Optimal Combustion Phasing Modeling and Control of a Turbulent Jet Ignition Engine

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Abstract—Combustion phasing control is very important for internal combustion engines to achieve high thermal efficiency with low engine-out emissions. Traditional open-loop map-based control becomes less favorable in terms of calibration effort, robustness to engine aging, and especially control accuracy for turbulent jet ignition (TJI) engines due to the increased number of control variables over conventional spark-ignition engines. In this paper, a model-based feedforward controller is developed for a TJI engine. A feedback controller is designed based on the linear quadratic tracking control with output covariance constraint (OCC). Since the TJI main-chamber combustion is influenced by the pre-chamber one, the proposed controller optimizes the control variables in both combustion chambers. The proposed feedforward and feedback controller reduces standard deviation of combustion phase by more than 22% over a group of baseline controllers through a series of dynamometer engine tests.

Index Terms—Turbulent jet ignition, combustion phasing control, optimal control

I. INTRODUCTION

In recent years, improving the thermal efficiency of internal combustion (IC) engines becomes increasingly important for powertrain researchers and engineers [1]. According to the analysis based on 2015 EPA certification data, a further 30% reduction in fuel consumption is required for gasoline engines to meet the US 2025 GHG regulations, assuming the improvement is originated solely from IC engines [2].

Turbulent jet ignition (TJI) combustion is a promising combustion technology that is able to achieve low engine out emissions and high thermal efficiency. The application of the TJI combustion technology ranges from engines for passage cars to commercial vehicles (such as delivering trucks and stationary power generators) with a thermal efficiency improvement of more than 25 percent [2]. Furthermore, the TJI technology was used in Ferrari’s Formula One racing car [3]. The history of TJI engines can be traced back to 1940s [4]. The TJI combustion system mainly consists of three parts: a large main-chamber, a small pre-chamber (a few percent of main-chamber volume), and a single- or multi-orifice nozzle connecting the two combustion chambers. Once the pre-chamber is ignited, the pressure difference between the two sides of nozzle drives the hot reacting products into the main-chamber. The hot reacting products from the pre-chamber are called turbulent jets. Since the turbulent jets contain significantly large ignition energy, the main-chamber can be ignited under ultra lean conditions. There are two major advantages for lean combustion. The first is the improvement of engine thermal efficiency. It is achieved by increasing the specific heat ratio of the air-fuel mixture inside the main-chamber with extra air [4]. Lean combustion can also reduce NOx (nitrogen oxides) formation due to the low combustion temperature [4]. Therefore, lean or diluted combustion attracts more researchers’ attention as the regulations on NOx emissions become increasingly stringent [5].

Another factor that directly affects the engine thermal efficiency is the combustion phase [6]. Among several critical combustion phasing locations of interest, CA50 (crank location when 50% of fuel is burned) is of primary interest, and thus, is widely used for combustion phasing analysis and control [7]. The ignition timing associated with the most efficient combustion is called maximum break torque (MBT) timing that can be achieved by regulating the CA50 location to its optimum. Once the CA50 is far away from that location, the efficiency decreases and the combustion also becomes less stable, leading to high coefficient of variation (COV) of the indicated mean effective pressure (IMEP). Therefore, combustion phasing control is important for both conventional spark ignition (SI) and TJI engines.

Different methods have been used for combustion phasing control. Generally, these methods can be divided into two groups. The first focuses on finding the optimal combustion phase. For example, references [8]–[10] use the extreme seeking control method to find the optimal spark timing and reference [11] uses an adaptive optimal control method similar to extreme seeking. The second group mainly focuses on combustion phase detection and use it for optimal combustion phase control. In reference [7], a combustion phasing model is developed based on an artificial neural network and used for closed loop control; reference [12] uses an adaptive control method based on a radial basis function network; and reference [13] utilizes the hypothesis test for combustion phasing control. Reference [14] developed an ionization based closed-loop combustion phase detection and control strategy. In this paper, the optimal CA50 values under different engine operational conditions are derived from the developed engine model and is regulated at its optimal value through feedforward and feedback control. Different combustion phasing control methods have been developed for a various IC engines, but not for TJI engines. For TJI engines, besides spark timing, pre-chamber control variables can also affect the combustion phasing in the main-chamber. This makes the open-loop map-based controller less accurate, because the main-chamber combustion phasing is subject to the characteristics of turbulent jets from the pre-
chamber. Therefore, model-based feedforward and feedback control are preferable. The increased number of control variables makes the control problem more complicated, but they provide additional degree of freedoms in control to improve the closed-loop system performance, which will be shown in the controller validation section of the paper.

The main contribution of this paper is two-fold: the development of a state-space TJI engine model for controller design and a combustion phasing controller, consisting of both feedforward and feedback controls, developed for the TJI engine based on the unique TJI combustion characteristics.

