration begins as usual with \( f(x_2) = 1 \), and proceeds according to the following steps.

1. Integration continues, point by point, until \( x_2 \) changes sign. The program must now remember the previous point, \( A \).

2. A shooting routine is used, adjusting the integration step size and integrating from \( A \) to \( B \), until \( B \) is within some tolerance of \( x_2 = 0 \).

3. The trajectory is now considered to be in the \((x_1, t)\) plane; thus it is a candidate for sticking. If it is in the sticking region, the algorithm decides whether it will leave the sticking region at curve \( C_1 \) or \( C_2 \), based on the location of the point in the \((x_1, t)\) plane. It then uses equations (6) and (7) to calculate the time of departure from the sticking region.

4. Using the current values of \( x_1 \) and \( t \), the algorithm determines whether the onset of motion is upward \((x_2 > 0)\) or downward \((x_2 < 0)\). If upward, it sets \( f = 1 \) and if downward, it assigns \( f = -1 \). It then starts over at step 1.

Data from this scheme will not have a constant sampling rate (integration time step). In order to use data processing programs which require a constant sampling rate, the data is interpolated into data with a constant sampling rate.

The numerical integration scheme is well defined through the discontinuity, provided that the time at which a stuck trajectory begins to slip can either be calculated, or determined as infinite (implying permanent sticking). Thus, in the numerical procedure, solutions exist, and are unique in the forward sense. This idea is revisited in section 5.5.
Figure 7: A numerical solution of the Coulomb oscillator with $\Omega = 1.25$, $\alpha = 1.9$, and $k = 1.5$; projection on the two-dimensional ($x, \dot{x}$) space.

4.2.3. Results

The numerical study of the Coulomb model concentrates on the parameter values $k = 1.5$, $\Omega = 1.25$, and $\alpha = 1.9$. The parameters are chosen not from experimentally measured values, but as values which produce behavior qualitatively similar to the experiment. Chaotic motions of this type can be seen for a large range of parameters.

The phase portrait of a chaotic numerical solution is shown in Figure 7. Sticking takes place at the trajectory cusps in the funnel structure. The system has three state variables: displacement $x$, velocity $\dot{x}$ and time $t \mod 2\pi/\Omega$ arising from the periodic excitation. A projection of the 3-D phase portrait is shown in Figure 8. The geometry of the attractor is the same as that of Figure 4.

Taking a slice of time, we can examine the Poincaré section. Again, its image appears to be one-dimensional. We can define a coordinate $s$ along the one-dimensional image, and plot the return map, as in Figure 9. The dynamics reduce to a one-dimensional map.

All of these features—the funnel structure, the 3-D attractor, the 1-D Poincaré image, and the tent-like return map—compare well with the experiment.

A bifurcation analysis using an increasing parameter $a$ has shown period-doubling to be the root to chaos. Lyapunov exponents, however, were not calculated, since the presence of the discontinuity interferes with its calculation from the equations
Figure 8: Numerical solution of the Coulomb friction attractor shown in 3-D with $x$, $t(\text{mod}2\pi/\Omega)$, and $\dot{z}$ as the radial, circumferential and longitudinal coordinates.
Figure 9: Return map on a coordinate $s$ defined along the Poincaré section for a numerical simulation of the Coulomb model.

of motion. A method for computing Lyapunov exponents in the presence of a discontinuity can be found in the literature\textsuperscript{37}.

4.3. Modeling with a Smooth Friction Law

We have seen how the multivalued discontinuity led to the presence of sticking regions. It turns out that a smooth friction law with a very steep slope in place of the discontinuity can produce nearly sticking effects. The convenience of using a smooth friction law is that we need not be concerned with the presence of discontinuities while performing the numerical integration. To this end, we choose a smooth function for $f(v)$ which has features approximating static and kinetic coefficients of friction,

$$f(v) = (\mu_k + (1 - \mu_k) \text{sech}(\beta v)) \tanh(\beta v),$$

and plug this friction function into the non-dimensional equation of motion (1).

For various parameter values we can observe period-one, period-two, and higher-period motions. We can also find chaotic motions with some similarities to those of the experiment. Period doubling is the observed route to chaos. We present results for the parameter values $\Omega = 1.3, a = 1.45, k = 1.5, \zeta = 0.015, \alpha = 50, \beta = 5, \mu_k = 0.7, \text{and } \mu_s = 1.$

The phase portrait, displayed in 2-D, has the familiar funnel structure (Fig-
Figure 10: A numerical solution of the smooth-friction oscillator with $\Omega = 1.3$, $a = 1.45$, and $k = 1.5$; projection on the two-dimensional $(x, \dot{x})$ space.

Figure 10. The 3-D phase flow shows a stretching and folding toroidal (Figure 11). A return map on the Poincaré section reveals a tent-like one-dimensional map (Figure 12). These features resemble those of the experiment and the Coulomb model.

True stick-slip cannot take place with the smooth friction model. Since there is no discontinuity, there are no true sticking regions. Instead of a discontinuity, there is a very steep slope at the origin of the friction function. Its effect is to strongly dampen motions in that region. As a result, "nearly sticking" motion seems to take place. This is visible at the tiny loops in the funnel of Figure 10, as compared to the cusps of Figure 7, which represent true stick. It is also visible in Figure 11 as the circular orbits near the zero-velocity plane, underneath the fold.

With the smooth function we can easily calculate a Lyapunov exponents. Since the smooth law is differentiable, the differential equation of motion is differentiable. Hence, the variational equations, which require differentiation of the vector field, can be computed. The largest Lyapunov exponent was calculated as $\lambda = 0.11$, which is greater than zero, indicating sensitivity to initial conditions and a loss of long-term predictability.$^{38}$
Figure 11: Numerical solution of the smooth-friction attractor shown in 3-D with $x$, $t(\text{mod}2\pi/\Omega)$ and $\dot{x}$ as the radial, circumferential, and longitudinal coordinates.
4.4. Modeling with a State-Variable Friction Law

In the previous section we described friction laws which depend only on velocity and displacement (via the normal load). In this section, we explore a friction law which depends on an additional "hidden" variable. Such friction laws have been dubbed state-variable friction laws, and have been used to model experimental data on the steady sliding of rocks\textsuperscript{32,33,34}.

Why introduce an extra state variable here\textsuperscript{7}? Notice that in the 3-D display of the experimental dynamics (Figure 3), orbits that start on the inside of the deformed torus travel through positive velocity, and fold onto the outside of the torus, passing through the zero-velocity plane before returning to zero velocity. On the other hand, in the previous simulations, trajectories that start on the inside of the deformed torus travel through positive velocity, and fold onto the outside of the torus, scaling onto the zero-velocity plane (sticking motion). The discrepancy could be due to experimental factors, such as the effect of higher modes in the beam, or filtering. Conversely, there might be some shortcomings in the simple friction laws. In what follows, the state-variable friction model will produce the feature of overshooting the zero-velocity plane prior to achieving sticking motion.

We choose a friction model based on experimental observations of friction measurements, without regard to the physics of friction\textsuperscript{23}. In this simulation, the
equations of motion have an additional state $\theta$, such that

$$\ddot{x} + x + n(x)\theta = a \cos(\Omega t), \quad \dot{\theta} = -\gamma(\theta - f(\dot{x})), \quad (9)$$

where $f(\dot{x})$ is a simple friction function, such as the Coulomb law. $\theta$ represents the instantaneous coefficient of friction, and for a constant $\dot{x}$ it asymptotically approaches a backbone function, represented by $f(\dot{x})$. Thus, the friction law is like a simple one, but with some inertia.

Equations (9) use a state-variable friction law similar to those of Ruina$^{33,34}$. The backbone is slightly different, and it allows for relative sliding to switch directions. (Ruina has proposed a modification to his law to accommodate sliding reversals). Also the right-hand side of Ruina's version of the second of equations (9) is multiplied by the velocity. Experiments with other frictional systems indicate that this type of velocity dependence is necessary to model certain friction phenomena$^{34}$. We are also assuming that the normal load $n(x)$ has no direct effect on the dynamics of $\theta$, contrary to the observations of others$^{39,40}$.

Equilibrium solutions of equations (9) for the unforced case with $n(x) = 1$ are $\ddot{x} = -\dot{\theta}$ and $\dot{\theta} = f(0)$. Thus, if $f(\dot{x})$ is multivalued at $\dot{x} = 0$, such as with the Coulomb law, the undriven oscillator will have infinitely many equilibria. Still, during oscillation, the friction force will not change discontinuously in time.

A simulation of the oscillator with skewed plates was performed by letting $n(x) = 1 + kx$, $x > -1/k$ and $n(x) = 0$, $x < -1/k$. For simplicity, we used $f(\dot{x}) = \tanh(50\dot{x})$ (a smooth approximation of the Coulomb law with $\mu_k = \mu_s = 1$). A 3-D phase portrait is shown in Figure 13 for the parameter values of $k = 1.5$, $\Omega = 1.25$, $a = 1.9$, and $\gamma = 10$. Note, however, that equations (9) represent a four-dimensional system. In the plot, motions starting from the inside of the torus flow through positive velocity and overshoot the zero-velocity plane before achieving "nearly" sticking motion, similar to the experimental oscillator.

The state-variable friction law in equations (9) is not guaranteed to be passive. The friction will do positive work on the system whenever $\theta$ and $\dot{x}$ differ in sign. No analysis has been performed in this regard. For the large motions simulated, this does not present any problems. However, for small motions, it could be that the model and experiment will have qualitatively different behaviors.

