Solutions to Practice Exam 2

1. a) The local oscillator frequency is \( f_{LO} = f_c + f_{IF} = 1120 + 2500 = 3620\text{kHz} \). The image frequency should be \( 3620 + 2500 = 6120\text{kHz} \), such that \( f_{image} - f_{LO} = f_{IF} \).

b) The assumption that \( \phi(t) \) is small provides the basis for using the PLL for demodulating FM signals. When \( \phi(t) \) is small, we can make the assumption that \( \sin(\phi(t)) \approx \phi(t) \). This approximation allows us to represent PLL by a linearized model, which enables us to perform frequency demodulation using PLL with output \( v(t) = \frac{k_f}{k_v} m(t) \).

c) Let \( y(t) = n(t) * h(t) \). Therefore, \( S_y(f) = |H(f)|^2 S_n(f) = \text{rect}(f) \frac{N_0}{2} \).

\[
S_y(f) = \begin{cases} 
\frac{N_0}{8}, & 1 \leq |f| \leq \frac{3}{2} \\
0, & \text{otherwise}
\end{cases}
\]

Note: Multiplying with \( \cos(2\pi f_c t) \) in the time domain is equivalent to convolution in the frequency domain. Since we are computing power spectral densities, it is equivalent to convolution with \( \frac{1}{4} [\delta(f - f_c) + \delta(f + f_c)] \).

2. a) \( E[X_1X_2] = E[X_1]E[X_2] = 4 \), using independence of the random variables.

b) \( E[Y] = E[2X_1 + X_2] = 2E[X_1] + E[X_2] = 6 \)

c) \( \sigma_Y^2 = E[Y^2] - E^2[Y] \)

\[
E[Y^2] = E[(2X_1 + X_2)^2] = E[4X_1^2] + E[4X_1X_2] + E[X_2^2] = 4(16 + 4) + 4(4) + (16 + 4) = 48 + 16 + 12 = 76
\]

\( \sigma_Y^2 = 76 - 36 = 40 \)

d) \( Y \) is a Gaussian random variable since it is a linear transformation of Gaussian random variables. Therefore,

\[
f_Y(y) = \frac{1}{\sqrt{2\pi(40)}} \exp\left(\frac{-(y-6)^2}{80}\right)
\]

e) \( P[Y \leq 2] = 1 - Q\left(\frac{2 - 6}{\sqrt{40}}\right) = 1 - Q\left(\frac{-4}{2\sqrt{10}}\right) = Q\left(\frac{2}{\sqrt{10}}\right) \approx 0.27 \)
Note: SNR is usually shown in terms of dB

\[ SNR = 10 \log_{10} \left( \frac{\text{Signal Power}}{\text{Noise Power}} \right) \]

Example: Consider SSB system shown below.
In general, the transmit power (power of transmitted signal \( s(t) \)) is constrained by FCC or by a certain designed battery life.

\[ m(t) \] has PSD \[ |M(f)|^2 \]

\[ M(f) = \begin{cases} 
0.002 & |f| \leq 1.5 \text{kHz} \\
0.001 & 1.5 \text{kHz} < |f| \leq 3 \text{kHz} \\
0 & |f| > 3 \text{kHz} 
\end{cases} \]

a) Find \( A_c \) such that transmitted power is 100mW

\[ S(f) \]
Power \[ \int |s(t)|^2 dt = \int |s(t)|^2 df \]

\[ = 2 \left[ 1500 \frac{0.001 A_C}{2} + 1500 \left( \frac{0.003 A_C}{2} \right)^2 \right] = 100 \text{mW} \]

\[ = \frac{1}{2} \left[ 1500 A_C^2 \cdot 10^{-6} + 1500 A_C^2 \cdot 9 \cdot 10^{-6} \right] = 100 \cdot 10^{-2} \]

\[ \approx (1500) A_C^2 \]

\[ A_C = 2.65/1 \]

b) Power of \( z(t) \) assuming there's no noise

\[ z(t) \]

\[ \left[ 1500 \left( \frac{0.003 A_C}{4} \right)^2 + 1500 \left( \frac{0.001 A_C}{4} \right)^2 \right] = \frac{1}{4} \cdot 100 \text{mW} = 25 \text{mW} \]

c) \( n(t) \) is AWGN \( S_n(t) = 0.5N_0 \)

\( N_0 = 0.0001 \text{mW/Hz} \)

PSD and power in demodulated output \( z(t) \) due to noise \( L_n(t) = 0 \)

\[ n(t) \cos(2\pi f_c t) \rightarrow \frac{1}{4} S_n(f - f_c) + \frac{1}{4} S_n(f + f_c) \]

\[ \frac{N_0}{4} \]

\[ \text{Power} = \frac{60000 \cdot 1000000}{4} = 1 \cdot 10^{-2} \]

\[ = 0.00015 \text{mW} \]
d) SNR at the output:
\[
\frac{25 \text{ mW}}{0.15 \text{ mW}} = 166.67 \approx 22.2 \text{ dB}
\]

e) Assume DSB (no filtering)

\[ Ac = ? \]
\[ P_t = 100 \text{ mW} \]
\[
4 \left[ 1500 \left( \frac{0.001 Ac}{2} \right)^2 + 1500 \left( \frac{0.003 Ac}{2} \right)^2 \right] + 100 \text{ mW}
\]
\[ 10.1500 \times 10^{-6} \left( \frac{Ac}{2} \right)^2 = 100 \times 10^{-3} \]
\[ Ac = 2.5 \text{ B} \]

Power in \( z(t) \)
\[
2 \left[ 1500 \left( \frac{0.002 Ac}{2} \right)^2 + 1500 \left( \frac{0.001 Ac}{2} \right)^2 \right]
\]
\[ 50 \text{ mW} \] (signal power)

noise power same \[ 0.00015 \text{ W} \rightarrow 0.15 \text{ mW} \]

\[ \text{SNR} = \frac{5.0}{0.15} = 33.33 \rightarrow 25.2 \text{ dB} \]

3 dB increase in SNR.

If you have the same transmit power constraint, DSB has higher
SNR, SSB is more spectrally efficient.