1. a) To show that \( \{\phi(t - k)\}_{k \in \mathbb{Z}} \) is an orthonormal family, use the equivalent relationship in the frequency domain \( \sum_k |\Phi(\omega + 2k\pi)|^2 = 1 \). It is clear that for \( \omega \in \left[-\frac{2\pi}{3} - 2n\pi, (2\pi)/3 - 2n\pi\right], \sum_k |\Phi(\omega + 2k\pi)|^2 = 1 \). The only thing left is to show that the equality holds in overlapping regions. Take for example, \( \omega \in \left[(2\pi)/3, (4\pi)/3\right] \),
\[
\Phi(\omega)^2 + \Phi(\omega - 2\pi)^2 = \theta(2 - \frac{3\omega}{2\pi}) + \theta(2 + \frac{3(\omega - 2\pi)}{2\pi})
\]
\[
= \theta(2 - \frac{3\omega}{2\pi}) + \theta(1 + (2 - \frac{3\omega}{2\pi}))
\]
\[
= \theta(2 - \frac{3\omega}{2\pi}) + \theta(1 - (2 - \frac{3\omega}{2\pi}))
\]
\[
= 1 \tag{1}
\]
Therefore, \( \{\phi(t - k)\}_{k \in \mathbb{Z}} \) is an orthonormal set and the question already states that it spans \( \mathcal{V}_0 \), so it is an orthonormal basis for \( \mathcal{V}_0 \).

b) To show that \( \phi(t) \) creates a MRA system, first we need to show \( \mathcal{V}_j \subset \mathcal{V}_{j+1} \). This is equivalent to saying that there exists a periodic function \( H(\omega) \in L_2([0, 2\pi]) \) such that \( \Phi(2\omega) = \frac{1}{\sqrt{2}} H(\omega) \Phi(\omega) \). Then choose \( H(\omega) = \sqrt{2} \sum_k \Phi(2\omega + 4k\pi) \). We also need to show \( \lim_{j \to \infty} \mathcal{V}_j = L_2 \). In this case, it is enough to show that if \( < f, \phi_{j,k} >= 0, j, k \in \mathbb{Z} \) then \( f = 0 \).
\[
< f, \phi_{j,k} >= 0 \quad \sum_{k \in \mathbb{Z}} F(2^m(\omega + 2k\pi)) \Phi^*(\omega + 2k\pi) = 0 \tag{2}
\]

c) We can find \( \Psi(\omega) = \frac{1}{\sqrt{2}} H_1(\omega) \Phi(\omega) \). \( H(\omega) \) can be found graphically from \( \Phi(\omega) \) and the corresponding recursion equation. Since \( H_1(\omega) = e^{-j(\omega + \pi)} H^*(\omega + \pi) \), the wavelet function in the frequency domain is:
\[
\Psi(\omega) = e^{-j(\pi + \frac{\pi}{2})} \sqrt{\frac{3|\omega|}{2\pi}} - 1 \quad 2\pi/3 \leq |\omega| < 4\pi/3
\]
\[
= e^{-j(\pi + \frac{\pi}{2})} \sqrt{2 - \frac{3|\omega|}{4\pi}} \quad 4\pi/3 \leq |\omega| < 8\pi/3
\]
\[
= 0 \quad \text{otherwise} \tag{3}
\]
d) 

2. For length 6 Daubechies filter, \( N = 6 \) and the number of vanishing moments \( K = 3 \). Therefore, \( H(\omega) \) can be written as
\[
H(\omega) = \sqrt{2}(\frac{1 + e^{-j\omega}}{2})^3 R(e^{-j\omega})
\]
\[ R(z)R(z^{-1}) = Q(z^{-1}) = P\left(\frac{2 - z - z^{-1}}{4}\right) \]

\[ P(y) = \sum_{k=0}^{2} \binom{2 + k}{k} y^k \]

(4)

Inserting \(\frac{2-z-z^{-1}}{4}\) for \(y\) gives the following polynomial for \(Q(z^{-1})\).

\[ Q(z^{-1}) = 3z^2 - 18z + 38z^{-1} + 3z^{-2} \]