4.5. Summary

Due to the multivalued discontinuity, the Coulomb law incorporates stick-slip in its behavior, while the smooth, simple friction law, and state-variable law approximate stick-slip. Each friction model is able to recreate qualitative features seen in the experimental system. These features include the strange attractor on a branched manifold, and the reduction of the dynamics to a one-dimensional map. The state-variable friction law was able to model the addition detail of trajectories overshooting the sticking region before approaching it. The experiment displayed this overshoot, while the simple friction models did not incorporate it.
Figure 13: Numerical solution revealing the attractor for the state-variable-friction model shown in 3-D with $x, t \mod 2\pi/\Omega$ and $z$ as the radial, circumferential, and longitudinal coordinates.
The Coulomb model and the smooth model suggest that the important contributors to the formation of the attractor on a branched manifold are the (near) discontinuity of the vector field, and the dependence of the friction on displacement as well as velocity. A negative slope is not necessary for obtaining chaotic motion in a friction oscillator. Such negative rate dependence is likely to be a material friction property, while the displacement dependence can easily be a property of the mechanical set-up.

In the next section, we discuss the geometric properties of stick-slip in this oscillator, and how the one-dimensional map arises.

5. The Geometry of Stick-Slip

In this section, we study the geometry of the vector field corresponding to equations (1), (2), and (3). We have already noted that an interesting feature of this ordinary differential equation of motion is that, because of the friction function, it is discontinuous and multivalued at \( \dot{z} = 0 \). It is multivalued in that \( f(0) \) can take on any value between \(-\mu_s\) and \(\mu_s\).

One approach to such a problem is to view it as a piecewise continuous system, examine the continuous pieces, and match them. Alternatively, we might look at the limiting behavior of continuous systems which in some sense converge to the discontinuous system\(^{41}\). This section focuses on the former approach.

The case of \( k = 0 \), and hence \( n(z) \equiv 1 \), can be solved for periodic motion\(^{15}\) and their stabilities\(^4\) by breaking the problem into piecewise linear equations and exploiting symmetry in \(z\). Unfortunately, by including \( k \neq 0 \), thereby causing \( n(z) \) to be active and nonconstant in equation (1), we lose the symmetry in \(z\). Hence, the calculation of periodic orbits and their stabilities is extremely difficult. We choose another approach for the analysis: we graphically examine the qualitative nature of the system. This type of analysis has been done in classic nonlinear-vibrations texts\(^{42,50}\) for two-dimensional autonomous systems.

Equation (1) with equation (2) is piecewise integrable, that is it is solvable in subregions of the state space. We geometrically observe the nature of the flow in each of these regions of solvability, and then see how these solutions interact at the boundary of the regions. The dynamics of the flow are viewed in terms of a map on the boundary between the regions. From a qualitative picture of this map, we can construct the attractor, and show that

1. the dynamical behavior reduces to a one-dimensional map,

2. the flow of the Coulomb oscillator may not be invertible (previously reported by Shaw\(^4\)),

3. the flow may reach its attractor in finite time, and

4. the attractor has dimension less than or equal to two.
Takens\textsuperscript{43} had observed these properties in constrained systems\textsuperscript{44}. These properties are made possible because the Coulomb friction law produces a discontinuous and multivalued vector field (the same mechanism responsible for stick-slip motion). Finite attraction time to "terminal attractors" has been exploited for the learning process in neural networks with non-Lipschitz components\textsuperscript{45}. Other properties may arise in systems with discontinuities. For example, in a problem modeling shock, Antman\textsuperscript{46} uncovered nonunique forward solutions. We will also look at the ramifications of stick-slip on experimental methods in section 7.

5.1. Piecewise Linear Equations

Again, we look at the special case of $\mu_s = \mu_k = 1$. Further, we neglect the viscous damping term $\zeta \dot{x}$ (recall that the experimental damping ratio was measured as $\zeta = 0.015$). Finally, out of interest, we will only look at the case where $k > 1$, for which the sticking region has a particular structure (see section 4.2.1). Equation (1) with equations (2) and (3) can be written for regions in which they are solvable:

$$\ddot{x} + (1 + k)x = -1 + a \cos(\Omega t), \quad \dot{x} > 0, \quad x > -\frac{1}{k}, \quad (10)$$

$$\ddot{x} + (1 - k)x = 1 + a \cos(\Omega t), \quad \dot{x} < 0, \quad x > -\frac{1}{k}. \quad (11)$$

For $x \leq -1/k$, the normal load becomes zero due to the no-contact condition, and we have

$$\ddot{x} + x = a \cos(\Omega t), \quad x < -\frac{1}{k}. \quad (12)$$

For $-1 < k < 1$, equation (10) has harmonic solutions. For $k > 1$, the solution of equation (11) consists of a driven saddle.

Let $D$ denote the $(x, t)$-plane ($\dot{x} = 0$) and consider the mappings associated with the flow of points based at $D$:

$$P^+ : \ominus \to D, \quad \dot{x} > 0,$$

$$P^- : \ominus \to D, \quad \dot{x} < 0,$$

where $P^+$ is the map that arises from the flow of equation (10), and $P^-$ is the map that arises from the flow of equation (11), and $\ominus \subset D$ and $\ominus \subset D$ denote the domains of $P^+$ and $P^-$, respectively. The goal is to geometrically describe the maps $P^+$ and $P^-$, and see how they interact with the sticking region. The idea is sketched in Figure 14. For simplicity, we will start the discussion by omitting equation (12), which arises from the no-contact condition. Thus we will discuss the mapping between the flow of equation (10) and equation (11). This analysis is of interest anyway, since regular and chaotic motions confined to the region $x > -1/k$ have been observed in some numerical integrations. The results will be completed by including equation (12) numerically.
Figure 14: Trajectories governed by each piecewise linear equation are associated with either the map $P^+: \oplus \rightarrow D$, for trajectories in $\dot{z} > 0$, or the map $P^- : \ominus \rightarrow D$, for trajectories in $\dot{z} < 0$. Some orbits get mapped into the sticking region $R$, where the motion remains constant until such time that the orbit is no longer in the sticking region. The curves $C_1$ and $C_2$ represent the boundaries, in $D$, between the sticking regions $R$ and the domains $\oplus$ and $\ominus$ of $P^+$ and $P^-$, respectively.
5.1.1. The sticking region—geometric interpretation

Analysis of equation (1) with equation (2) for \( \dot{x} = 0 \) leads to the sticking regions. In section 4.2, the sticking region was found by analyzing equations (4) for fixed points and using the multivaluedness of \( f(x_2) \) at \( x_2 = 0^{15,4,16} \). However, we will describe the sticking regions using a geometric viewpoint which will help set the mood of this analysis.

If we write equations (4) in extended phase space for \( \dot{x} > 0 \), we have

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1 + k)x_1 - 1 + a \cos(\Omega t), \quad x_2 > 0 \\
i &= 1.
\end{align*}

By looking at the sign of \( \dot{x}_2 \) adjacent to the \((x, t)\)-plane \( D \), we find regions where the flow of equations (13) is upward and regions where the flow is downward. The curve (in the plane \( D \)) dividing the regions, called \( C_1 \), is given by \( \dot{x}_2 = 0 \), which yields

\[ C_1 : \quad x_1 = \frac{-1 + a \cos(\Omega t)}{1 + k}. \]

We can do the same for the case of \( \dot{x} < 0 \). The equations in extended phase space are

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1 - k)x_1 + 1 + a \cos(\Omega t), \quad x_2 < 0 \\
i &= 1.
\end{align*}

The regions for upward and downward flow on the \((x, t)\)-plane \( D \) for equations (14) are separated by a curve \( C_2 \), which is given by \( \dot{x}_2 = 0 \), or

\[ C_2 : \quad x_1 = \frac{1 + a \cos(\Omega t)}{1 - k}. \]

Since the flows of equations (13) and (14) meet at \( D \), we try to match the flow above \( D \) with the flow below \( D \). There are some regions where the flow of equations (13) is directed from \( \dot{x} > 0 \) toward \( D \), and simultaneously the flow of equations (14) is directed from \( \dot{x} < 0 \) toward \( D \), producing a conflict in the flow directions. (Similarly, there are regions where \( \dot{x} > 0 \) and \( \dot{x} < 0 \) are directed away from \( D \).) These regions of conflict are the sticking regions \( R \), as shown in Figure 15. Regions where the flow directions agree and are upward (toward \( x \)) are labeled \( \oplus \), and regions where the flow directions are downward (toward \( x \) ) are labeled \( \ominus \). In the region “above” (defined as \( \{(x, t) : x > x^{(1)} \) and \( x > x^{(2)} \) for \( (x^{(1)}, t) \in C_1 \), and \( (x^{(2)}, t) \in C_2 \) ) both \( C_1 \) and \( C_2 \), the flows of both equations (13) and (14) are directed toward the \((x, t)\)-plane. Hence, orbits hitting this region are trapped in \( D \), \( x \) remaining fixed, as time evolves until the moment that the direction of flows
Figure 15: The sticking regions when the no-contact condition is observed. $\oplus$ and $\ominus$ indicate upward and downward flows. $R$ indicates the sticking region. $B^+$ labels the portion of $C_1$ with positive slope, bordering $\oplus$, and $B^-$ labels the portion of $C_2$ with positive slope, bordering $\ominus$. $\Omega = 1.25, a = 1.9$, and $k = 1.5$.

is in agreement. The multivaluedness of $f(x_2)$ provides a bridge between the flow of equations (13) and the flow of equations (14), and allows this trapping to take place. In the region "below" both $C_1$ and $C_2$, $x_1 \leq -1/k$, and due to the no-contact condition, there are no sticking regions.

If we apply the no-contact condition, the direction of flow for $x < -1/k$ is then governed by equation (12). The curve $C_3$ separating regions of upward and downward flow in equation (12) is given by

$$C_3 : x_1 = a \cos(\Omega t), \quad x_1 < -\frac{1}{k}.$$ 

The curves $C_1$, $C_2$, and $C_3$ all intersect at the same points:

$$x_1 = -\frac{1}{k},$$

$$t = \frac{1}{\Omega} \arccos\left(-\frac{1}{ak}\right).$$
5.2. Qualitative Mapping of Regions

We would like to see how the active regions (nonsticking regions) map under $P^+$ and $P^-$, and how they interact with the sticking regions. Again, results are presented for parameter values of $a = 1.9, \Omega = 1.25$, and $k = 1.5$.

