Characterizing Wave Behavior in a Beam Experiment by Using Complex Orthogonal Decomposition

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Complex orthogonal decomposition (COD) is applied to an experimental beam to extract the dispersive wave properties using response measurements. The beam is made of steel and is rectangular with a constant cross section. One end of the beam is free and is hung by a soft elastic cord. An impulse is applied to the free end. The other end is buried in sand to absorb the wave as it travels from the impact site on the free-end; this effectively prevents reflections of the wave off the buried end and emulates a semi-infinite beam. The beam response is measured with an array of accelerometers, whose signals are integrated to obtain an ensemble of displacement signals. Acceleration responses are also compared in the frequency domain to predictions from the Euler-Bernoulli model. COD is applied to the displacement ensemble to obtain complex modal vectors and associated complex modal coordinates. The spatial whirl rates of nearly harmonic modal vectors are used to extract the modal wave numbers, and the temporal whirl rates of the modal coordinates are used to estimate the modal frequencies. The dispersion relationship between the frequencies and wave numbers compare favorably to those of the theoretical infinite Euler-Bernoulli beam.

1 Introduction  
Complex orthogonal decomposition (COD) [1] is in the family of output only modal decomposition methods related to proper orthogonal
decomposition (POD). POD was first used in mechanics by Lumley to study turbulence in fluid dynamics [2]. Berkooz et al. [3] reviewed the properties of POD for turbulence. POD has been applied to structural dynamics [4–6, 30], and specifically to extract modal parameters [7–10]. POD is equivalent to singular value decomposition [9, 11–13], and has several different variations for the extraction of different parameters of interest. These variations include mass-weighted reduced-order proper orthogonal decomposition (MWPOD) [14, 15], Ibrahim time-domain decomposition [16], smooth orthogonal decomposition (SOD) [15, 17, 18], state-variable modal decomposition (SVMD) [15, 19] and, the topic of this paper, COD. To see the connection between POD and COD, next we will explain POD followed by COD.

First to perform POD, the analyst captures measurement signals (for example displacements) at different points simultaneously on the structure during its response to some type of excitation, for example an impulse [8]. Next an ensemble matrix, \( X \in \mathbb{R}^{M \times N} \), is created from the sensor measurements, where \( M \) is the number of sensors and \( N \) is the number of time samples taken. Specifically, \( X = [x_1^T x_2^T \cdots x_M^T]^T \), where \( x_i = [x_i(0) \ x_i(\Delta T) \ x_i(2\Delta T) \ \cdots \ x_i((N-1)\Delta T)]^T \), such that each row of the ensemble matrix is the sampled data from one sensor and each column is a single time sample. Often the mean is subtracted from each sensor signal. Next a correlation matrix, \( R = \frac{XX^T}{N} \), is computed, such that \( R \in \mathbb{R}^{M \times M} \). Finally, an eigenvalue problem is formulated as \( Rv = \lambda v \). The eigenvectors are called proper orthogonal modes (POMs). For lightly damped linear vibration systems with a mass matrix proportional to the identity these converge to linear normal modes [7, 9]. The eigenvalues are called proper orthogonal values (POVs) and relate to modal energy [3, 10]. MWPOD, SOD, SVMD, and COD all involve variations on this algorithm. Each method extracts different modal parameters.

Complex orthogonal decomposition uses simultaneous measurements much like POD. However we must extend these signals into complex analytic form. Such signals do not have negative frequency content, and can be created by manipulation in the frequency domain or by using the Hilbert transform [1, 20]. There are several parameter estimation algorithms that take place in the complex domain, which have been used in the field of electromagnetics, and use response ensembles and eigenvalue problem to estimate parameters, including MUSIC [21] and ESPRIT [22]. The details of the COD algorithm are presented in the next section. Thus far COD has been applied to nematode posturing [23], whiting fish locomotion [24], and discerning traveling and standing modal waves [1]. COD has also been applied to extract the dispersion relation from a simulated beam [25].