This paper is organized as follows. The next section briefly describes the TJI engine configuration studied in this paper. Section III presents the proposed state-space engine model; the controller is then designed based on the developed state-space model in Section IV. The experimental validation results of both proposed model and control strategy are included in Section V. Conclusions are provided in the last section.

II. TARGET ENGINE DESCRIPTION AND SPECIFICATIONS

Fig. 1 shows the basic architecture of the single cylinder Dual-Mode TJI (DM-TJI) engine studied in this paper. Each cylinder has two combustion chambers, pre and main ones. The pre-chamber is located on top of the main-chamber. Two chambers are connected through six small orifices between them. The engine has two fuel injectors, one for pre-chamber and one for main-chamber in the intake port (see Fig. 1). A distinctive feature of this DM-TJI engine is the inclusion of an air injector in the pre-chamber. The air injector is located at the opposite side of the pre-chamber fuel injector. When air injector is turned on, the pressurized air is injected against the fuel jets to reduce the amount of fuel deposited on the pre-chamber wall. More importantly, the air injector is also used to control the air-to-fuel ratio (AFR) in the pre-chamber to be close to stoichiometry and to improve the pre-chamber combustion characteristics, especially under heavy exhaust-gas-recirculation. For TJI engines, the combustion is firstly initiated in the pre-chamber by the spark plug, and then the hot reacting products in the pre-chamber are pushed into the main-chamber through the connecting orifices between two chambers and ignite the lean mixture in the main-chamber. The engine specifications are listed in Table I. Both chambers use EPA LEV-II liquid gasoline.

III. NONLINEAR STATE-SPACE TJI ENGINE MODEL

During engine operations, the combustion in the current engine cycle is influenced by the residual gas produced from the previous cycle and certain fuel left on the pre-chamber wall due to fuel impingement of the fuel injection event in the pre-chamber despite the inclusion of the air injection [15]. The fuel film left from the previous cycle on the pre-chamber wall changes the AFR, especially in the pre-chamber. Both residual gas in both pre- and main-chambers and fuel film on the pre-chamber wall are the primary sources of the engine combustion cycle-to-cycle dynamics. The states of the state-space engine model thus need to describe the properties of the residual gas in both chambers and the fuel film on the pre-chamber wall. Fig. 2 shows the key combustion events that will be used for model development.

A. Main-chamber State-space Model

In the current TJI engine, the air-fuel mixture in the main-chamber is usually lean. Therefore, there is always oxygen left after combustion in the residual/exhaust gas. If all the fuel is assumed to be burned completely, the residual/exhaust gas can be divided into two parts: the first part has the same composition as air and the second part only has the reaction products with no oxygen. In other words, the second part of the mixture has the same composition as the residual/exhaust gas for an engine under stoichiometric combustion condition, assuming complete combustion. In the rest of the paper, the second part of the mixture is referred to as complete stoichiometric products (CSP). The CSP dilutes the charge rather than air; the oxygen concentration is reduced which will slow down the combustion reactions [16].
The major assumptions used for developing the main-chamber state-space model are stated below.

1) It is assumed all the fuel burned completely during the combustion. This is reasonable because the AFR control can be adopted to make sure adequate AFR in the main-chamber.
2) The main-chamber pressures at intake valve close (IVC) and intake valve open (IVO), $P_{main}^{IVC}$ and $P_{main}^{IVO}$, can be approximated by the intake and exhaust manifold pressures, $P_{Int}$ and $P_{Exh}$, respectively.
3) The mass of the gas inside the main-chamber is assumed to be constant when both the intake and exhaust valves are closed.

According to the above assumptions, the properties of the residual gas in the main-chamber can be fully described by the following two states:

1) The mass of CSP left from last cycle at IVC: $m_{CSP}^{IVC}(k)$, where $k$ is the engine cycle index.
2) The mass of residual gas left from last cycle at IVC: $m_{r}^{IVC}(k)$, including the CSP and residual air.

The total mass inside the main-chamber at IVC, $m_{main}^{IVC}(k)$, and its average temperature, $T_{main}^{IVC}(k)$, can be calculated based on the intake and exhaust manifold pressures and the mass of residual gas; see Appendix A.

During the compression, combustion and expansion processes, the mass of the gas inside the main-chamber is constant. The temperature during combustion can be simplified to be a combination of polytropic process and heat release process [17]. Therefore, the temperature at the end of combustion (EOC) can be represented by:

$$T_{EOC}^{main}(k) = T_{IVC}^{main}(k) \left( \frac{V_{IVC}^{main}(k)}{V_{EOC}^{main}(k)} \right)^{n-1} + \frac{Q_{LHV} m_{f, fresh}^{main}(k)}{m_{IVC}^{main}(k) C_v}$$

(1)

where $n$ is the polytropic index and $Q_{LHV}$ is the fuel lower heating value. $T_{EOC}^{main}$ can be calculated based on the ideal gas law.