Referring to Figure 15, we consider the mapping of the region $\oplus$ via $P^+$, and the mapping of the region $\ominus$ via $P^-$. If $R$ is the sticking region, then $\oplus \cup \ominus \cup R = D$. Motions in $R$ either stay in $R$ forever, or, through the evolution of time, exit $R$ into $\oplus$ or $\ominus$ via the map $S : R \rightarrow B^+ \cup B^-$, where $B^+$ and $B^-$ are part of the boundaries of $\oplus$ and $\ominus$ as shown in Figure 15. (The sets $B^+$ and $B^-$ are the components of $C_1$ and $C_2$, respectively, through which sticking orbits originating in $R$ must pass as they exit $R$.) Therefore, the system can be understood through the mappings of $\oplus$ and $\ominus$. Certainly $P^+(\oplus) \cap \oplus = \emptyset$, and $P^-(\ominus) \cap \ominus = \emptyset$. It is also likely that $P^+(\oplus) \cap R \neq \emptyset$, and $P^-(\ominus) \cap R \neq \emptyset$.

We want to qualitatively describe the images $P^+(\oplus)$ and $P^-(\ominus)$.

The dynamics of the Coulomb oscillator can be described by successive mappings of $\oplus$ or $\ominus$, under $P^+, P^-$, and $S$, when appropriate. Figure 16 shows the computer-generated sequence of mappings of $\ominus$ for a particular set of parameter values. The process accounts for no-contact condition. Within one period of time, the entire region of initial conditions has been crushed into a set of curves. This smashing is not asymptotic! It occurs suddenly in the sticking regions. The attractor lies in the images of $B^+$ and $B^-$. Many initial points will be condensed onto the attractor in finite time.

A three-dimensional blob of initial conditions condenses into a two-dimensional blob as it flows into the sticking region. Thus volumes of phase space are collapsed during a stick. This dimensional collapse corresponds to the heavy energy dissipation discussed in section 3.

The result in Figure 16 suggests that we only need to understand mappings of $B^+$ and $B^-$ to understand the long-term behavior of the entire system. When we study the system in this way, we are implicitly assuming that all orbits eventually pass through the sticking region, exiting onto $B^+$ and $B^-$. We must therefore ask, under what conditions do all orbits pass through the sticking region?

**Fact:** If the inverse image and the forward image of $\ominus$ (or $\oplus$) do not intersect in $D$, then all orbits will pass through the sticking region.

**Proof:** Suppose $P^-(\ominus) \cap P^{+1}(\ominus) = \emptyset$. If a point $r \notin P^{+1}(\ominus)$, then $r_1 = P^+(r) \notin \ominus$. Since $P^-(\ominus) \cap \ominus = \emptyset$, $r_1 \in R$. On the other hand, $r \in P^{+1}(\ominus)$, and $r \in \ominus$, then $r_1 = P^+(r) \in \ominus$, and $r_2 = P^-(r_1)$. By the hypotheses that $P^-(\ominus) \cap P^{+1}(\ominus) = \emptyset$, we know that $r_2 \notin P^{+1}(\ominus)$. Since $r_2 \notin P^{+1}(\ominus)$, we have seen in the above argument that $P^+(r_2) \in R$.

In other words, if we follow any point $s$ starting in $\ominus$, its mapping $r = P^-(s)$ will be in the image of $\ominus, P^-\ominus).$ If $P^-\ominus does not intersect the preimage of $\ominus$, then $r$ is not in the preimage of $\ominus$. If $r$ is not already in $R$, then its mapping $P^+(r)$ will certainly be in $R$, and the motion will stick. In the case when the initial $r \in R$,
Figure 16: Successive mappings of \( \Theta \). Within one period of excitation, the entire set of points has condensed to a line. \( \Omega = 1.25, a = 1.9 \), and \( k = 1.5 \).

the motion begins trivially in the sticking region.

This fact provides a sufficient condition for all trajectories to eventually pass through the sticking regions. We need no proof for the images and preimages of \( \Theta \) because all orbits in \( \Theta \) either map to \( R \) or to \( \Theta \).

It can be shown numerically that, for the parameters of focus, \( P^- (\Theta) \) and \( P^- (\Theta) \) do not intersect\(^6\).

The sequence of mappings of the boundary of \( \Theta \) can be viewed as a one-dimensional tent-like map. Depending upon the degree of stretching and folding, we may have periodic or chaotic dynamics. The dynamics of this oscillator has been shown to be associated with a one-dimensional single-humped map (see section 4.2). Unlike the usual case where the one-dimensional map is an approximation which exploits a strong stable foliation, this one-dimensional map arises exactly.

5.9. Construction of the Attractor

The sequence of mappings of \( B^- \) can be viewed by wrapping \( t(\text{mod} \frac{2\pi}{\Omega}) \) around and back to itself (in \( S^1 \)). These mappings can then be extrapolated into a flow in \((x, \dot{x}, t(\text{mod} \frac{2\pi}{\Omega}))\)-space. The resulting template (Figure 17) resembles the images of Figures 3, 4, and 8, displayed in the same space. The dynamics is in a branched manifold. Whether the branched manifold produces a strange attractor or a periodic

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Figure 17: The attractor is constructed by identifying periodicity in the time variable, and extrapolating the previous map sequence into a flow.

attractor depends upon the parameter values. Whether all the motions go through the sticking region also depends upon parameter values.

Templates of the Lorenz attractor and the Rössler attractor are prime examples of branched manifolds. (Guckenheimer and Holmes demonstrate the construction of the Lorenz branched manifold, and further geometric analysis.) Their motions are strongly asymptotically attracted to the branched manifolds. The semiflow on the branched manifold is an approximation, and does not fully represent the detailed behavior of the system.

In the Coulomb oscillator, however, the motion is condensed suddenly (not asymptotically) onto the branched manifold. This noninvertible condensation, due to the discontinuity and multivaluedness of the Coulomb friction law, gives the Coulomb oscillator the opportunity to have a strange attractor in a two-dimensional manifold. The dynamics in the branched manifold fully represents the long-term behavior of the system.

When analyzed using constrained equations, chaos in Lorenz- or Rössler-like attractors is also confined to a two-dimensional manifold. The forced belt-driven oscillator should also be confined to a two-dimensional manifold.

The entire attracting set for this system consists of the attracting branched-manifold, and of the permanent-sticking region which surrounds this toroidal structure. The permanent-sticking region exists on $D$ for all $x_1$ greater than the maximum of $\frac{-1+\alpha}{1+\kappa}$ and $\frac{1-\alpha}{1-\kappa}$. These values represent the maximum values of curves $C_1$ and $C_2$. All points landing in this part of the sticking region are stuck forever.

The dynamics on branched manifolds may be viewed via one-dimensional maps, i.e. maps of the form $s_{n+1} = g(s_n)$. A map that would account for both the dynamics
on the branched manifold, and the permanently sticking motions, would consist of
a component \( (s_n \geq 0) \) resembling a single-humped map, and a component \( (s_n < 0) \)
coinciding with the identity line, respectively. The component coinciding with the
identity line produces an infinite locus of fixed points for all \( s_n < 0 \). Motions on
\( s_n > 0 \) are dynamic, and may have periodic or chaotic attractors. Motions that get
mapped to \( s_n < 0 \) are trapped there forever.

5.4. The Limit of Large Excitations

Let us look at the geometry as the excitation amplitude \( a \) becomes large. Letting
\( x = az \), the approximate equations for large \( a \) are given as

\[
\begin{align*}
\ddot{z} + (1 + k)z &= \cos \Omega t, \quad \dot{z} \geq 0, \quad (15) \\
\ddot{z} + (1 - k)z &= \cos \Omega t, \quad \dot{z} \leq 0, \quad (16) \\
\ddot{z} + z &= \cos \Omega t, \quad z < 0.
\end{align*}
\]

Hence, as \( a \to \infty \), the constant term of each piecewise continuous system becomes
insignificant; the system approaches piecewise linearity. Therefore, the form of the
mapping of each boundary in Figure 16 approaches a limiting form.

With the scaled coordinate \( z \), the curves defining the sticking regions approach
the form

\[
C_1 : z = \frac{\cos \Omega t}{1 + k}, \\
C_2 : z = \frac{\cos \Omega t}{1 - k},
\]

as \( a \to \infty \). This is true for all \( k \neq 1 \). The ratio of the amplitudes of these curves
is \( r = (1 + k)/(1 - k) \). The parameter value \( k = 1.5 \) leads to \( r = 5 \). For the case of
\( k = 1 \), the curves defining the sticking regions are vertical lines. Hence, the sticking
regions are vertical strips.

Thus, the geometry of the sticking regions approaches a limiting geometry in
the same scaled coordinate \( z \) for which the mappings of the boundaries approach a
limiting form. Thus, the entire system dynamics approach a limiting dynamics as
\( a \to \infty \). This will qualitatively be of the form of the branched manifold constructed
in the previous section. This limiting dynamics is in the scaled coordinate \( z \). In
the real coordinate \( x \), the dynamics grows with \( a \). System parameters determine
whether all trajectories exhibit sticking behavior, whether the attractor is chaotic,
or whether all motions go into the permanent-sticking region.

In the case of \( k = 0 \), \( C_1 \) and \( C_2 \) coincide, and equations (15) and (16) are the
same. Thus, the system approaches the behavior of a frictionless system.
5.5. Existence and Uniqueness of Solutions

Existence and uniqueness for systems with simple discontinuous friction laws has been addressed in various contexts. References to this topic go back to Painlevé in 1895. With robot linkages and sliding bars with Coulomb friction \( f = \mu \text{sign}(v) \), the normal load, and hence the friction, is dependent on the acceleration. When the Coulomb model is used, such systems have been shown to have the possibilities of both multiple solutions and no solutions.

