ECE 802-606 Homework 5 Solutions

1. Prove Moyal’s formula.

\[ | \int s_1(t)s_2^*(t)dt |^2 = 2 \pi \int \int W_1(t, \omega)W_2(t, \omega)dt \omega \] (1)

Expanding the left hand side of the equality gives:

\[ | \int s_1(t)s_2^*(t)dt |^2 = \int \int s_1(v)s_1^*(u)s_2^*(v)s_2(u)dvdu \] (2)

Let’s make the following change of variables, \( t = \frac{v+u}{2}, \tau = v - u \), then \( v = t + \tau/2, u = t - \tau/2 \). The Jacobian determinant is

\[ J = \begin{bmatrix} \frac{\partial v}{\partial t} & \frac{\partial v}{\partial \tau} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial \tau} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1 & -1/2 \end{bmatrix} = -1 \] (3)

Therefore, \( dvdu = |J|dtd\tau = dt \omega \). The integral becomes

\[ | \int s_1(t)s_2^*(t)dt |^2 = \int \int \int W_1(t, \omega)e^{j\omega t}W_2^*(t, \omega')e^{-j\omega' \tau}dtd\omega d\omega' \]

\[ = 2\pi \int \int \int W_1(t, \omega)W_2^*(t, \omega')\delta(\omega - \omega')dtd\omega d\omega' \]

\[ = 2\pi \int \int \int W_1(t, \omega)W_2^*(t, \omega)dtd\omega \] (4)

where the last equality follows from the fact that Wigner distribution is real.

2. \( s(t) = A_1e^{j\omega_1t} + A_2e^{j\omega_2t} \),

a) Find the instantaneous frequency

First we need to compute Wigner distribution of the signal. From results developed in Cohen

\[ W(t, \omega) = A_1^2 \delta(\omega - \omega_1) + A_2^2 \delta(\omega - \omega_2) + 2A_1A_2\delta(\omega - \frac{\omega_1 + \omega_2}{2})cos((\omega_2 - \omega_1)t) \] (5)

The instantaneous frequency is defined as:

\[ < \omega >_t = \frac{\int \omega W(t, \omega)d\omega}{|s(t)|^2} \] (6)

where \( |s(t)|^2 = A_1^2 + A_2^2 + 2A_1A_2cos((\omega_2 - \omega_1)t) \). Therefore, the instantaneous frequency is:

\[ < \omega >_t = \frac{A_1^2\omega_1 + A_2^2\omega_2 + 2A_1A_2cos((\omega_2 - \omega_1)t) \frac{\omega_1 + \omega_2}{2}}{A_1^2 + A_2^2 + 2A_1A_2cos((\omega_2 - \omega_1)t)} \] (7)
b) We want to find the condition on $A_1, A_2, \omega_1, \omega_2$ such that the instantaneous frequency lies in between the frequencies of the complex exponentials.

$$\frac{\omega_1 + \omega_2}{2} = \frac{A_1^2\omega_1 + A_2^2\omega_2 + 2A_1A_2\cos((\omega_2 - \omega_1)t)\frac{\omega_1 + \omega_2}{2}}{A_1^2 + A_2^2 + 2A_1A_2\cos((\omega_2 - \omega_1)t)}$$

(8)

Simplifying the expression gives:

$$A_1^2\omega_1/2 + A_2^2\omega_2/2 + A_1^2\omega_1/2 + A_2^2\omega_2 = A_1^2\omega_1 + A_2^2\omega_2$$

(9)

This gives the condition that the instantaneous frequency will be in between the frequencies of the complex exponentials given that $A_1 \neq A_2$ and $\omega_1 = \omega_2$.