From [18], the main-chamber rate of combustion can be calculated by

$$\begin{align*}
x_b(\theta) &= 1 - \exp \left\{ -a \left[ \frac{\theta - \theta_{ign}(\theta)}{\Delta \theta_d} \right]^{m+1} \right\} \\
\theta_{ign}(\theta) &= \theta_0 - \int_{\theta_0}^{\theta} \left[ b(\theta) - 1 \right] d\theta
\end{align*}$$

(2)

where $x_b$ is the mass fraction burned; $\theta$ represents the current crank angle position; $\theta_0$ is the start of ignition; and $\Delta \theta_d$, $a$ and $m$ are calibration parameters. $b(\theta)$ is determined by the following equation

$$b(\theta) = \beta \cdot m_{tur}^{+}(\theta) + 1$$

$$m_{tur}^{+}(\theta) = \begin{cases} m_{tur}(\theta) \geq 0 \\ 0 \end{cases} \quad m_{tur}(\theta) < 0$$

(3)

where $m_{tur}(\theta)$ is the mass flow rate of the turbulent jets from the pre- to main-chamber; and $\beta$ is a calibration parameter used to simulate the increased burn rate caused by the turbulent jets. The above two equations are able to simulate the effects of the pre-chamber combustion to the main-chamber combustion.

The crank angle position at EOC, $\theta_{EOC}$, can be obtained by substituting $x_b(\theta_{EOC}) = 0.9$ into the first equation in Eqn. (2)

$$\frac{\theta_{EOC} - \theta_{ign,b}}{\Delta \theta_d} = \left( -\frac{\ln(0.1)}{a} \right)^{\frac{1}{m+1}} = Sol$$

(4)

Then $\theta_{EOC}$ can be derived by using Eqns. (2) and (3) as follows,

$$\theta_{EOC} = \theta_{ign} + \Delta \theta_d \cdot \text{Sol} - \int_{\theta_{ign}}^{\theta_{EOC}} \beta m_{tur}(\theta) d\theta$$

(5)

where $\theta_0$ can be approximated by spark timing; and the total mass of the turbulent jet during combustion, $m_{tur}$, will be calculated in the next subsection.

After combustion, the total CSP mass can be calculated by

$$m_{CSP}^{main}(k) = m_{r,CSP}(k) + m_{f,fresh}^{main}(k)(1 + \varphi_s)$$

(6)

where $\varphi_s$ is the stoichiometric AFR; and $m_{f,fresh}^{main}(k)$ is the mass of fresh fuel in the main-chamber. After the combustion, the gas in the main-chamber polytropically expands until exhaust valve open (EVO).

$$T_{EVO}^{main}(k) = T_{EOC}^{main}(k) \left( \frac{V_{EOC}^{main}(k)}{V_{EVO}^{main}(k)} \right)^{n-1}$$

(7)

And $P_{EVO}^{main}$ can be calculated by the ideal gas law. After EVO, the gas in the main-chamber expands until its pressure is equal to the exhaust manifold pressure. $P_{IVO}^{main}$ can be approximated by $P_{Exh}$. Then the mass of residual gas in the main-chamber can be calculated by:

$$m_{IVO}^{main} = \frac{P_{Exh}(k+1)V_{IVO}^{main}}{R \left( \frac{m_{IVO}^{main}(k)}{P_{IVO}^{main}(k)} \right)^{n-1} T_{EVO}^{main}(k)}$$

(8)

From IVO to EVC, part of the gas in the exhaust manifold flows back into the engine cylinder due to the pressure difference between intake and exhaust manifolds. This part of the residual gas is calculated based on the widely used Fox model [19].

$$m_{backflow}^{main}(k+1) = C_i \left( \frac{P_{IVO}^{main}}{P_{Exh}} \right)^{-0.87} \sqrt{P_{Exh} - P_{IVO}} m_{IVO}^{main}(k+1)$$

(9)

Since $m_{IVC}^{main}(k+1)$ cannot be determined without knowing the mass of residual gas, its value is approximated by

$$m_{IVC}^{main} = \frac{P_{IVO}^{main}(k+1)V_{IVC}^{main}}{RT_{IVO}^{main}(k)}$$

(10)

The total mass of residual gas left for next cycle can be calculated by

$$m_{r}^{main}(k+1) = m_{IVC}^{main}(k+1) + m_{backflow}^{main}(k+1)$$

(11)

The content in the residual gas is the same as that at EOC. Therefore, the CSP left for the next cycle is

$$m_{r, CSP}^{main}(k+1) = m_{r}(k+1) \frac{m_{CSP}^{main}(k)}{m_{IVC}^{main}(k)}$$

