ECE 802-606 Homework 1
due September 11, 2003
Reading: Chapters 1 and 2 from Burrs et al.

1. Consider the interval $[-1, 1]$ and the basis functions \{1, t, t^2, t^3, \ldots\} for $L_2[-1, 1]$. The inner product on this vector space is defined as, $<f, g> = \int_{-1}^{1} f(t)g^*(t)dt$. You can show that these functions do not form an orthonormal basis. Given a finite or countably infinite set of linearly independent vectors $x_i$, we can construct an orthonormal set $y_i$ with the same span as $x_i$ as follows:

- Start with $y_1 = \frac{x_1}{\|x_1\|}$.
- Then, recursively set $y_2 = x_2 - \frac{<x_2, y_1>}{\|x_2 - <x_2, y_1>\|} y_1$.
- Therefore, $y_k = \frac{x_k-v_k}{\|x_k-v_k\|}$, where $v_k = \sum_{i=1}^{k-1} <x_k, y_i> y_i$.

This procedure is known as the Gram-Schmidt orthogonalization and it can be used to obtain an orthonormal basis from any other basis.

a) Write a MATLAB function that will implement the Gram-Schmidt algorithm. Your input will be a set of linearly independent functions, and your output will be a set of orthonormal functions that span the same space. Include a copy of your MATLAB code.

b) Apply your algorithm on Legendre polynomials and plot the first five orthonormal basis functions you get.

2. Define $<v, w>$ for $v = (v_1, v_2)$ and $w = (w_1, w_2) \in \mathbb{R}^2$ as

$$<v, w> = (v_1, v_2) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$  \hspace{1cm} (1)

a) Show that $<v, v> = 0$ for all vectors $v = (v_1v_2)$ with $v_1 + 2v_2 = 0$.

b) Is this a valid inner product?

3. Project the function $f(x) = x$ onto the space spanned by $\phi(x), \psi(x), \psi(2x), \psi(2x-1) \in L_2[0, 1]$ where

$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 1, & 0 \leq x < 1/2 \\ -1, & 1/2 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Sketch the projection of $f(x)$ to this space.

4. a) Inner products are preserved through unitary transforms. Show that the inner product of two functions is preserved through Fourier transform, i.e.

$$\int_{-\infty}^{\infty} f(t)h^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)H^*(\omega)d\omega.$$  \hspace{1cm} (2)
b) For \( h_T(t) = \frac{\sin(\pi t/T)}{(\pi t/T)} \), prove that \( \{h_T(t - nT)\}_{n \in \mathbb{Z}} \) is an orthogonal basis of the space \( U \) of functions whose Fourier transforms are bandlimited in \([−\pi/T, \pi/T]\) and if \( f(t) \in U \) then \( f(nT) = \frac{1}{T} < f(t), h_T(t - nT) > \).

Hint: \( H_T(\omega) = \begin{cases} T & -\pi/T < \omega < \pi/T \\ 0 & \text{otherwise} \end{cases} \)

c) Show that the scaling function \( \phi(t) = \frac{\sin(\pi t)}{\pi t} \) creates a multiresolution analysis system, i.e. define \( V_0 = \text{span}\{\phi(t - k)\} \) and show that \( V_j \subseteq V_{j+1} \).

5. For the given signal \( p(t) \), represent \( p(t) \) as a linear combination of Haar scaling functions, i.e. determine the Haar scaling coefficients.
   a) Is the solution unique?
   b) Determine the length 4 representation of the signal \( p(t) \) with 4 non-overlapping scaling functions.
   c) Determine a representation of the same signal as a combination of Haar wavelet and scaling functions.

6. Let \( f(t) = e^{-t^2/10} (\sin(2t) + 2\cos(4t) + 0.4\sin(t)\sin(50t)) \). Discretize the function \( f \) over the interval \( 0 \leq t \leq 1 \) with 256 points. Use \( n = 8 \) as the top level in your wavelet analysis. Using MATLAB wavemenu function, analyze this signal with Haar wavelets. Plot the resulting approximations at each level and compare with the original signal.