In a follow up to the simulated beam study, this paper reports on the application of COD to an experimental beam to extract the geometric dispersion relationship. Examples of other studies in extracting wave parameters are [30–34]. In the following, Section 2 gives a detailed overview of COD. Section 3 provides a brief background of waves in an Euler-Bernoulli beam. Section 4 describes the experiment and data processing. Section 5 shows the results of COD applied to a propagating wave in a beam, with interpretations.

## 2 Complex Orthogonal Decomposition

Complex orthogonal decomposition is an extension of POD with a very similar computation, with the notable exception that COD requires an analytic signal. First a structure is instrumented, for example with accelerometers distributed evenly on the structure. Next the structure is excited, in this experiment with an impact hammer, and the sensors are sampled simultaneously. Once the measurement data is acquired the data is arranged in a measurement ensemble \( X \) as described in section 1, and then the analytic form of \( X \) is computed to get \( Z \).

To compute \( Z \), the analytic form of \( X \), first we take the fast Fourier transform (FFT) of each row of \( X \) from the time domain to frequency domain to
get \( \tilde{X} \). The FFT ensemble in the discrete frequency domain can be defined to roughly cover the spectrum from approximately \(-\omega_{ny}\) to \(\omega_{ny}\), where \(\omega_{ny}\) is the Nyquist frequency. Each row of \( \tilde{X} \) is the FFT of a sensor signal and each column is a frequency sample, and the elements of \( \tilde{X} \) are \( \tilde{X}_{ij} \) with \( i = 1, \ldots, M \) and \( j = -(N - 1)/2, \ldots, N/2 \) (for example in the case that \( N \) is even). Then the negative spectrum is nullified, and the positive spectrum is doubled, such that the elements of \( \tilde{Z}_{ij} \) of \( \tilde{Z} \) are

\[
\tilde{Z}_{ij} = \begin{cases} 0 & \text{if } j < 0 \\ 2\tilde{X}_{ij} & \text{if } j \geq 0 \end{cases}.
\]

(1)

The complex analytic ensemble is obtained, using the inverse FFT (IFFT), as \( Z = \text{IFFT}(\tilde{Z}) \).

Now that the measurement ensemble data has been converted into analytic signals, the correlation matrix \( R \) is computed as \( R = \frac{Z Z^H}{N} \), where superscript \( H \) denotes the Hermitian operation (conjugate transpose). Once the correlation matrix is computed the eigenvalue problem is formulated such that \( R\mathbf{v} = \lambda \mathbf{v} \). \( \mathbf{v} \) is a complex orthogonal mode (COM) and \( \lambda \) is the corresponding complex orthogonal value. Indeed, the correlation matrix can also be formed in the frequency domain as \( R = Z Z^H / N \), to produce the same COMs [25].

The complex modal coordinate (COC) ensemble, \( Q \), is computed such that \( Q = V^{-1}Z \), where each column of \( V \) is an eigenvector \( \mathbf{v} \) and \( V^{-1} = V^H \), if normalized, due to orthogonality of the Hermitian eigenvalue problem. Each row \( \mathbf{q}_i^T \) of \( Q \) is a sampled COC, such that \( Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_M]^T \), \( Q \in \mathbb{C}^{M \times N} \). For nearly harmonic COCs, the rate of change of the phase with respect to the time sample is equal to the frequency, \( \omega \), such that the instantaneous frequency of the \( i^{th} \) modal coordinate is \( \omega_i = \frac{d}{dt} \{\angle \mathbf{q}_i\} \), where, in the sampled case, the time derivative is applied numerically. Similarly, the COMs are complex and the rate of change of the phase of nearly harmonic COMs with respect to the spatial position, \( x \), defines a local wave number of the \( i^{th} \) mode as \( k_i = \frac{d}{dx} \{\angle \mathbf{v}_i\} \). Using the COMs and \( Q \), the mean wave number, \( k_0 \), and frequency, \( \omega_0 \), can be extracted for each mode \( i \).

Before applying COD to the beam experiment we review the Euler-Bernoulli beam model.