(12)
B. Pre-chamber State-Space Model

In the pre-chamber, the AFR of the air-fuel mixture is always kept close to stoichiometry to have good ignitability. Therefore, there is usually no air left after combustion in the pre-chamber. The fuel film mass on the pre-chamber wall is also very important because it determines how much fuel will vaporize into the gas mixture before ignition. As a result, the states selected for pre-chamber are:

1) the mass of residual gas left from last cycle at IVC, \( m_{\text{pre},g}^\text{IVC}(k) \), with the same composition as CSP.
2) the mass of fuel film on the pre-chamber wall at IVC, \( m_{\text{pre},f,\text{film}}^\text{IVC}(k) \).

The total mass in the pre-chamber and its temperature at spark timing can be calculated based on the gas properties in the pre-chamber at IVC; see Appendix B.

The combustion in the pre-chamber is very fast, so a constant volume combustion model is used to approximate the combustion process below.

\[
T_{\text{EOC}}^\text{pre}(k) = T_{\text{SPK}}^\text{pre}(k) + \frac{m_{\text{fuel,\text{vap}}}(k)}{m_{\text{SPK}}(k)} \cdot Q_{\text{LHV}} \tag{13}
\]

\[
P_{\text{EOC}}^\text{pre}(k) = \frac{m_{\text{SPK}}(k) \cdot R T_{\text{EOC}}^\text{pre}(k)}{P_{\text{pre}}(k)} \tag{14}
\]

where the subscript \( \text{SPK} \) denotes the properties at spark timing; \( V_{\text{pre}} \) is the pre-chamber volume; and \( m_{\text{fuel,\text{vap}}}(k) \) is the total vaporized fuel in the pre-chamber.

Then the gas in the pre-chamber expands until its pressure is similar to the main-chamber one. The associated time is defined as end of expansion (EOE). Then, \( T_{\text{EOE}}(k) \) can be obtained.

If spark timing is controlled around the optimum value, \( P_{\text{pre}}^\text{main, \text{EOE}} \) can be approximated by the motoring pressure at top dead center. Then the mass at EOE, \( m_{\text{pre}}(k) \), can be calculated by the ideal gas law. The mass flow rate of the turbulent jet after EOE is usually very small, so \( m_{\text{tur}}(k) \) can be calculated by

\[
m_{\text{tur}}(k) = m_{\text{SPK}}(k) - m_{\text{EOC}}^\text{pre}(k) \tag{15}
\]

After EOE, it is assumed that all the gas, pushed into pre-chamber, does not mix with the gas in the pre-chamber and then is pushed out as main-chamber pressure decrease. Therefore, the gas in the pre-chamber is first polytropically compressed and then expanded to IVC. Then, the mass of the residual gas can be calculated based on the ideal gas law.

\[
m_{\text{pre}}(k+1) = \frac{P_{\text{int}}(k+1)}{R (\frac{T_{\text{EOC}}^\text{pre}(k)}{P_{\text{pre}}^\text{EOC}(k)})^{\frac{T_{\text{EOC}}^\text{pre}(k)}{R T_{\text{EOC}}^\text{pre}(k)}}} \cdot T_{\text{EOC}}^\text{pre}(k) \tag{16}
\]

The fuel film on the pre-chamber wall keeps vaporizing during the entire engine cycle. The vaporization rate can be calculated by Eqn. (17).

\[
\dot{m}_{\text{f,\text{vap}}}(k) = \frac{m_{\text{pre}}(k)}{\delta_{f} \rho_{f}} \cdot M_{f} \cdot h_{m} \cdot \frac{P_{s0}}{R_{m} T_{f}} \cdot \exp \left( -\frac{\Delta H}{R_{m} T_{f}} \right) \tag{17}
\]

where \( T_{f} \) is the fuel film temperature; \( P_{s0} \) is the fuel saturated vapor pressure constant; \( \Delta H \) is the fuel molar enthalpy of vaporization; \( \delta_{f} \) is the fuel film thickness; and \( \rho_{f} \) is the density of the liquid fuel. More details about this equation can be found in [15].

