ECE 457 EXAM 2
April 4, 2003

- No textbooks, notes or HW solutions.
- One page of hand-written notes.
- Calculators are allowed.
- Exam is 50 minutes.
- To maximize your score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.
- Good luck.


1. [5] An ergodic random process \( n(t) \) has mean equal to 1, and \( E[n^2(t)] = 5 \). What’s the total power of \( n(t) \)?

2. [5] A white noise process with power spectral density \( S_s(f) = 5W/Hz \) is passed through a filter, \( H(f) = \text{rect}(f/2) \). What’s the power of the output noise?

3. [10] Determine whether the following statements are true or false.
   
a) If two random variables \( X \) and \( Y \) are uncorrelated, then they are independent.
   
   False
   
   b) \( S(f) = \frac{f}{f^2 + 16} \) is a valid power spectral density function.
   
   False
c) If $X(t)$ is a stationary random process, $E[X(t)]$ is a constant. \[ \text{True} \]

d) If the power of a bandpass noise process $n(t)$ is 10 Watts, then the power of the baseband equivalent process $n_c(t)$ is 5 Watts. \[ \text{False} \]

e) A phase-lock-loop can be used for FM demodulation. \[ \text{True} \]

4. [5] A radio station transmits AM signal with a carrier frequency of 1500kHz. When an inexpensive radio receiver, which has a poor selectivity in its RF-stage bandpass filter, is tuned to 1500 kHz, the signal is heard loud and clear. The same signal is also heard (not as strong) at another dial setting. State at what frequency you will hear this station.

Note: Intermediate frequency (IF) for an AM receiver is 455kHz. Assume that the local oscillator frequency is, $f_{LO} = f_c + f_{IF}$.

\[ 1500 + 910 = 2410 \text{ kHz} \]

5. [10] A certain continuous random variable has the cumulative distribution function

\[ F_X(x) = \begin{cases} 
0, & x < 0 \\
Ax^3, & 0 \leq x < 10 \\
B, & x > 10 
\end{cases} \]

Find the proper values for $A$ and $B$.

\[ B = 1 \]

\[ A \left(\frac{10}{1000}\right)^3 = \frac{1}{1000} \]

\[ A = \frac{1}{1000} \]
Part B – Show all your work to receive full credit.

1. [30] Two independent random variables $X$ and $Y$ have means and variances as given below:
   \[
   m_x = 2, \sigma_x^2 = 3, m_y = 1, \sigma_y^2 = 5
   \]

   A new random variable $Z$ is defined as $Z = 3X - 4Y$.
   
   a) [6] Find the mean of $Z$.
   
   b) [8] Find the variance of $Z$.
   
   c) [6] If $X$ and $Y$ are both Gaussian, write the density function for $Z$.
   
   d) [10] Find $P(1 \leq Z < 5)$.

   a) $E[Z] = 3E[X] - 4E[Y] = 6 - 4 = 2$

   b) $\sigma_Z^2 = E[Z^2] - (E[Z])^2$


   $= 9(3 + 4) - 24(2)(1) + 16(1 + 1)$ 

   $= 92 - 48 + 96$ 

   $= 136$

   $\sigma_Z^2 = 136 - 4 = 132$

   c) $f_Z(z) = \frac{1}{\sqrt{2\pi} \sqrt{132}} \exp\left(-\frac{(z - 2)^2}{2 \cdot 132}\right)$

   d) $P(1 \leq Z < 5) = F_Z(5) - F_Z(1)$

   $= \left(1 - \Phi\left(\frac{5 - 2}{\sqrt{132}}\right)\right) - \left(1 - \Phi\left(\frac{1 - 2}{\sqrt{132}}\right)\right)$

   $= \Phi\left(-\frac{1}{\sqrt{132}}\right) - \Phi\left(\frac{3}{\sqrt{132}}\right)$

   $= 1 - \Phi\left(\frac{1}{\sqrt{132}}\right) - \Phi\left(\frac{3}{\sqrt{132}}\right)$

   $= 1 - \Phi(0.0966) - \Phi(0.29) = 0.0004 - 0.382 = 0.182$
Extra Page for Question 1:
2. [35] Consider the DSB transmitter and receiver shown below. The message signal has bandwidth $W$, the predetection filter is a bandpass filter centered at $f_c$ with bandwidth equal to $2W$ and the postdetection filter is a low pass filter with bandwidth $W$. Unlike the coherent demodulation considered in class, the local oscillator in the receiver has a phase error equal to an unknown constant $\phi$. $n(t)$ is white Gaussian noise with power spectral density, $S_n(f) = \frac{N_0}{2}$.

![Diagram of DSB transmitter and receiver]

For the following questions, express your answers in terms of $A_e, m(t), N_0, \phi, W$.

a) [6] Find an expression for the SNR at the output of the predetection filter.
b) [18] Find an expression for the SNR at the output of the system.
c) [6] Find the detection gain for this receiver.
d) [5] Interpret your results, i.e. how does the SNR change with $\phi$?

Hints:

$\cos(a)\cos(b) = \frac{1}{2}(\cos(a - b) + \cos(a + b))$

$\sin(a)\cos(b) = \frac{1}{2}(\sin(a - b) + \sin(a + b))$

$\cos^2(a) + \sin^2(b) = 1$
Extra Page for Question 2:

a) Signal at the output of predetection filter:

\[ S_{\text{signal}} = \frac{Acm(t) \cos(2\pi fct)}{N_0/2} \]

Noise at the output of pred.

\[ n_{\text{bp}}(t) \]

Noise power:

\[ (2) \left( \frac{N_0}{2} \right) \cdot 2W = 2N_0W \]

\[ \text{SNR}_{\text{pre}} = \frac{Acm^2(t)}{2N_0W} \]

b) Signal at the output of post-detection:

\[ S_{\text{signal}} = Acm(t) \cos(2\pi fct) \cos(2\pi fct + \phi) \]

\[ = Acm(t) \cos(\phi) + Acm(t) \cos(4\pi fct + \phi) \]

Won't pass through LPF.

Noise at the output of post detection:

\[ n_{\text{bp}}(t) = n_c(t) \cos(2\pi fct) + ns(t) \sin(2\pi fct) \]

\[ \left( n_c(t) \cos(2\pi fct) + ns(t) \sin(2\pi fct) \right) \left( 2\cos(2\pi fct + \phi) \right) \]

\[ = n_c(t) \cos(\phi) + in_c(t) \cos(4\pi fct + \phi) + ns(t) \sin(\phi) + ns(t) \sin(4\pi fct + \phi) \]

Won't pass through LPF.

Signal power:

\[ Acm^2(t) \cos^2(\phi) \]

Noise power:

\[ \frac{n_c^2(t) \cos^2(\phi) + ns^2(t) \sin^2(\phi)}{n_c^2(t) = ns^2(t) = n_{\text{bp}}^2(t) = 2N_0W} \]

\[ 2N_0W \left( \cos^2(\phi) + \sin^2(\phi) \right) = 2N_0W \]

\[ \text{SNR}_{\text{post}} = \frac{Acm^2(t) \cos^2(\phi)}{2N_0W} \]
c) Detection Gain: \[ \frac{(SNR)_{post}}{(SNR)_{pre}} = 2 \cos^2 \phi \]

d) As \( \phi \) increases, SNR decreases.
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