### 3 Background: Euler-Bernoulli Beam

Applying a traveling wave solution to the Euler-Bernoulli beam equation provides a theoretical framework for the experimental results. The equation of a uniform Euler-Bernoulli beam is

\[
\rho A \ddot{y}(x,t) + EI \frac{d^4}{dx^4}[y(x,t)] = 0,
\]

(2)

where \( x \) is the position along the beam, \( t \) is time, \( y(x,t) \) is the transverse bending displacement, \( \rho \) is the mass density, \( A \) is the area of beam’s cross section, \( E \) is the modulus of elasticity, and \( I \) is the area moment of inertia of the beam’s cross section about the neutral axis. An infinite beam has boundary conditions of bounded displacement and slope.

Following Graff [26], inserting a traveling wave solution in complex harmonic form,

\[
y(x,t) = e^{i(\kappa x - \omega t)},
\]

(3)

into Eq. (2), and solving for \( \omega \), leads to the geometric dispersion relationship \( \omega = ak^2 \), where

\[
a = \sqrt{\frac{EI}{\rho A}}.
\]

(4)

The phase velocity is then defined as \( c = \omega/k = ak \). Thus, for the Euler-Bernoulli beam, the speed of an individually propagating harmonic wave increases linearly with its wavenumber. The group velocity is \( c_g = d\omega/dk = 2ak \) for the beam.

In the previous study [25], the COD was applied to a simulated response to an initial Gaussian displacement distribution starting at rest [26]. The
extracted traveling wave modes could be related to a form that resembled Eq. (3), from which $\omega$ and $k$ were extracted as discussed above, and were shown to be consistent with the dispersion relation. This analysis was robust to added sensor noise.

In this work, we apply this analysis to an experiment that emulates a semi-infinite beam with an impulse excitation at the end. Transverse vibrations of a semi-infinite Euler-Bernoulli beam with an end impact have been addressed by Graff [26] by means of Laplace and Fourier transforms, resulting in an analytical expression of the displacement as a function of time. Büßow [29] also treated the problem with slightly different boundary conditions to obtain the displacement in the frequency domain. In the Appendix, we include an analysis to express the displacement and acceleration responses as functions of space and frequency, which can be used when evaluating the response data. We keep in mind that our beam in the sand pit was set up with a focus on extracting the beam’s dispersion behavior. Although the constraint of the sand on the beam deflection at large $x$ could cause some deviation between the responses of the ideal theory of [26], [29], and the Appendix, from the experimental beam, even in the unconstrained regions of the beam, we would expect some consistency in some aspects of the response behavior.

4 Experiment
4.1 Setup

The test specimen was a rectangular steel beam with a constant cross section. In effort to emulate a semi-infinite free beam, the beam was suspended with elastic cords, such that one end was free, and the other was embedded in a sand pit, as done by Önsay and Haddow [27]. The sand absorbs the wave and prevents reflections off the buried end of the beam. A schematic of the experiment is shown in Fig. 1. In this case, the sandbox was filled with unpacked coarse sand. The beam had a rectangular cross section with a width of 0.0698 m and a thickness of 0.0045 m. The length of the beam was 2.04 m. The unburied part of the beam measured 1.43 m, such that approximately 0.609 m was buried in the sand. The density and modulus of elasticity of the beam were 7870 kg/m$^3$ and 200 GPa, respectively, based on published values for steel. From the geometry and material properties the theoretical value from Eq. (4) is $a = 6.548$ m$^2$/s.

The beam was sensed with 31 accelerometers placed at a distance $\Delta x = 0.0458$ m apart over a distance of $L = 1.4198$ m and was sampled at $f = 25,000$ Hz using a National Instrument’s PXI data acquisition system. The beam was struck with a PCB model 086C80 mini impact hammer, lightly such that bending deflections were not visible to the naked eye.