3. Prove that the spectrogram can be written as convolution of Wigner distribution of the signal with Wigner distribution of the window.

$$P_{SP}(t, \omega) = \int \int W_s(t', \omega')W_h(t' - t, \omega - \omega')dt'd\omega'$$

(10)

To prove this equality, we’ll use Moyal’s formula proved in the previous question.

$$P_{SP}(t, \omega) = \frac{1}{2\pi} \left| \int s(u)h(u - t)e^{-j\omega u}du \right|^2$$

(11)

Using the formulation from the previous question, let $s_1(u) = s(u)$, and $s_2^*(u) = h(u - t)e^{-j\omega u}$. Therefore, using Moyal’s formula, the spectrogram can be written as

$$P_{SP}(t, \omega) = \int \int W_{s_1}(u, \omega')W_{s_2}(u, \omega')dud\omega'$$

(12)

where $W_{s_1}(u, \omega') = W_s(u, \omega')$ since $s_1(u) = s(u)$. $W_{s_2}(u, \omega')$ is the Wigner distribution of $h^*(u - t)e^{j\omega u}$ and can be expressed in terms of Wigner distribution of the window.

$$W_{s_2}(u, \omega') = \int h(u - t - \tau/2)h^*(u - t + \tau/2)e^{j\omega(u + \tau/2)}e^{-j\omega(u - \tau/2)}e^{-j\omega'\tau}d\tau$$

$$= \int h(u - t - \tau/2)h^*(u - t + \tau/2)e^{j(\omega - \omega')\tau}d\tau$$

$$= W_h^*(u - t, \omega - \omega')$$

$$= W_h(u - t, \omega - \omega')$$

(13)

Therefore,

$$P_{SP}(t, \omega) = \int \int W_s(u, \omega')W_h(u - t, \omega - \omega')dud\omega'$$

(14)

4. The linear chirp signal is $e^{j\omega_0t + t\beta^2/2}$.

The frequency marginal is computed for the spectrogram with a length 64 window. Note that the instantaneous energy for this signal will be equal to 1 for all time since we used an analytic signal. Similarly, the time marginal is a constant for all time, and that constant is determined by the window’s energy.
5. Pseudo Wigner distribution is given as $W_{PS}(t, \omega) = \frac{1}{2\pi} \int h(\tau)s(t+\tau/2)s^*(t-\tau/2)e^{-j\omega\tau}d\tau$
a) For pseudo Wigner distribution to satisfy the time marginal, the following equality should hold

$$\frac{1}{2\pi} \int \int h(\tau)s(t + \tau/2)s^*(t - \tau/2)e^{-j\omega\tau}d\tau d\omega = |s(t)|^2$$

$$\int h(\tau)s(t + \tau/2)s^*(t - \tau/2)\delta(\tau)d\tau = |s(t)|^2$$

$$h(0)|s(t)|^2 = |s(t)|^2$$  \hspace{1cm} (15)

Therefore, pseudo Wigner distribution will satisfy the time marginal when $h(0) = 1$.

b) Similarly, we can get a condition on the window to get the correct frequency marginal. Pseudo Wigner distribution can be written in the frequency domain as $\int h(\tau)s(t+\tau/2)s^*(t-\tau/2)e^{-j\omega\tau}d\tau = \int H(\omega-\theta)W_s(t, \theta)d\theta$. For the frequency marginal
to be satisfied:

\[
\int \int H(\omega - \theta) W_s(t, \theta) d\theta dt = \int H(\omega - \theta) W_s(t, \theta) dt d\theta = \int H(\omega - \theta) |S(\theta)|^2 d\theta
\] (16)

The last equality will equal to $|S(\omega)|^2$ if and only if $H(\omega - \theta) = \delta(\omega - \theta)$. Therefore, $H(\omega) = \delta(\omega)$. In the frequency domain, the window is a delta function, which corresponds to $h(t) = 1 \ \forall t$.

c) The only window that will satisfy both of these constraints is $h(t) = 1 \ \forall t$. This is equivalent to not windowing the local autocorrelation function and thus yields the Wigner distribution. Therefore, there is no window selection for pseudo Wigner distribution that will give the correct marginals.