Substituting \( \dot{m}_{\text{f,\text{vap}}}(k) = m_{\text{pre}}(k+1) \) into Eqn. (17) and solving the differential equation by assuming that \( T_{f} \) is a constant, the mass of the fuel film right before fuel injection during the next engine cycle can be calculated by

\[
m_{\text{pre}}^\text{f,\text{film}}(k+1) = \left[ m_{\text{pre}}^\text{f,\text{film}}(k) + m_{\text{pre}}^\text{pre}(k+1) \right] \exp \left( -a_{\text{f,\text{vap}}(k+1)} \Delta t(\Omega) \right) \tag{18}
\]

where \( \Delta t(\Omega) \) is the time duration of each engine cycle; \( \Omega \) is engine speed; \( m_{\text{pre}}^\text{f,\text{film}}(k) \) is the fuel film mass right before the fuel injection in engine cycle \( k \), and \( m_{\text{pre}}^\text{pre}(k+1) \) is the mass of the injected fuel impinged on the wall. The parameter \( a_{\text{f,\text{vap}}} \) in Eqn. (18) is

\[
a_{\text{f,\text{vap}}} = \frac{M_{f} h_{m,s,s} P_{s0}}{\delta_{f} \rho_{f}} \cdot \frac{P_{s0}}{R_{m} T_{f}} \cdot \exp \left( -\frac{\Delta H}{R_{m} T_{f}} \right) \tag{19}
\]

where \( T_{f}(t) \) is approximated by \( a_{\text{f,\text{vap}}}(T_{\text{EOC}} + T_{\text{IVC}}^\text{pre}) \) (a calibration parameter); \( h_{m,s,s} \) replaces \( h_{m} \) in Eqn. (17) as a calibration parameter in the state-space model.

C. CA50 Calculation

The parameter, \( \Delta \theta_{d} \), in Eqn. (5) is mainly determined by the main-chamber gas properties. According to the experiment results, \( \Delta \theta_{d} \) can be approximated by a linear function of \( P_{\text{int}}, \Omega, \lambda\), and \( x_{\text{main},\text{CSP}} \), where \( \lambda \) is the normalized air-fuel ratio in the main-chamber and \( x_{\text{main},\text{CSP}} \) is the main-chamber CSP mass fraction at IVC. Then, \( \theta_{\text{CA50}} \) can be finally obtained by

\[
\theta_{\text{CA50}} = \theta_{\text{SPK}} + \Delta \theta_{d} + \theta_{\text{CSP}}(P_{\text{int}} - P_{\text{int,0}}) + b(\Omega - \Omega_{0}) + c(\lambda - \lambda_{0}) + d(x_{\text{main},\text{CSP}} - x_{\text{main},\text{CSP,0}}) + e(m_{\text{tur}} - m_{\text{tur,0}}) \tag{20}
\]

where the parameters, \( P_{\text{int}}, \Omega_{0}, \lambda_{0}, \), and \( m_{\text{tur,0}} \) represent the nominal operational values. A linear Least-Squares fitting process is used to determine the unknown parameters, \( \Delta \theta_{d}, a, b, c, d \) and \( e \), in Eqn. (20).

Considering all the equations above, we finally got the nonlinear state-space equation in the following form.

\[
x(k+1) = f(x(k), u(k))
\]

\[
y(k) = h(x(k), u(k))
\]

where the function \( f \) and \( h \) are shown in the Appendix C, \( x(k) \) is the state vector; \( u(k) \) is the control input vector; and \( y(k) \) is the output vector defined below.

\[
x(k) = [m_{\text{pre}}^\text{pre}(k), m_{\text{pre}}^\text{f,\text{film}}(k), m_{\text{main}}^\text{main}(k), m_{\text{r,\text{CSP}}}^\text{main}(k)]^{T}
\]

\[
u(k) = [\theta_{\text{SPK}}(k), m_{\text{pre}}^\text{pre}(k), m_{\text{pre}}^\text{f,\text{film}}(k), m_{\text{main}}^\text{main}(k), P_{\text{int}}, \Omega(k)]^{T}
\]

\[
y(k) = [\theta_{\text{CA50}}(k)]
\]

IV. CONTROLLER DEVELOPMENT

It can be found that the burn duration in the main-chamber is determined by the gas properties in both combustion chambers. In other words, CA50 can be controlled by spark timing and other variables in the pre-chamber. In this paper, three variables are used to control the main-chamber CA50. They are spark timing, \( \theta_{\text{SPK}} \); pre-chamber fuel and air injection amounts, \( m_{\text{pre}}^\text{f,\text{film}} \) and \( m_{\text{pre}}^\text{f,\text{film}} \). The other parameters,
where \( m_{\text{fuel}}^\text{main}(k), P_{\text{Int}}(k) \) and \( \Omega(k) \), are considered as disturbances. Fig. 3 shows the basic architecture of the control system. The feedforward controller is a model-based controller. The feedback controller uses output constraint control method to regulate the CA50 to the desired value. The main-chamber AFR is controlled by the main-chamber fuel injector using a map-based feedforward control as a function of intake manifold pressure.