4.2 Data Processing

The data included 100 samples before the hammer impact and 300 samples after the impact for a total of 400 samples. We aim to integrate the acceleration data to obtain velocity and displacement. Numerical integration can be problematic because an integration constant is introduced, and low-frequency noise is amplified and can cause the integrated signal to drift. To reduce these effects the following steps were taken. First the data was filtered forward and backward with a high-pass filter with a cutoff frequency of 100 Hz. Second, the mean of each sensor’s time history was subtracted from its samples using Matlab’s "detrend" with a "constant" modifier. Third, any linear trends were removed using the same command as above with a "linear" modifier. Fourth, the signals were translated on the time axis such that the sample before the hammer impact had a time and force value of zero. The first and last 100 samples were truncated leaving the start of the impact plus 200 samples.
Next, to get velocities, the signals were numerically integrated using the "cumtrapz" command. The means were subtracted from the velocities, which were then high pass filtered, and integrated once more to get displacements. The means were subtracted from displacements and the displacements were filtered for a final time. These $N = 200$ samples of displacements were then used for COD.

The minimum detectable wavenumber defined by the span of sensors, $L$, is $k_{\text{min}} = \frac{2\pi}{L} = 4.4$ rad/m. The maximum detectable wavenumber is defined by a spatial Nyquist criterion as $k_{\text{max}} = \frac{\pi}{\Delta x} = 68.6$ rad/m. Based on the sampling rate $f = 25,000$ Hz and time record of $N/f = 0.008$ s, the maximum detectable frequency is $f_{\text{max}} = 12,500$ Hz and the minimum detectable frequency is $f_{\text{min}} = 125$ Hz.

Making use of the theoretical dispersion relation $\omega = ak^2$, or $k = \sqrt{2\pi f/a}$, we find that the temporal sampling parameters correspond to theoretical wavenumber limits of $k_{\text{max}} = 109$ rad/m and $k_{\text{min}} = 10.9$ rad/m. The approximate total wavenumber limits are thus $k_{\text{max}} = \min(k_{\text{max}}, k_{\text{max}}) = 68.6$ rad/m, and $k_{\text{min}} = \max(k_{\text{min}}, k_{\text{min}}) = 10.9$ rad/m. Thus, the upper limit on extractable wavenumbers (and hence frequencies) is determined by the spatial sampling interval Nyquist criterion, and the lower limit on extractable wavenumbers is determined by the length of the time record.

5 Results and Discussions

The acceleration histories in Fig. 2 show the ensemble signal energy decay as the response propagates into the sand pit, presumably with a small damping effect in the exposed beam as well. Fig. 3 shows the acceleration history of sensors 1, 16, and 31. The plot shows that the wave hits sensor 1 first with a profile that resembles a pulse. The dispersive wave hits sensor 16 next, with high frequencies arriving ahead of the low frequencies, as higher frequencies propagate faster in dispersive beams. The frequencies spread out more by the time the response reaches sensor 31. The residual high frequency wiggle is due to the pulse reflecting back and forth through the width of the beam. The frequency of this ringing matched the first mode of vibrations across the width of the beam, when viewed as a plate. These trends were also observed by Önsay and Haddow [27].

Various hammer tips were used in the exper-
The accelerations were filtered and integrated twice, according to the process of section 4.2, to produce the displacements in Fig. 4, where sensors 1, 16, and 31, only, are shown for clarity. In reality, the displacement should start at zero until the wave reaches the sensor location. However, despite the high-pass filtering and removal of means and linear trends together, the signals were left with some low-frequency distortion. This distortion, however, is outside of the frequency range used for the calculation of the dispersion frequencies and wavenumbers, and therefore does not affect the modal dynamics nor the characterization of dispersion within this range of calculation. Nonetheless, we can see some key features in Fig. 3 carrying over to Fig. 4, showing the high frequency components of wave displacement arriving in sensors 16 and 31 ahead of lower-frequency components, consistent with beam dispersion.

The FFTs of accelerometers signals for sensors 1, 16, and 31 are shown as the black (dark) solid
The acceleration response is likewise \( \hat{Y}(x, \omega) \) for the ideal, and the associated input \( \hat{F}(\omega) \) can be considered as a unit impulse response function in the frequency domain, that is, a frequency response function between the displacement \( y(x, t) \) and the end input, such that for the case of an ideal impulse at the endpoint, where \( \hat{F}(\omega) = \hat{F} \), then \( \hat{Y}(x, \omega) \) results in the Eqn. (6).