The model studied here does not have the influence of acceleration on the frictional force. Existence and uniqueness of such vector fields has been studied by Filippov. Our concern is actually with the choice of the friction model, namely in modeling the discontinuity. The geometric discussions above lead to a simple criterion on whether the discontinuous function will admit solutions.

The multivaluedness of \( f(x) \) at \( x = 0 \) is necessary for the existence of solutions in the sticking region. If \( f \) were a single-valued function, and, say, \( f(0) = 0 \), what would happen? A trajectory in the sticking region in \( D \) would be governed by the equations (1) and (2) with \( f(0) = 0 \). This reduces to a case where there is no static friction, and has the form of equation (12). An orbit in \( D \) would be, with probability one (unless it lies on the curve defined by \( x_1 = a \cos \omega t \), in which case it would be in equilibrium for an instant, and then no longer be on the curve), directed out of \( D \). Such a trajectory, if it were to exist, would have to immediately leave \( D \). But it cannot leave \( D \) since the flow outside \( D \) is directed toward \( D \). So the trajectory does not exist. The multivaluedness of \( f \) provides a bridge for the flow across \( D \) through the sticking region. The discontinuous \( f(v) \) can have a multivalued bridge which exceeds the connection between \( +1 \) and \( -1 \) (\( \mu_s > \mu_k \)). Then the sticking regions are larger, but trajectories can still remain directed in \( D \) while in the sticking region.

Thus, solutions to equation (1) subject to the commonly and loosely defined friction function \( f(\dot{x}) = \text{sign}(\dot{x}) \) will not exist in the neighborhood of regions in the discontinuity where the sense of the adjacent vector fields are in conflict. However, in defining a multivalued friction law through the ideas of static friction, solutions exist everywhere.

Generally, the flow as defined on the discontinuity must be compatible with the piecewise flows adjacent to the discontinuity. Filippov's solution construction ensures such compatibility.

The numerical procedure outlined in section 4.2.2 effectively implements the existence criterion by checking for sticking, and evaluating the time at which slip begins. Such an algorithm assumes that the path of the trajectory is well defined from stick until slip.

Trajectories described in this geometrical setting are unique in the forward sense. We have already pointed out that sticking trajectories are noninvertible. This could be interpreted as nonunique in backward time.
Figure 18: A modified horseshoe map includes condensation of the bent part into a line. This line is then reinjected into the map. The attractor has a portion of fractal dimension \(1 + \frac{\log 2}{\log 3}\), interlaced with a portion of dimension one.

5.6. Nonuniform Dimension and Topology

We know that stick-slip can lead to collapse in the dimension of phase space\(^4,35,6\). However, once the dimension has collapsed, can the attracting dynamics reestablish the higher dimensional character?

Suppose we were to have nonlinearities in addition to the friction, and the mappings \(P^+\) and \(P^-\) were to correspond to nonlinear flows. Imagine that the mapping of \(\oplus\) were to stretch and fold onto \(\ominus\) like a horseshoe, with some parts overhanging into \(R\). A schematic is a modified Smale horseshoe map\(^53\) shown in Figure 18. The part of the image normally lost from the square and thereafter ignored is instead compressed to a line, as in the sticking region, and reinjected into the map. We still have the fractal invariant set \(\Lambda\) of dimension \(d_\Lambda\) of the Smale horseshoe, in addition to a set of dimension \(d_\star\), corresponding to iterations of the map of the compressed reinjected lines. All orbits will end up on one of these two sets. Infinite iterations of the reinjected line will also produce a set of dimension \(d_\Lambda\) and a set of dimension \(d_\star\). Thus, if an orbit passes through the squashed line, it passes through a one-dimensional object. With further iterations, it may approach an object of dimension \(d_\Lambda\). The composite attractor contains components of dimension one interlaced with a component of dimension \(d_\Lambda\). The two components have different dimension and topology.

Conceivably, such events might take place in a flow if the dynamical system were to have a sticking region, and a horseshoe which leaves portions of its image in the sticking region. In attempting to cook up such an example, we modify the Duffing equation, which has horseshoes, by adding a localized dry-friction damper. The hope is that this example produces a flow a fractal set due to the stretching and folding of the horseshoe, in conjunction with a non-fractal set due to the smashing
that takes place in the sticking region.

Figure 19 shows a Poincaré map of a two-well oscillator with an applied dry damper. The equation for this oscillator is

\[ \ddot{x} + 0.1\dot{x} - x + x^3 + F(x, \dot{x}) = A \cos 2.1t, \tag{17} \]

where \( F(x, \dot{x}) \) represents the friction force, and \( A = 1.2 \). The friction function emulates a damper located roughly in the range \( x \in [-0.25, 0.25] \) by the function \( F(x, \dot{x}) = n(x)f(\dot{x}) \), where

\[ n(x) = \frac{1}{2}(\tanh(50(x - \frac{1}{4})) - \tanh(50(x + \frac{1}{4}))), \]

and \( f(\dot{x}) \) is defined as in equation (2). Physically, in this system, motions can momentarily stick in the damper and experience condensation, or they can oscillate freely near either well, or pass between wells without sticking, and undergo folding as in the Duffing system. The Poincaré section appears to have regions where the dimension is one, off which branch segments of higher dimension. Since iterations of a one-dimensional region are mixed with the higher-dimensional portion, it is difficult to separate regions and calculate correlation dimensions. For a Duffing oscillator with a damper approximated by a continuous friction law, and \( A = 1.1 \), we were able to obtain the correlation dimension on portions of its Poincaré map. A calculation on a selected set of points from its Poincaré section yields the correlation dimension on a region of the section. We calculated a correlation dimension of 0.98 on the central region which looks like a heavy, crisp line, and a correlation dimension of 1.3 on the right and left lower lobes of the section.

Because of the Coulomb damper, true sticking motion takes place, accompanied with condensation of orbits. The low-dimensional portion of its Poincaré section in Figure 19 is presumably not Cantored, with a capacity of one. In such case, an orbit passes through portions of the attractor with varying dimension and varying topology (Cantored and not Cantored). Neighborhoods of an orbit experience an evolution in the topology of the surrounding attractor.

This is preliminary, and a more careful quantification of nonuniform topologies in the attractor is currently under consideration.

5.7. Convergence of Limit Sets

This section has emphasized the matching of piecewise flows from a geometric viewpoint. Another possible route for studying nonsmooth systems might be in examining smooth systems which become nonsmooth in some limit\(^4^1\).

If we integrate equation (17) with a smooth friction function, e.g. \( \dot{f}(\dot{x}) = \tanh(\alpha \dot{x}) \), with \( \alpha = 50 \), the Poincaré section (Figure 20) looks strikingly similar to that of the discontinuous vector field. We suspect the same trend would take place based for Coulomb and smooth models of the mass-spring system based on results from section 4.
Figure 19: A Poincaré section taken as a slice in time of a Duffing oscillator with friction applied when $-0.25 < x < 0.25$. The coefficient of friction is represented by the Coulomb law with $\mu_s = \mu_k$. 
Figure 20: A Poincaré section taken as a slice in time of a Duffing oscillator with friction applied when \(-.25 < x < .25\). The friction is modeled by \(\tanh(50z)\).

The smooth damper does not produce true sticking motion, since the lack of a multivalued discontinuity does not produce sticking regions. There is no condensation (smashing is not complete) when the motion is "almost sticking." Thus, the low-dimensional portion of the Poincaré section of Figure 20 is actually a Cantored set of dimension slightly greater than one.

We would expect that as the parameter \(a\) increases, and the smooth friction function \(f\) better approximates the discontinuity, that the attractors in Figures 19 and 20 would become more similar. This leads us to consider sequences of vector fields,

\[
\dot{x} = g^n(x, t),
\]

where \(g^n(x, t) \rightarrow g(x, t)\) in some way as \(n \rightarrow \infty\). It seems that for certain convergent sequences of vector fields, the sequence of \(\omega\)-limit sets converges. (It is important to look for the convergence of \(\omega\)-limit sets, and not the convergence of solutions. If the limiting \(\omega\)-limit set has a strange attractor, then two orbits which are not identical will generally separate because of a positive Lyapunov exponent.)

If the \(\omega\)-limit sets of the sequence of vector fields converge, the result may not represent the \(\omega\)-limit set of the vector field

\[
\dot{x} = g(x, t),
\]

using a usual idea of convergence. In fact, if \(g(x, t)\) is not Lipschitz, it could be that
no solution exists over the region of $x$ to be considered.

In the Duffing oscillator with an applied dry damper, we can approximate the Coulomb law $f(\dot{x})$ with $\bar{f}(\dot{x}) = \tanh(\alpha \dot{x})$, and obtain good numerical results. Similarly, using $\bar{f}(\dot{x})$ in equation (1) also yields good numerical results, as did the smooth law used in section 4.3. As $\alpha \to \infty$, the function $\bar{f}(\dot{x})$ converges to

$$\begin{align*}
\bar{f}(\dot{x}) &= 1, \quad \dot{x} > 0, \\
\bar{f}(\dot{x}) &= 0, \quad \dot{x} = 0, \\
\bar{f}(\dot{x}) &= -1, \quad \dot{x} < 0,
\end{align*}$$

in the sense that $\bar{f}(0) = 0$ for any $y$, and for any small $\epsilon$, there is an $\alpha$ such that $|\bar{f}(y) - \bar{f}(y)| < \epsilon$. But if we replace $f(\dot{x})$ with $\bar{f}(\dot{x})$ (which is equal to $\text{sign}(\dot{x})$) in equation (1), we lose the existence of a solution in the sticking region, as discussed in section 5.1.1. Thus, we have an example in which the the limiting behavior of a sequence of vector fields is not the same as the behavior of the limiting vector field. The behaviors of the sequence of smooth vector fields, corresponding to a sequence of increasing $\alpha$ values, (seem to) approach the behavior of a multivalued vector field, which, in the usual sense, is not the limiting vector field.