### A. Feedforward Control

The feedforward controller calculates the value of pre-chamber air and fuel injection amounts and spark timing based on \( m_{\text{main}}^\text{main}(k), P_{\text{Int}}(k) \) and \( \Omega(k) \). The pre-chamber air injection quantity is determined by a pre-calibrated map. The pre-chamber fuel injection quantity and spark timing is obtained by solving

\[
\begin{align*}
\lambda_{\text{SPK}}^\text{pre} &= 1 \\
\theta_{\text{CA50}} & = \theta_{\text{CA50, desired}}
\end{align*}
\]

where \( \lambda_{\text{SPK}}^\text{pre} \) is the normalized air-fuel ratio in the pre-chamber at spark timing. The detailed expression of these equations can be easily obtained based on the discussion in Section III, as follows

\[
\begin{align*}
\theta_{\text{SPK}}^\text{pre} &= \frac{\varphi_s}{m_{\text{main}}^\text{main,SPK} + m_{\text{inj,air}}^\text{pre} + (1-k)m_{\text{inj,fuel}}^\text{pre} + m_{\text{vap}}^\text{pre} \Delta \theta_{\text{SPK}} + \left( m_{\text{film},m}^\text{pre} + m_{\text{inj,fuel},i}^\text{pre}\right)[1 - \exp(-a_{\text{vap}}\Delta \theta_{\text{SPK}})]}
\end{align*}
\]

where

\[
\begin{align*}
\lambda_{\text{SPK}}^\text{pre} &= \varphi_s \\
\theta_{\text{CA50}}^\text{pre} &= \theta_{\text{SPK}}^\text{pre} + \Delta \theta_0 + a(P_{\text{Int}} - P_{\text{Int0}}) + b(\Omega - \Omega_0) \\
&+ c(\lambda - \lambda_0) + d(x_{\text{CSP}}^\text{main} - x_{\text{CSP0}}^\text{main}) - e(m_{\text{ur0}} - m_{\text{Spk}}^\text{pre}[1 - \left( \frac{P_{\text{POE}}^\text{pre} V_{\text{pre}}^\text{pre}}{R}\right) \frac{1}{2} (m_{\text{Spk}}^\text{pre} T_{\text{Spk}}^\text{pre} + m_{\text{fuel,vap}}^\text{pre} Q_{\text{LHV}}^\text{pre} C_v^\text{pre})^{-\frac{1}{2}})]
\end{align*}
\]

where

\[
\begin{align*}
m_{\text{Spk}}^\text{pre} &= m_{\text{r}}^\text{pre} + m_{\text{inj}}^\text{pre} + m_{\text{main,SPK}}^\text{pre}
\end{align*}
\]

Since Eqn. (23) is a highly nonlinear equation of \( m_{\text{inj,fuel}}^\text{pre} \) and \( \theta_{\text{SPK}}^\text{pre} \), it is difficult to solve the equations analytically. Numerical methods can be used, but the computational effort would be too high for real time engine control. By investigating Eqn. (23), it can be found that there are only two highly nonlinear terms that are functions of the two unknown variables. The other terms can be determined once the feedforward control inputs and \( m_{\text{inj,air}}^\text{pre} \) are given. The two terms are

\[
\begin{align*}
&f_1(P_{\text{Int}}^\text{pre,SPK}, m_{\text{inj,air}}^\text{pre}, \theta_{\text{SPK}}^\text{pre}) \\
&f_2(P_{\text{Int}}^\text{pre,SPK}, m_{\text{inj,air}}^\text{pre}, m_{\text{inj,fuel}}^\text{pre}, \theta_{\text{SPK}}^\text{pre})
\end{align*}
\]

Since the expression of \( f_1 \) and \( f_2 \) can be easily derived based on results in Section III, they are not shown here. Then, \( m_{\text{inj,fuel}}^\text{pre} \) and \( \theta_{\text{SPK}}^\text{pre} \) can be solved for any desired \( \theta_{\text{CA50}}^\text{pre} \) by replacing the two terms with their linear approximations.

\[
\begin{align*}
&f_1 = \frac{\partial f_1}{\partial \theta_{\text{SPK}}^\text{pre}} (\theta_{\text{SPK}}^\text{pre} - \theta_{\text{SPK0}}^\text{pre}) + f_1(\theta_{\text{SPK0}}^\text{pre}) \\
&f_2 = \frac{\partial f_2}{\partial m_{\text{inj,fuel}}^\text{pre}} (m_{\text{inj,fuel}}^\text{pre} - m_{\text{inj,fuel0}}^\text{pre}) + f_2(m_{\text{inj,fuel0}}^\text{pre}, \theta_{\text{SPK}}^\text{pre})
\end{align*}
\]

(27)

The values of \( m_{\text{inj,fuel}}^\text{pre} \) and \( \theta_{\text{SPK}}^\text{pre} \) come from the state estimator in the feedback controller.