However, Figure 6 shows that the impact is not ideal, and the associated input \( \hat{F}_{exp}(\omega) \) is confined to a finite bandwidth. The response to this nonideal impact is \( \hat{Y}_{exp}(x, \omega) = \hat{H}(x, \omega) \hat{F}_{exp}(\omega) \). The acceleration response is likewise \( \hat{A}_{exp}(x, \omega) = \hat{A}(x, \omega) \hat{F}_{exp}(\omega) / \hat{F} \), where \( \hat{A}(x, \omega) \) due to an ideal impact is given as Eqn. (6) in the Appendix.

We have evaluated \( |\hat{A}(x, \omega)| \) at the sensor locations \( x_i \), and at the frequencies of the FFT, and then obtained \( |\hat{A}(x_i, \omega)| = |\hat{A}(x_i, \omega)| |\hat{F}(\omega)| / \hat{F} \) with a normalized input signal ("normalized" to optimize the accuracy for sensor 15). The results are the solid lines in the plots of Figures 5a - 5c. Thus, relative to the normalized input, the results are in good qualitative agreement. The magnitude of the plot for sensor 1 is very sensitive, as the theory predicts a very steep change in response amplitude as \( x \) gets small. These plots show that, although the beam is embedded in a sand pit down stream of the wave, the exposed region of the beam behaves, in the bandwidth of our excitation, similarly as a if it were a semi-infinite beam. It also shows that the analysis of the Euler-Bernoulli beam under an end impact provides useful predictions of behavior in the frequency and space domains, particularly if the input is quantified.

COD was applied, and some of the COMs (Fig. 7) and COCs (Fig. 8) are shown for illustration purposes. The COMs depicted in Fig. 7 show the real and imaginary parts of selected extracted complex modes, normalized such that each modal vector has unit amplitude. The plots are parameterized in the sensor location index. Thus, there are 31 real and imaginary ordered pairs plotted and connected with straight lines. A perfect whirl would indicate a purely spatially harmonic wave mode whose real and imaginary parts are 90 degrees out of phase. Thus, the spatially whirling extracted modes resemble harmonic wave modes, and the spatial whirling rate can then be used to estimate the complex modal wave number, as long as the spatial sampling interval is small enough to accurately estimate a wavenumber.

The COCs plotted in Fig. 8 show the real and imaginary parts of corresponding complex modal coordinates. The normalization of the complex modes, and the COC modal energy (the COVs), define the amplitude of the complex modal coordinates. These plots are parameterized in the time index. Thus, there are 200 points plotted in each graph, connected with lines, such that the graphs appear rather smooth for the lower frequency modes. In this time parameterization, the typical trend is that the modal coordinate begins as a nearly harmonic oscillation, and then decays as the modal component travels off of the measurement zone. Examples of the real parts as functions of time are shown in Figs. 9 and 10. Since the higher frequencies travel faster through the beam, the higher frequency modal coordinates have shorter durations of
nearly harmonic oscillation. This trend was also observed in decompositions of numerically simulated waves [25]. In the experiment, the modes are not pure harmonics, and there is noise and modeling error that contribute to some modal pollution apparent as a lower amplitude, lower frequency oscillation after the strong harmonic has left the measurement zone. The interval of the strong nearly harmonic oscillation is then used to extract the modal coordinate temporal whirl rate, which represents the frequency of the mode. The mechanism for how experimental error leads to modal pollution was analyzed previously [23].

The extracted geometric dispersion relationship is shown in Fig. 11. In this figure, values of $\omega$ and $k$ are plotted in the ranges for which the mode shapes and modal coordinates exhibited intervals of well defined whirls.