(5)

The roots of this polynomial are found to be \(2.7127 + 1.4439j, 2.7127 - 1.4439j, 0.2873 + 0.1529j, 0.2873 - 0.1529j\). Since we are interested in designing the maximum phase filter, we choose the roots that are outside the unit circle, \(2.7127 + 1.4439j, 2.7127 - 1.4439j\). Now, we can write \(R(e^{-j\omega}) = A(e^{-j\omega} - r_1)(e^{-j\omega} - r_1^*)\), where \(r_1\) equals to \(2.7127 + 1.4439j\). \(A\) comes out to be 0.1993. Therefore,

\[ H(z) = \sqrt{2}\left(\frac{1 + z^{-1}}{2}\right)^3(0.1993)(1 - (2.7127 + 1.4439j)z^{-1})(1 - (2.7127 - 1.4439j)z^{-1}). \]

(6)
Writing $H(z)$ and reading off the coefficients gives, $h(0) = 0.0352$, $h(1) = -0.08545$, $h(2) = -0.135$, $h(3) = 0.4598$, $h(4) = 0.8069$, $h(5) = 0.33267$. The scaling and wavelet functions are time reversed versions of the minimum phase one.

3. Frequently signals have a bias, which is a polynomial part added to a bounded rapidly oscillating part, such as $f(t) = a + bt + ct^2 + g(t)$, where $g(t)$ is a sinusoidal signal. Suppose that we have 1024 samples of $f$ on the interval $[-1, 1]$. In order to analyze this signal with Daubechies family of wavelets, the smallest length filter we should choose is $N = 6$ which has 3 vanishing moments. This is due to the fact that the polynomial is of order 2, and using wavelets with more than 3 vanishing moments will ensure that the wavelet is orthogonal to the polynomial part of the signal. Based on the decomposition in wavemenu, the polynomial part of the signal lives in $V_6$. The sinusoidal part of the signal lives in $W_6, W_7, W_8, W_9, W_{10}$.

4. a) The basic recursion equation for $\phi(t)$ is given as $\phi(t) = \sum_n h(n) \sqrt{2}\phi(2t - n)$. If the derivatives of $\phi(t)$ exist, $\phi^k(t) = \sum_n h(n) \sqrt{2}^k \phi^k(2t - n)$. We can change the order
of the derivative and the summation since $h(n)$ is a FIR filter, the summation is over a finite index. Therefore, $\phi^k(t)$ satisfies a recursion type relationship, but if we look closely we see that the sum of these new filter coefficients $\sum_n h(n)2^k \neq \sqrt{2}$, so it doesn’t exactly satisfy the basic recursion equation. b) For $\phi(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$, $\phi'(t) = \begin{cases} 1 & -1 \leq t \leq 0 \\ -1 & 0 \leq t \leq 1 \end{cases}$ The coefficients in this case are $h(-1) = \frac{1}{\sqrt{2}}$, $h(0) = \sqrt{2}$, $h(1) = \frac{1}{\sqrt{2}}$.

5. The Coiflet conditions are given by $\int \phi(t) dt = 1, \int t^k \phi(t) dt = 0$ $1 \leq k < p$. By following the same approach presented in class for the proof of vanishing moments theorem,

$$\Phi^{(k)}(0) = (-j)^k \int t^k \phi(t) dt = 0 \quad 1 \leq k < p$$ (7)

The basic recursion equation in the frequency domain is $\Phi(\omega) = \frac{1}{\sqrt{2}} H(\omega/2) \Phi(\omega/2)$.

$$\Phi'(0) = \frac{1}{\sqrt{2}}(H'(0)\Phi(0) + H(0)\Phi'(0) = 0 \rightarrow H'(0) = 0$$

$$\Phi^{(k)}(0) = \sum_{l=0}^{k} \binom{k}{l} H^{(l)}(0)\Phi^{(k-l)}(0) \frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} H^{(k)}(0) = 0 \rightarrow H^{(k)}(0) = 0, \ 1 \leq k < p$$ (8)