5.8. Summary

This section presented a geometric way of describing the dynamics of a discontinuous, multivalued Coulomb oscillator. The technique is a three-dimensional extension of that previously employed in classical texts on two-dimensional autonomous systems.

The analysis reconstructs an attractor similar to that seen in numerical integrations. During the reconstruction, it was shown that infinitely strong contraction takes place during sticking motion. Because of this condensation, a one-dimensional map can describe the long-term dynamics exactly. This is unlike the usual case in which a one-dimensional map is an approximation which exploits a strong stable foliation.

Some other properties of the behavior which result from the discontinuous and multivalued nature the vector field (the same mechanism which gives rise to stick-slip motion) are

- the flow may not be invertible
- the flow may reach the attractor in finite time
- strange attractors may have dimension less than or equal to two

While there may be some debate over whether discontinuities really exist in physical systems, certainly near discontinuities exist, and to the resolution of measurements, they may be indistinguishable from actual discontinuities.
It seems that oscillators undergoing stick-slip have the potential for having strange attractors with nonuniform topology, in the sense that a trajectory's neighborhood in an attractor might vary in its topology while the orbit flows. The case examined briefly here suggested that the orbit could traverse through an attractor which is locally Cantored in some regions, but not in others.

While the geometric perspective on matching piecewise-smooth flows was emphasized, it is also conceivable that one might examine nonsmooth systems as a limit of a sequence of smooth systems. Visual evidence suggests that the sequence of limit sets of smooth systems can converge to the limit set of a nonsmooth system. However, the vector field itself might converge to a subtly different nonsmooth vector field. Issues such as existence might be heeded.

6. Symbol Dynamics

Chaotic motion in maps can be characterized by symbol sequences. Since stick-slip motion can reduce a 3-D flow to a 1-D map, it may be worthwhile to consider symbol dynamics. Stick-slip motion provides a natural basis for producing symbol sequences during the motion, where, for example, S represents motion which is sticking, and N represents motion which is not sticking (slipping). The resulting symbol sequence can be used to characterize the dynamics. In this section, we apply symbol dynamics to characterize chaos using binary autocorrelation functions and macroscopic Lyapunov exponents\textsuperscript{54}. We continue the study by comparing the bifurcation sequence, and its associated symbol sequences, of the Coulomb oscillator, with the "universal" bifurcation sequence of "standard" one-dimensional maps\textsuperscript{55}. Universal behavior refers to behavior that is consistent for all parameter ranges in a given class of systems\textsuperscript{9}.

6.1. The Binary Autocorrelation and Macroscopic Lyapunov Exponent

Singh and Joseph\textsuperscript{54} have proposed a technique for extracting quantitative information from a binary symbol sequence. First it is necessary to represent the symbol sequence $u(k)$ as a string of 1's and -1's. These values are chosen so that the expected mean of a random sequence of equally likely symbols is zero. As the trajectory passes through the Poincaré section for the $k$th time, if it is not sticking, we set $u(k) = 1$. If it is sticking, we set $u(k) = -1$. A binary autocorrelation function on such a symbol sequence is defined as

$$r(n) = \frac{1}{N} \sum_{k=1}^{N} u(k+n)u(k),$$

for $n = 0, 1, 2, \ldots$, and $N \gg n$. If the sequence is chaotic, the autocorrelation should have the property $r(n) \to 0$ as $n \to \infty$.

If the sequence becomes uncorrelated, an estimate of the largest Lyapunov exponent can be obtained by using the binary autocorrelation function. The largest
Lyapunov exponent can be defined as

\[ \lambda = \frac{1}{N} \sum_{n=1}^{N} \log_2 \frac{d(n)}{d_o(n - 1)}, \]  

where \( d_o(n - 1) \) is the difference between two nearby trajectories at the \((n - 1)\)th iterate, and \( d(n) \) is the distance between them after one iteration. Since the binary sequence is uncorrelated, we can estimate \( d_o(n - 1) \) as the expected distance \( \bar{d}_o(n - 1) \) between two randomly chosen points in the same symbol region. In our example, we measure the distance using coordinate \( s \) on the Poincaré plot. Two points chosen from the sticking region have an expected distance \( \bar{d}_o(n - 1) = 1/3 \). Two points from the nonsticking region have the same expected distance. If \( u(n - 1) \) and \( u(n) \) are in the same region, their iterates will either stay in that region, be in different regions, or both be in the other region. One defines

\[ \alpha = \log_2 \frac{\bar{d}(n)}{d_o(n - 1)}, \]

where \( \bar{d}(n) \) is the expected distance between two points when each is chosen from separate regions. For our problem, again using \( s \) to measure distances, \( \bar{d}(n) = 1 \) and \( \alpha = \log_2 3 \). Replacing \( d_o(n - 1) \) and \( d(n) \) in equation (18) with their expected values defines the macroscopic Lyapunov exponent, \( \lambda_m \). The exponent is dubbed macroscopic because the distances involved are not infinitesimal. Through the derivation of Singh and Joseph\textsuperscript{54}, the macroscopic Lyapunov exponent can be written in terms of the binary autocorrelation function:

\[ \lambda_m = \frac{1}{2} \alpha[1 - r(1)^2]. \]

Application of these ideas to a symbol sequence derived from the tent map yields a rapidly decaying autocorrelation and a Lyapunov exponent \( \lambda_t = 0.787516 \) for a string of 100000 symbols, and an exponent of \( \lambda_t = 0.787705 \) for a string of 2048 symbols. Its exact “microscopic” exponent, calculated using \( \log_2 \), is \( \lambda_{te} = 1 \). While the tent map has uniform stretching in the sense that the slope has a magnitude of two for all \( z \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \), the difference between the microscopic and macroscopic values comes from the fact that the global behavior of the map includes folding. Application to the logistic map yields a rapidly decaying binary autocorrelation function, and a macroscopic Lyapunov exponent of \( \lambda_t = 0.791578 \) for a sequence of 100000 symbols, and \( \lambda_t = 0.791116 \) for 2048 symbols, compared to an exact microscopic exponent of \( \lambda_{te} = 1 \).

Binary sequences of length 2048 were obtained from the experimental oscillator of section 3 and from a numerical simulation using the smooth friction law of section 4.3. The macroscopic Lyapunov exponent for the experimental oscillator was calculated to be \( \lambda_{exp} = 0.79055 \). That of the smooth-law simulation was computed as \( \lambda_{s1} = 0.79219 \). In section 4.3, the largest microscopic Lyapunov exponent for
the smooth-law flow was estimated numerically\textsuperscript{38}. It can be related to that of the Poincaré map via \( \lambda_{\text{flow}} = \lambda / T \), where \( T \) is the driving period. This calculation of the exponent for the Poincaré map from the equations of motion produced \( \lambda_{x2} = 0.77 \).

The actual computation of the macroscopic Lyapunov exponent can be carried out for symbol sequences from either a deterministic or random source. Thus it could not be used to distinguish chaos from noise. However, using symbol dynamics thusly can lead to an estimate of the order of magnitude of the largest Lyapunov exponent. In the case of the friction oscillator, such a calculation can be based on data concerning stick and slip. Such 'yes' and 'no' data could conceivably be obtained, for example, by microphone.

6.2. The Bifurcation Sequence and Universality

One-dimensional maps of a "standard" form undergo a "universal" bifurcation sequence when the bifurcation parameter is varied. However, the map arising from the Coulomb model does not have "standard" form. Nonetheless, in this section the bifurcation sequence of the Coulomb model is compared to that of the standard one-dimensional maps to see if it exhibits "universal" behavior. All observed components of the bifurcation sequence fit the universal sequence, although some universal events are not witnessed.

From a mathematical viewpoint, universal behavior of standard maps has been studied in detail. Thus, if a dynamical system exhibits universal behavior, then much is already known about the system. In terms of system identification, examining whether behavior is universal might give a clue as to whether the unknown system fits a standard class of systems. On the other hand, for parametric system identification, where differences in behavior with respect to system parameters is of importance, interest might be focused on nonuniform behavior.

6.2.1. Bifurcations of one-dimensional maps

One-dimensional maps of the type

\[
x_{n+1} = \lambda g(x_n),
\]

where the function \( g(x) \) satisfies certain assumptions, have been studied extensively\textsuperscript{9,56,57,58,10}. When considering universal sequences of periodic orbits, the critical assumptions are that

1. \( g(0) = g(1) = 0 \),

2. \( g(x) \) is smooth with a unique maximum at \( x_0 \), and \( \lambda > 0 \), and

3. \( g(x) \) has a negative Schwartzian derivative for \( x \in [0,1] - \{x_0\} \),

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where the Schwartzian derivative for a function $g(x)$ is defined as

$$D_s g(x) = \frac{g''(x)}{g'(x)} - \frac{3}{2} \left( \frac{g''(x)}{g'(x)} \right)^2.$$  

For discussions on metric universality, i.e., Feigenbaum numbers, we can relax the above assumptions, and only assume $g''(x_0) < 0$.

The dynamics of maps which satisfy assumptions 1-3 undergo a universal sequence of bifurcations, where the bifurcation parameter is $\lambda$. For $\lambda = \lambda_1$ sufficiently small, such that $\bar{x} = \lambda_1 g(\bar{x})$, and $|\lambda_1 g'(\bar{x})| < 1$, $\bar{x}$ is a stable periodic point of $\lambda g(x)$. As $\lambda$ increases, a periodic cycle remains until $\lambda = \lambda_2$, at which the periodic point loses stability and a stable periodic cycle of period two is born. A stable cycle of period two exists until $\lambda = \lambda_4$, where the period two loses stability and a period four is born. This period-doubling sequence continues, producing stable periodic cycles of period $2^n$, $n \to \infty$, as $\lambda$ approaches a limiting value $\lambda_\infty$. Given any $\lambda$ such that $\lambda_n < \lambda < \lambda_0$, there exists an infinite number of unstable periodic cycles and a stable cycle of period $n$. The stable cycle undergoes a similar period-doubling sequence as above, to a limiting value of $\lambda_\infty$. Typically, on a bifurcation diagram, windows of relatively low period $n$ are identifiable to the eye.