The pre-chamber AFR needs to be maintained around the stoichiometry for good ignitability and stable combustion in the pre-chamber. For the same main-chamber condition, the AFR in the pre-chamber is determined by all three control inputs, but only two are independent. If two of them are given, the remaining one can be calculated according to the first equation in Eqn. (23). As a result, \( m_{\text{inj,air}}^\text{pre} \) is not used by feedback controller but calculated based on the other two controller outputs, \( \theta_{\text{SPK}}^\text{pre} \) and \( m_{\text{inj,fuel}}^\text{pre} \), in the ‘Air Compensation’ block of Fig. 3.

### B. Output Covariance Constraint Control

The nonlinear state-space engine model was first linearized by Matlab around the operational condition with IMEP=6 bar, \( \lambda=1.85 \), and \( \Omega=1500 \) rpm. The linearized TJI engine model can be expressed as

\[
\begin{align*}
\tilde{x}(k+1) &= \tilde{A}x(k) + \tilde{B}u(k) + \tilde{G}v(k) \\
\tilde{y}(k) &= \tilde{C}x(k) + \tilde{D}u(k) + \tilde{H}v(k) + \tilde{v}(k)
\end{align*}
\]

(28)

with the values of the matrices:

\[
\begin{align*}
\tilde{A} &= \\
\tilde{B} &= \\
\tilde{C} &= \\
\tilde{D} &=
\end{align*}
\]

(29)

where \( \tilde{u}(k) = [\theta_{\text{SPK}}^\text{pre}(k), m_{\text{inj,fuel}}^\text{pre}(k)]^T \) is the control input; and \( \tilde{v}(k) = [\tilde{m}_{\text{fuel}}(k), \tilde{P}_{\text{Int}}(k), \tilde{\Omega}(k)]^T \) is considered as disturbance. The overhead tilde denotes the deviation of the corresponding parameter from its reference value, for example,
$x(k) = x_0(k) + \tilde{x}(k)$, given $x_0(k)$ the reference value around which the model is linearized.

For the current control hardware, the value of CA50 takes one engine cycle to compute and transmit to the engine controller. Therefore, the measured CA50 is delayed by one engine cycle, resulting one more state,

$$x_d(k + 1) = \tilde{y}(k),$$  (30)

Accordingly, the actual plant model is

$$x_p(k + 1) = A_p x_p(k) + B_p u(k) + G_p w_p(k)$$
$$y_p(k) = C_p x_p(k) + v_p(k)$$  \hspace{1cm} (31)

where

$$A_p = \begin{bmatrix} \bar{A} & 0^{1 \times 1} \\ C & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix},$$
$$G_p = \begin{bmatrix} G & 0^{4 \times 1} \\ \bar{H} & 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0^{1 \times 4} & 1 \end{bmatrix}$$  \hspace{1cm} (32)

and the augmented state vector and the disturbance input vector are

$$x_p(k) = \begin{bmatrix} \tilde{x}(k) \\ x_d(k) \end{bmatrix}, \quad w_p(k) = \begin{bmatrix} \tilde{w}(k) \\ \bar{w}(k) \end{bmatrix}.$$  \hspace{1cm} (33)

The controller is designed so that the CA50 tracks the desired value $r(k)$. The tracking error is defined as

$$e(k) = r(k) - y_p(k).$$  \hspace{1cm} (34)

To eliminate the steady-state tracking error, an integral action is introduced by defining the integration of the tracking error as

$$e_i(k + 1) = e_i(k) + T_e e(k).$$  \hspace{1cm} (35)

For simplicity, $T_e$ is set to 1. The augmented state vector $x_i(k) = \begin{bmatrix} x_p(k) \end{bmatrix}, e_i(k)$ leads to the following augmented state equation

$$x_i(k + 1) = A_i x_i(k) + B_i u_i(k) + d r(k) + G_i w_i(k)$$
$$y_i(k) = C_i x_i(k)$$
$$z_i(k) = M_i x_i(k) + v_i(k)$$  \hspace{1cm} (36)

where

$$A_i = \begin{bmatrix} A_p & 0^{5 \times 1} \\ -C_p & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0^{5 \times 1} \\ 1 \end{bmatrix},$$
$$G_i = \begin{bmatrix} G & 0^{5 \times 1} \\ 0 & 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0^{1 \times 5} & 1 \\ C_p & 0 \end{bmatrix}$$  \hspace{1cm} (37)

and the vector $z_i(k)$ is the noisy measurements of $y_i(k)$. Therefore, $M_i = C_i$.

Suppose that the plant is controlled by a strictly proper output feedback stabilizing control law given by

$$x_c(k + 1) = A_c x_c(k) + F_z z_i(k)$$
$$u_i(k) = K_c x_c(k) + K_r r(k)$$  \hspace{1cm} (38)

Then, the closed-loop system becomes

$$x(k + 1) = A x(k) + D_r r(k) + D w(k)$$
$$y_i(k) = C x(k)$$
$$u_i(k) = C_u x(k) + K_r r(k)$$  \hspace{1cm} (39)

where $x(k) = \begin{bmatrix} x_i(k) \\ x_c(k) \end{bmatrix}$, $w(k) = \begin{bmatrix} w_i(k) \\ v_i(k) \end{bmatrix}$. The closed-loop system matrices $A$, $C$, $D$, and $D_r$ can be easily obtained based upon the above equations.