In Fig. 11 it can be seen that the experimental data show good agreement with theory. From theory we have $\omega = ak^2$ where $a = 6.548 \text{ m}^2/\text{s}$. A least squares fit of the the geometric dispersion relation using COD extracted $k$ and $\omega$ leads to a value of $a_{fit} = 6.4431 \text{ m}^2/\text{s}$, which gives an underestimation of 1.61%. Based on regression error analysis [28], the extracted valued of $6.44 \text{ m}^2/\text{s}$ leads to a mean squared error of $6.1667 \times 10^4 \text{ rad/s}$, and its 95% confidence
interval is \( a = [6.2638, 6.26224] \text{ m}^2/\text{s} \). When the data processing laid out in section 4.2 was done without the high-pass filtering, i.e. where the data just had the means subtracted followed by integration repeated as needed to get displacements, the least squares estimate of \( a \) was \( a_{\text{fit}} = 6.8 \text{ m}^2/\text{s} \), which was an overestimation of 3.8%. The phase velocity is shown in Fig. 12, comparing the equation \( c = ak \) with \( c = \omega/k \) computed from COD-extracted \( \omega \) and \( k \). The solid line is theory and the circles are COD extracted data points. Similarly, the group velocity is shown in Fig. 13, from values that were computed using forward differences on the COD extracted \( \omega \) and \( k \) from Fig 11.

Several things are worth noting at this point. We find that the COVs increase with decreasing modal wavenumber (decreasing frequency). This may be sensible if we consider that the impulse excitation is slightly stronger for lower frequencies (see Fig 6). Furthermore, the higher frequencies propagate faster according to the beam theory, such that lower frequency modes are active in the measurement zone for longer time intervals. The excitation bandwidth and modal activity duration contribute to higher mean squared amplitudes (COVs) for lower modes.

We also see that the extracted data in Figure 11 falls within the theoretical limits of \( k_{\min}, k_{\max}, \) and \( \omega_{\max} \) determined by sampling parameters. Spatial resolution has a great effect: approximately the lowest 1/3 of the COMs (those with highest COVs and lower wavenumbers) have good whirl properties with high enough spatial sampling resolution to allow for good \( k \) extraction. The second COM shown in Fig. 7 had the best circular whirling. As the COVs decrease the COMs become less circular in the complex plane. This is partly because the spatial resolution becomes coarse as the modal wave number increases. However the COCs are effected to a lesser degree by resolution limitations than the COMs, which makes sense because the spatial sampling distance and time record were found to be the parameters that limited the accessible frequency range, and the temporal sampling was thus abundantly fast, such that the accessible modal coordinates were smoothly sampled.

5.1 Using COCs to Extract Modal Amplitudes

It may be of interest to determine the amplitude of the wave traveling through the beam as a function of frequency. To achieve this the fast Fourier transform (FFT) of the the COCs were computed and the maximum magnitude and its frequency was recorded to derive COD extracted \( |\tilde{A}_{\text{cod}}(x, \omega)| \) shown as the red circles in Fig. 14. In order to compare this with theory, \( \tilde{A}(x, \omega) \) was computed for each sensor location \( i \) and then each \( \tilde{A}(x_i, \omega) \) was multiplied by the FFT of the modal impact hammer signal shown in Figure 6 to get \( \tilde{A}(x_i, \omega) \) where \( i = 1, \cdots, 31 \). A composite was created such that \( U(x, \omega) = \frac{\sum \tilde{A}(x_i, \omega)}{M} \) was computed to have one
Fig. 14: COD extracted modal amplitude vs frequency (○) compared to theory

composite function called $U(x, \omega)$ and is shown as the solid line in Fig. 14. $U(x, \omega)$ is simply the average of the scaled theoretical acceleration for each sensor. The plot of the $A_{\text{cod}}(x, \omega)$ (circles) and $U(x, \omega)$ (line) is shown in Figure 14 and show great agreement.

6 Conclusion
Experiments were performed on a thin beam suspended in the sand to emulate a semi-infinite beam. The beam was instrumented with accelerometers, excited with an impulse, and the measured responses were integrated into displacement signals and then analyzed using COD to extract the underlying complex modes. True to beam theory, higher frequency wave components traveled faster, and thus remained active for shorter segments of the time record, than lower frequency components. The measured acceleration responses also agreed qualitatively with theoretical responses to the measured input pulse.