The bifurcation sequence of the map (19) exhibits universal behavior, that is behavior common to any function $g(x)$ which satisfies the assumptions stated above. As the bifurcation parameter increases, there will be a bifurcation sequence of stable periodic orbits. From this sequence, we could construct a sequence of the period lengths of these stable cycles. It is a universal property that this sequence of period lengths is the same for all such maps. For each stable periodic cycle, there exists a parameter value $\hat{\lambda}$ such that one point $p_0$ of the periodic sequence lies at $x_0$. The value of a periodic point $p_0(\lambda)$ is continuous in $\lambda$, and there is a region $(\hat{\lambda} - \delta_1, \hat{\lambda} + \delta_2)$, for some $\delta_1 > 0, \delta_2 > 0$, such that the periodic cycle remains stable, and $p_0$ stays near $x_0$.

A symbol sequence for the periodic cycle can be defined by whether the $i^{th}$ iterate in the cycle is to the right of $x_0$ (R) or to the left of $x_0$ (L). Since $p_0$ is arbitrarily close to $x_0$, we assign the symbol C to $p_0$. For a cycle of period $m$, the remaining $m - 1$ iterates of $p_0$ are assigned the symbols R and L. For example, if the periodic cycle had a period of five, the symbol sequence might be CRLLL, pertaining to iterates of the point located very near $x_0$. The convention in the literature\textsuperscript{57,58} is to drop the symbol C. Therefore, a symbol sequence for a periodic cycle of period $m$ is defined as the symbols of the $m - 1$ iterates of $p_0$. In our example, the period-five symbol sequence is defined by the four symbols RLLL. A second universal property is that the symbol sequences of each of these stable periodic orbits is the same for all such maps.

Metric universality exists for maps which merely satisfy $g''(x_0) < 0$. In this case, the period-doubling sequence occurs according to Feigenbaum's ratio\textsuperscript{9,56}. If $\lambda_n$ is the parameter value for a cycle of period $m2^n$, then Feigenbaum's ratio is

$$\delta = \lim_{n \to \infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} = 4.6692 \ldots.$$  

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6.2.2. The one-dimensional map underlying the Coulomb-friction oscillator

In section 4.2.3, we saw how a one-dimensional map arises from a return map on a variable $s$ defined on the Poincaré image of the oscillator with Coulomb friction. Let us call this one-dimensional map $F(s)$.

We also saw, in section 5.1.1, how sticking regions can be described by observing the directions of the piecewise continuous vector fields at the discontinuity $D$, defined in the state space by $\dot{x} = 0$. When both vector fields agree to flow through $D$, the flow may pass through the discontinuity. When both vector fields point toward $D$, there is a stable sticking region $R$. Flows are trapped in $R$ until time evolves such that the orbits are on either of the boundaries, $B^+$ or $B^-$, of the sticking region $R$. A map describing this action would be $S : R \rightarrow B^+ \cup B^-$. $S$ is singular since it takes a two-dimensional region $R$ and maps it into a finite union of curves.

For the parameter case studied ($a = 1.9, \Omega = 1.5,$ and $k = 1.5$), the entire two-dimensional region $\Theta$ collapses into a one-dimensional curve. This geometrically illustrates the singularity which produces the one-dimensional map $F(s)$. The singularity is in the mapping $S$ in the sticking region. All motions pass through the sticking region.

When all motions pass through $R$, knowledge of the mappings of the boundaries $B^+$ and $B^-$ is sufficient to understand the attracting set.

An analytical expression for $F(s)$ would consist of three components. One component would involve orbits passing through the boundary $B^-$ and their intersection with the plane $D$, represented by $P^-(B^-)$. Finding $y = P^-(x)$ requires the solution of transcendental equations. The mapping $P^-(B^-)$ lies partly in $R$ and partly in $\Theta$. Thus, the second component of the analytical expression of the one-dimensional map would be a logical operation. The third component would then be to either solve for the time at which trajectories in $R$ leave the sticking region at $B^+$ or $B^-$, or else to solve the transcendental equation representing those orbits which map to $D$ via $P^+$.

In short, the analytical description of $F(s)$ is compounded with transcendental equations and logical operations. Because of this complexity, we do not produce such an analytical expression, and our work is done largely from a geometric standpoint.

The map $F(s)$ (Figure 9) may not satisfy all the assumptions in the above discussions. Perhaps most importantly, $F(s)$ does not fit the form of equation (19). Varying $a$ in equation (1) actually alters the resulting shape function $g$ as well as the magnitude $\lambda$. This is because the orientations of the sticking-region boundaries (section 5) are dependent on $a$.

The "instantaneous" shape function $g$ satisfies the first assumption (if the coordinate $s$ is rescaled), but not necessarily the second (taking "smoothness" in the context of the Schwartzian derivative, i.e. up to three derivatives) and third.

Although the map $F(s)$ (and hence $g$) cannot be determined explicitly, implicit relationships make it possible to calculate derivatives. This is precisely what Shaw did when calculating stabilities of periodic motions in a similar oscillator. To this end, we refer to Figure 16. As an orbit goes through the one-dimensional map, it
can undergo one of three series of compositions.

**Case 1.** Imagine an initial point, \((x_0, t_0)\), in region \(A_1\), as it evolves until time \(t_0\) at the boundary \(S(A_1)\) of the sticking region. Note that \(x_0\) and \(t_0\) are directly related. The corresponding flow goes through negative velocity and returns to the sticking region at time \(t_1\), in the region \(P^-(S(A_1))\), which can be determined by solving the transcendental equation \(\dot{x}^-(t_1; t_0) = 0\), where \(x^-(t_1; t_0)\) represents the \(x\) component of the flow corresponding to \(P^-\), and is the solution to equation (11). If \(\frac{\partial x^-}{\partial t_1} \neq 0\), then \(t_1\) is implicitly a function of \(t_0\), i.e. \(t_1 = p(t_0)\). Finally, \(x_1 = x^-(t_1; t_0)\) completes the one-dimensional mapping.

**Case 2.** Trajectories in Figure 16 based at some point \((x_1, t_1)\) in region \(A_2\) leave the sticking region at time \(t_2\) in region \(B_1 = S(A_2)\). Flowing with positive velocity, the orbit returns to the sticking region at time \(t_3\), in region \(P^+(B_1)\), at a value of \(x_2\), which can be determined from the transcendental equation \(\dot{x}^+(t_3; t_2) = 0\), where \(x^+(t_3; t_2)\) represents the \(x\) component of the flow corresponding to \(P^+\), and is the solution to equation (10). This stuck trajectory leaves the sticking region on the boundary \(S(A_1)\), and flows through negative velocity back to the sticking region in \(P^-(S(A_1))\), completing the mapping at a point \(x_3\) and time \(t_4\), which can be determined by the transcendental equation \(\dot{x}^-(t_4; t_3) = 0\). Then, \(x_3 = x^-(t_4; t_3)\).

**Case 3.** Some motions can be based at a point \((x_1, t_1)\) in region \(A_3\). Here, the point \((x_1, t_1)\) lies in the slipping region, and continues to flow in the same manner as \((x_2, t_3)\) does in the case above. The sticking phase between times \(t_1\) and \(t_2\) does not occur.

Orbits based in region \(A_3\) can involve complications due to the condition of \(n(x) = 0\) for values of \(x < -1/k\).

These possibilities constitute the 1-D mapping, \(F(s)\), for this oscillator. The function \(F(s)\) is not expressible in closed form. However we can, in most instances, calculate the derivatives of \(F(s)\), and examine the Schwartzian derivative. For example, in Case 1., the map starts at \(x_0\). We can calculate \(t_0 = \arccos(x_0(1 - k))/\Omega = \alpha(x_0)\). Then, \(t_1 = p(t_0)\), as an implicit function, and the next iterate of the map is \(x_1 = x^-(t_1; t_0)\). Dependence on the initial condition \(x_0\) is expressed through \(t_0\). The derivative of the implicit map \(F: x_0 \rightarrow x_1\) is

\[
F' = \left[ \frac{\partial x^-}{\partial t_1} \frac{dp}{dt_0} + \frac{\partial x^-}{\partial t_0} \right] \frac{d\alpha}{dx_0},
\]

where \(\frac{dp}{dt_0} = -\frac{\partial x^-}{\partial t_0} / \frac{\partial x^-}{\partial t_1}\). \(F'\) can be regarded as a function of \(t_1, t_0,\) and \(x_0\). In this description, subsequent derivatives have the form

\[
F^{(n+1)} = \left[ \frac{\partial F^{(n)}}{\partial t_1} \frac{dp}{dt_0} + \frac{\partial F^{(n)}}{\partial t_0} \right] \frac{d\alpha}{dx_0} + \frac{\partial F^{(n)}}{\partial x_0}.
\]

The first derivative of \(F\) for Case 2. is

\[
F' = \left[ \frac{\partial x^+}{\partial t_3} \frac{\partial q}{\partial t_2} + \frac{\partial x^+}{\partial t_2} \frac{\partial r}{\partial t_1} \right] \frac{\partial r}{\partial x_1} \left[ \frac{\partial x^-}{\partial t_1} \frac{dp}{dt_0} + \frac{\partial x^-}{\partial t_0} \right] \frac{d\alpha}{dx_0},
\]

(21)
where \( r(x_1) = \arccos(x_1(1 + k))/\Omega \).

If we consider the transition between mapping as in Case 1., and mapping as in Case 2., we will look at the orbits which pass through the intermediate points \((x_2, t_2)\), with \( x_2 = 1/(1 + k) \) and \( t_2 = 2\pi/\Omega \). This orbit could be classified as either Case 1 or 2, depending on whether, by definition, the action of \( r(x_2) \), and \( x^+(t_3; t_2) \) takes place. We can numerically compute the derivatives to show that the Schwarzian derivative is not smooth at this transition. Thus, the 1-D map will not satisfy “standard” hypotheses, and we need not expect to observe a universal bifurcation sequence.

We can also see that the maximum of \( F(s) \) is smooth, at least for some values of \( a \). This is relevant to discussions on metric universality (Feigenbaum’s number). Visual evidence is in Figure 9.

To discuss the maximum of \( F(s) \), we must first locate it. To this end, we look at the image \( P^+(B_1) \) in Figure 16. Close inspection indicates that this image has a local minimum in \( x \) at \( x = z \) for some value of \( t \), \( t \approx 5.5 \) in the figure). Additionally, for the given parameters, this point \( z \) denotes the minimum value of \( s \) in some Poincaré sections. In the Poincaré section at \( t = 6.5 \), for example, defining a coordinate \( \hat{s} \) such that \( 0 \leq \hat{s} \leq 1 \) and \( \hat{s} = 1 \) at \( z \) shows that the point \((z, t = 6.5) \in D \) corresponds to the maximum value of \( \hat{s} \). In a Poincaré section at \( t = t_2 \), the point \((z, t_2) \in D \) represents \( s_0 \), locating the local maximum in the underlying map.

Since a small neighborhood \( V \in B_1 \) of the orbit passing through this critical point is governed solely by a function \( P^+(V) \) which maps a curve monotonely increasing in \( x \) to a curve with a smooth minimum in \( x \) (at \( x = z \)), the point \( s_0 \) represents a smooth maximum in the underlying map.

We should point out that if \( P^+(B_1) \) were to lack a local maximum in \( x \), then the point \( s_0 \) would correspond to sticking orbits passing through the local maximum of the curve \( B^+ \) (Figure 15), and thus \( s_0 \) would represent a boundary between two functions active in the one-dimensional map (involving \( P^+ \) and \( P^- \)). In general, such a map would not have a smooth maximum. Whether this occurs for all values of \( a \) is not known.

6.2.3. Bifurcation analysis for the Coulomb oscillator

Since the equation of motion has a discontinuity at \( \dot{x} = 0 \), the plane in \((x, \dot{x}, t)\)-space defined by \( \dot{x} = 0 \) is a natural place to make a Poincaré section. In this Poincaré section we plot \( x \) for the bifurcation diagram shown in Figure 1. The bifurcation diagram includes trajectories which bounce off the underside \( (x < 0) \) of the \((x, t)\) plane. (Some trajectories meet the \((x, t)\) plane from below, stick, and then return below the plane. This corresponds, for example, to motion near the outer edge of the attractor in Figure 8.)

The method used to compare the bifurcation sequence in the Coulomb friction model to the standard one-dimensional maps is as follows: We compute and plot a bifurcation diagram, which has periodic windows. We identify periodic orbits.
Figure 21: A bifurcation diagram shows period doubling to be the route to chaos. The bifurcation parameter, the driving amplitude $a$, is increasing in this plot.
visible to a parameter increment $\Delta a$ of $\Delta a = 0.0005$. In doing so, we look for stable periodic orbits of period less than eight. The infinitely many higher periods are difficult to find because the ranges of $a$ on which they exist are narrow. We compare the sequence of period lengths of stable periodic cycles found in the Coulomb oscillator with the universal sequence of stable periodic cycles. If a period five, say, appears in the bifurcation diagram for a parameter value $a = \hat{a}$, then we observe the Poincaré section (from a slice in time) of the motion with the parameter set to $\hat{a}$. In the Poincaré section the orbit will consist of five points. From this Poincaré section, we can determine the symbol sequence of the periodic points in terms of the sticking and slipping regions. We assign the symbol $S$ to points which are sticking, and the symbol $N$ to points which are not sticking (slipping). We also compare the symbol sequence of the stable periodic cycles found in the Coulomb oscillator with the universal symbol sequences of stable periodic cycles. Finally, we estimate Feigenbaum numbers from the Coulomb oscillator data.

This sequence of $N$'s and $S$'s will be analogous to the symbol sequence for the associated map, and can be translated into a sequence of $R$'s and $L$'s, respectively. This translation is exact when the Poincaré section is taken at $t (\mod \frac{2\pi}{k})$ corresponding to $t_1$. This is because the maximum of the underlying map occurs at the critical point $s_0$ corresponding to $z$, which is the local minimum (in $x$) of curve $P^+(B_1)$. At a phase corresponding to $t_1$, this point cleanly separates sticking motions from slipping motions, as well as left from right.

However, when $t \neq t_1$, some other value of $s \neq s_0$ separates points that are sticking from points that are slipping. As an extreme case, when $t = 6.5$ in Figure 16, all points are in the sticking region (a symbol sequence would trivially consist only of $S$'s). In the neighborhood of $t_1$, the approximation of the $N$'s and $S$'s as $R$'s and $L$'s is reasonable due to the smoothness of the curve $P^+(B_1)$. Deviation of the $N$'s and $S$'s from the $R$'s and $L$'s represents an error in observation of behavior, rather than an effect on universality. Our Poincaré section was taken at $t (\mod \frac{2\pi}{k}) \approx 0.625$. Based on Figure 16 viewed at $t \approx 5.6$, error in the symbol dynamics should be small.

6.2.4. Results

A comparison of the sequence of periodic orbits, and their symbol sequences, for the friction oscillator and the logistic map is in Table 1. For the logistic map, $x_{n+1} = \lambda g(x_n)$ with $g(x) = x(1 - x)$. The values for the logistic map were obtained\textsuperscript{57} for periodic orbits of period seven or less. Two higher-period events were added since they were incidentally found in the friction oscillator.

The bifurcation sequence of a Coulomb friction oscillator, in some sense, resembles the universal sequence of standard one-dimensional maps. The observed periodic orbits, their period lengths and symbol sequences, of the Coulomb friction oscillator lie in the sequence of the standard maps, although several events remain undetected. In other words, every event in the oscillator is also in the universal
Table 1: The observed sequence of period length of stable periodic cycles in the Coulomb oscillator, and their symbol sequences, are compared to the sequence of stable periodic cycles in the logistic map. Each cycle listed is the first of an infinite period-doubling sequence, except those marked with a *, which arise from the previous cycle via period doubling. For the Coulomb oscillator, S indicates points which are sticking, and N indicates points which are not sticking. The symbol 0 represents a point which was so close to the boundary between N and S that it was not distinguishable. Each periodic cycle has one point which is very close to this boundary. We label such points with the symbol C. The listed symbol sequence of a cycle of period m consists of the m – 1 iterates of the point labeled C. Only cycles up to period seven are included in the table. Some of these cycles in the Coulomb oscillator were not found. Incidents of a period eight and a period ten were accidentally found and included.

<table>
<thead>
<tr>
<th>Osc.</th>
<th>Eqs. (1)-(3)</th>
<th>Log. Map $x_{n+1} = \lambda x(1 - x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>Symbol Seq.</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>N</td>
<td>1.36</td>
</tr>
<tr>
<td>4*</td>
<td>NSN</td>
<td>1.38</td>
</tr>
<tr>
<td>8*</td>
<td>NSNNNSN</td>
<td>1.3925</td>
</tr>
<tr>
<td>10</td>
<td>NSNNNSNSN</td>
<td>1.4064</td>
</tr>
<tr>
<td>6</td>
<td>NSNNN</td>
<td>1.415</td>
</tr>
<tr>
<td>7</td>
<td>NSNNNN</td>
<td>1.45415</td>
</tr>
<tr>
<td>5</td>
<td>NSNN</td>
<td>1.4737</td>
</tr>
<tr>
<td>7</td>
<td>NSNNNSN</td>
<td>1.4909</td>
</tr>
<tr>
<td>3</td>
<td>NS</td>
<td>1.535</td>
</tr>
<tr>
<td>6*</td>
<td>NS*NS</td>
<td>1.551</td>
</tr>
<tr>
<td>7</td>
<td>NSSN</td>
<td>1.5973</td>
</tr>
<tr>
<td>5</td>
<td>NSS</td>
<td>1.695</td>
</tr>
<tr>
<td>4</td>
<td>NSSS</td>
<td>1.833</td>
</tr>
<tr>
<td>7</td>
<td>RLLL</td>
<td>3.9689769</td>
</tr>
<tr>
<td>6</td>
<td>RLLL</td>
<td>3.9777664</td>
</tr>
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<td>7</td>
<td>RLLL</td>
<td>3.9847476</td>
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<td>5</td>
<td>RLLL</td>
<td>3.9902670</td>
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<td>RLLL</td>
<td>3.9945378</td>
</tr>
<tr>
<td>6</td>
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<td>3.9975831</td>
</tr>
<tr>
<td>7</td>
<td>RLLLL</td>
<td>3.9993971</td>
</tr>
</tbody>
</table>
sequence of events (but not vice versa), and the order of events in the oscillator
does not contradict the order in the universal sequence. Furthermore, the sym-
bol sequences associated with each observed periodic cycle in the Coulomb friction
oscillator is the same as in the corresponding periodic cycle in the standard one-
dimensional maps.

There are two possible explanations for the fact that some bifurcation events
were not detected. One possibility is that these events took place on a parameter
window smaller than the resolution at which we chose to search. The other possi-
bility is that the bifurcation sequence of the oscillator in fact may not match the
universal sequence of the standard maps. Such deviation could occur since the map
which arises from the Coulomb oscillator does not satisfy all of the assumptions re-
quired for universal behavior in the standard maps. Thus, it is possible that we are
observing a nonuniversal bifurcation sequence in the Coulomb oscillator. Coffman
et al.59 previously observed nonuniversal behaviour in nonstandard one-dimensional
maps. In that study, the nonuniversal events also occurred in the universal sequence,
but not according to the universal order.