The lower-frequency extracted complex modes resembled harmonic complex waveforms. The associated modal coordinates were dominantly harmonic during a time interval dictated by the wave speed of the corresponding component of the wave form, and the length of the measurement zone on the beam. The nearly harmonic nature of the decomposed mode shapes and modal coordinates contained information on modal wavenumber and frequencies, which could be estimated.

With this approach, we extracted the dispersion characteristics over a frequency and wavenumber interval, as well as the amplitude of the waves traveling through the beam. The results were consistent with Euler-Bernoulli beam theory, and with a previous analysis of simulated response data.

The work shows that COD is a fast and simple tool that can be used to extract the geometric dispersion relationship between the frequency, phase velocity, or group velocity, the wave number, and wave amplitudes for waves traveling in a semi-infinite uniform structure.

Appendix
The semi-infinite Euler-Bernoulli beam excited with a impulse at $x = 0$ described by $F(0, t) = \hat{F} \delta(t)$ is solved using the Fourier transform. The partial differential equation (PDE) and boundary conditions (BCs) are listed below:

\[
pA \ddot{y}(x, t) + EI \frac{\partial^4}{\partial x^4} [y(x, t)] = 0 \\
\dot{y}(0, t) = 0 \\
\ddot{y}(0, t) = \frac{\hat{F} \delta(t)}{EI}
\]

with two additional conditions of bounded displacements and waves that are only allowed to travel in the direction of the positive $x$-axis.

Taking the Fourier transform of the PDE and BCs yields

\[
-\omega^2 \hat{Y}(x, \omega) + \alpha^4 \hat{Y}'''' = 0 \\
\hat{Y}''''(0, \omega) = 0 \\
\hat{Y}(0, \omega) = \frac{\hat{F}}{EI}
\]

where $\alpha = \frac{EI}{pA}$, while remembering the bounded displacements and one-directional wave condition.
Now we have an ordinary differential equation of the form

\[ \dddot{Y} - \beta \ddot{Y} = 0 \]

where \( \beta = \frac{\omega^2}{\alpha^4} \). Substituting in the standard trial solution \( \ddot{Y} = e^{rx} \) and solving for \( r \) yields

\[ r = \pm \frac{\omega^{1/2}}{\alpha}, \pm i\frac{\omega^{1/2}}{\alpha}, \]

which leads to

\[ \ddot{Y}(x, \omega) = A_1 e^{\frac{\sqrt{\omega}}{\alpha} x} + A_2 e^{-\frac{\sqrt{\omega}}{\alpha} x} + A_3 e^{i\frac{\sqrt{\omega}}{\alpha} x} + A_4 e^{-i\frac{\sqrt{\omega}}{\alpha} x}. \]

Since the displacements are bounded, \( A_1 = 0 \), and since waves can only travel in the positive \( x \) direction, \( A_3 = 0 \), which can be seen by considering that the inverse Fourier transform combines these terms with \( e^{ix} \). Applying the other two boundary conditions, \( A_2 = A_4 \) are determined, resulting in

\[ \ddot{Y}(x, \omega) = \frac{\hat{F}}{\sqrt{EI(\rho \lambda^3/4)\omega^{3/2}(i-1)}} (e^{-\frac{\sqrt{\omega}}{\alpha} x} + e^{-i\frac{\sqrt{\omega}}{\alpha} x}). \]

This provides the frequency domain description of the displacement response. A loosely described derivation can be found in [29].

The acceleration response in the frequency domain is thus

\[ \dddot{A}(x, \omega) = \frac{-\omega^2 \hat{F}}{\sqrt{EI(\rho \lambda^3/4)\omega^{3/2}(i-1)}} (e^{-\frac{\sqrt{\omega}}{\alpha} x} + e^{-i\frac{\sqrt{\omega}}{\alpha} x}). \]

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