In the investigation of metric universality, the parameter values in the initial
period-doubling sequence were measured to be compared with Feigenbaum's ratio.
We measured the parameter value \(a_2\) at the first period-doubling bifurcation, and
\(a_4, a_8,\) and \(a_{16}\) at the subsequent bifurcations. We then compared
\[
\frac{a_4 - a_2}{a_8 - a_4}
\]

and
\[
\frac{a_8 - a_4}{a_{16} - a_8}
\]
to Feigenbaum's ratio of \(4.669\ldots\). The estimates are \(r_1 = 4.70 \pm 0.25\) and \(r_2 =
4.67 \pm 0.65\). In the estimates, the parameter increment was smaller than that of the
rest of this investigation. The uncertainty is in the numerical integration. Within
the scope of the error, it would not be unusual for the ratio to converge nonuniformly.
Higher bifurcations involve smaller window sizes, leading to larger errors in the
calculations of the \(r_i\).

6.3. Summary

Symbol dynamics has been used to characterize the dynamics of the Coulomb
friction oscillator through binary autocorrelations, macroscopic Lyapunov expon-
teaus, and bifurcation sequences.

Variation of a parameter in the continuous-time oscillator does not necessarily
correspond to variation of a single parameter in the underlying map. Because
of this, and because of other considerations, such as the Schwartzian derivative,
the map may not fit the description of the "standard" maps. Nonetheless, the
bifurcation sequence of the oscillator has been compared to that of the standard
one-dimensional maps.
The observed periodic orbits, their period lengths and symbol sequences, fit into the universal sequence. However, several universal events remain undetected. This may be because the necessary assumptions are not all met, and that the oscillator does indeed exhibit nonuniversal behavior. However, consideration must be given to the size of the increment in the bifurcation parameter, $\Delta a$. If a periodic window is smaller than $\Delta a$, the window may not be observed. Therefore, a statement regarding the nonexistence of a periodic window cannot be made, since $\Delta a$ can always be made smaller.

7. Stick-Slip and the Reconstruction of Phase Space

When running experiments, it is not typically feasible to measure all of the active states in an experimental system. To compensate for this, there are methods for estimating the behavior of the full state space from a small number of measured observables. In experimental observation, these methods involve phase-space reconstructions. The reconstruction of the full state space can be combined with other analytical ideas, such as fractal dimensions and Lyapunov exponents, singular systems analysis, and false nearest neighbors, for the purposes of nonlinear prediction, modeling, or simply to estimate bounds on the size of the system. However, the methods for reconstructing phase space have been developed for smooth systems.

The phase space reconstruction, is usually done by the method of delays (described below). Stick-slip, however, can cause the method of delays to fail! For example, stick-slip can lead to a singularity when mapping the observed time history into a higher dimensional space. This is not surprising since stick-slip is a nonsmooth phenomenon, and the validity of the method of delays has been proven only for smooth phenomena. (Takens’ embedding theorem states that, if basic hypotheses are satisfied, the method of delays of an observable produces an trajectory in the reconstructed phase space which is diffeomorphic to the trajectory in the real phase space. A stick-slip system does not have the smoothness required for Takens’ embedding theorem.)

This failure can be visualized by imagining a discretized stick-slip history with sampled displacements $x_n$. Suppose, for example, we were to reconstruct the phase space in three dimensions. According to the method of delays, we build vectors $(x_n, x_{n+k}, x_{n+2k})$. However, during a sticking interval, it is possible that the points $x_n = x_{n+k} = x_{n+2k} = x_{\text{stick}}$, and that the two points $(x_n, x_{n+k}, x_{n+2k})$ and $(x_{n+1}, x_{n+1+k}, x_{n+1+2k})$, for example, might both be the same point as $(x_{\text{stick}}, x_{\text{stick}}, x_{\text{stick}})$. Thus, when we plot the reconstructed vectors, many of them pile up on the identity line. (If $k$ were large, however, $x_n$, $x_{n+k}$, and $x_{n+2k}$ could span a time interval greater than the sticking time, and this problem is avoided. However if $k$ is too large, the delayed coordinate may become statistically independent of the reference coordinate.)

In such case, the reconstructed phase space is fundamentally different than the
real phase space. The map which takes the real phase-space manifold \( M_r \) into the reconstructed manifold \( M_r \) is not invertible, and thus not a diffeomorphism. The dimensionality study fails. Furthermore, this kind of study is often performed on systems for which the model is unknown. If a nonsmooth event such as stick-slip occurs without notice, poor results may be unsuspiciously obtained. This translates to inaccurate models and poor characterizations and predictions. For example, correlation integrals did not produce the expected straight-line characteristic in log-log plots versus the box size for the system upon which this chapter focuses. Popp and Stelter have also observed this in the friction-induced chaos of a belt-driven disk\(^\text{17}\).

There are two problems to address here. First, we would like to detect when a reconstruction problem of this nature occurs in a "black-box" experiment. Next, when there is a reconstruction problem, we need to make adjustments to relieve it, \( \) so that the rest of the dimensionality study may continue.

An example of this problem is illustrated in a numerical study of equation (1) with equations (2) and (3). The result of using the observable \( x \) is used to reconstruct the phase space by the method of delays is shown in Figure 22. The region of phase space trajectories between curves \( AB \) and \( CD \) are sticking. All of these points are collapsed onto the line segment \( EF \) (which is in fact on the identity), in the reconstructed phase space, during the action of the reconstruction. Thus, much information is lost.

In this system, we can remedy the situation with a reconstruction based on \( \text{two} \) variables. If we choose \( x \) and \( t(\text{mod} 2\pi / \Omega) \) as the observables, and perform delays on \( x \), the time variable serves to unfold these trajectory points which are otherwise collapsing onto the identity line. Thus no collapse occurs. A computation of false nearest neighbors\(^\text{60}\) has a healthy characteristic for the latter case, but not the former (Figure 23).

We can use a near-neighbors method to identify the collapsing problem. In the above examples, we saw how a stick-slip observable can collapse if the delay index \( k \) is not too large. Thus, as we increase the delay \( k \), the number of incidences of collapse should decrease. A plot of the number of points whose nearest neighbors lie within some prescribed distance vs. \( k \) is shown in Figure 24. The expected result shows that the collapsing events occur for certain intervals of \( k \) centered around harmonics of the driving period of the system. This is indicated by the \( + \) symbols, where there are large numbers of near neighbors for intervals of \( k \). Adding \( t(\text{mod} 2\pi / \omega) \) as an additional observable removes this feature, as seen by the \( \circ \) symbols.

We are seeking other tools for identifying this type of event, and will apply them to systems in which the stick-slip is not so visually obvious. The goal is to reveal signs of the event when it might not be recognized visually. For example, singularities in reconstructions are likely to effect redundancy computations as well, since they quantify the "sharpness of probability distributions" in the delay-coordinate space\(^\text{63}\). Iterates of singular points in reconstruction spaces are nonunique. Thus, we expect redundancy analysis to diagnose singularity events in reconstructions.
Figure 22: An illustration of the collapse which effectively takes place during a phase space reconstruction. The sticking region between AB and CD collapses singularly into EF.
Figure 23: The number of false nearest neighbors vs. the time delay and the embedding dimension for reconstructions with (a) displacement as the only observable, and (b) displacement and time $t (\text{mod} \ 2\pi / \Omega)$ as the two observables. The plot in (a) does not converge since, as the embedding dimension increases, points are removed from the line of collapse, causing them to appear as false nearest neighbors. The plot in (b) has a healthy characteristic since the time observable unfolds the line of collapse in the reconstruction. It indicates that two or three delay coordinates are necessary to unfold the attractor.
Figure 24: The number of points whose nearest neighbors lie within a distance of $\epsilon = 5\epsilon - 6$ for different values of delay $\tau$. When $\tau$ is such that there is collapsing in the phase space reconstruction, we have many more near neighbors than otherwise. The + symbols represent single-observable reconstructions. The o symbols indicate the number of points with $\epsilon$-near neighbors for reconstructions from two observables, $x$ and $t(\text{mod} 2\pi/\omega)$. 
We have also observed that correlation sums do not scale uniformly with the size of the reference balls. Whether they produce a distinctive feature indicative of singularities in the reconstruction is also under investigation.

8. Conclusion

In this chapter, we have studied forced single-degree-of-freedom oscillations with stick-slip friction, mostly through a particular example. An important phenomenon of such oscillators is that stick-slip causes a collapse in phase-space. For a threedimensional system, this leads to one-dimensional map dynamics. This aspect of the dynamics has been observed experimentally, and numerically with a Coulomb friction model. The phenomenon occurs approximately for smooth friction laws.

The geometry of the flow was studied to show how the dimension collapses when phase flow interacts with sticking regions. With this geometric perspective, it can be seen how the flow is noninvertible. Attractors can be reached in finite time. Chaos can occur in a 2-D manifold. The dimensional collapse seems not to condemn the attractor to a lower-dimensional entity. Nonlinear piecewise-smooth flows can apparently stretch and fold a collapsed volume to an extent which reestablishes a higher-dimensional quality, resulting in a nonuniform topology for the attractor.

Since oscillations with stick-slip can be represented by discrete maps. One-dimensional maps have been studied extensively. Symbol dynamics provides a useful tool. Stick and Slip are natural ways to define a symbol sequence. We have used symbol sequences for this oscillator to compute binary autocorrelations and estimate the order of magnitude of the maximum Lyapunov exponents. We have also examined the bifurcation sequence in terms of symbol dynamics.

Finally, we discussed problems that stick-slip can pose for reconstructing phase space from a sampled observable. A remedy has been presented for the oscillator of focus. The long-range goal is to be able to identify a reconstruction problem in a “black box” nonsmooth system, and also to extend the current reconstruction techniques to accommodate the nonsmoothness.

Stick-slip oscillators represent a class of mechanical systems. The ideas presented in this chapter should shed light on understanding the nonlinear dynamics of such systems. Hopefully, such understanding will facilitate research and development in areas including robotics, automotive squeak, rail-wheel dynamics, micromachines, and earthquake engineering.

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10. References