Consider the closed-loop system above. Let $W_i$ and $V_i$ denote positive symmetric matrices with dimension compatible to the process noise $w_i$ and $v_i$. Defined $W = \text{block diag} \begin{bmatrix} W_i & V_i \end{bmatrix}$ and $X$ the closed-loop controllability Gramian from the weighted disturbance input $W^{-1/2} w$. Since $A$ is stable, $X$ is the positive semi-definite solution of the following Lyapunov equation

$$X = AXA^T + DWD^T.$$  \hspace{1cm} (40)

Rewrite the performance output vector $y_i$ into $y_i := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where $y_j = C_j x$ for $j = 1, 2$.

Then, the output covariance constraint (OCC) problem is to find an output feedback controller for the plant that minimizes the OCC cost

$$J_{OCC} = \text{trace} \ R C_u X C_u^T,$$  \hspace{1cm} (41)

subject to the output covariance constraints

$$Y_j = C_j X C_j^T < Y_j, \quad j = 1, 2,$$  \hspace{1cm} (42)

where $Y_j > 0$ ($j = 1, 2$) are given.

The OCC problem is actually a linear quadratic (LQ) control problem with a special choice of output-weighting matrix $Q$. The value of $Q$ is obtained by the iteration algorithm proposed in [20]. According to [21] and [20], the feedback controller is in the following form.

$$x_c(k + 1) = (A_i + B_i K_c - F M_p)x_c(k) + F_z z_i(k)$$
$$u_i(k) = K_c x_c(k) + K_r r(k)$$  \hspace{1cm} (43)

Note that

$$F = A_i K M^T_i (V + M_i K M^T_i)^{-1},$$  \hspace{1cm} (44)

where $K$ is the solution to the algebraic Riccati equation

$$K = A_i [K - K M^T_i (V_i + M_i K M^T_i)^{-1} M_i K] A_i^T + D_i W_i D_i^T;$$  \hspace{1cm} (45)

control gain $K_c$ is provided by

$$K_c = -(R + B_i^T S B_i)^{-1} B_i^T S A_i,$$  \hspace{1cm} (46)

where $S$ is the solution to the algebraic Riccati equation

$$S = A_i^T (S - B_i (B_i^T S B_i + R)^{-1} B_i^T S) A_i + C_i^T Q C_i;$$  \hspace{1cm} (47)

and

$$K_r = (B_i^T S B_i + R)^{-1} B_i^T F_1,$$  \hspace{1cm} (48)

where

$$F_1 = -(I - A_i^T + A_i^T S F_3 F_2)^{-1} A_i^T S F_3 F_2 d$$  \hspace{1cm} (49)

and

$$F_2 = B_i^T R^{-1} B_i, \quad F_3 = (I + F_2 S)^{-1}.$$  \hspace{1cm} (50)

The OCC problem has several different interpretations. In a stochastic point of view, first assume that $w_i$ and $v_i$ are uncorrelated zero-mean white noises with intensity matrices $W_i > 0$ and $V_i > 0$. Let $E$ be the expectation operator.
and define \(E_\infty[\cdot] := \lim_{k \to \infty} E_k[\cdot]\), it is easy to see that

OCC problem minimizes \(E_\infty[u_\ell(k) R_u(k)]\) subject to the OCCs \(Y_j := E_\infty[y_j(k)y_j^T(k)] < \bar{Y}_j\), \(j = 1, 2\). These constraints can be interpreted as constraints on the variance of the performance variables or lower bounds on the residence time of the performance variables [22].

In a deterministic point of view, first define the \(\ell_\infty\) and \(\ell_2\) norms as follows

\[
\|y_j\|_{\ell_\infty}^2 := \sup_{k \geq 0} y_j^T(k) y_j(k)
\]

\[
\|w\|_{\ell_2}^2 := \sum_{k=0}^{\infty} w^T(k) w(k),
\]

and the \(\ell_2\) disturbance set

\[
W := \\{ w \in R^{n_w} \text{ s.t. } \|W^{-1/2} w\|_{\ell_2}^2 \leq 1 \},
\]

where \(W > 0\) is a real symmetric matrix. Then, for any \(w \in W\), the following inequalities hold

\[
\|y_j\|_{\ell_\infty}^2 \leq [Y_j], \quad j = 1, 2,
\]

\[
\|w\|_{\ell_2}^2 \leq [C_u X C_u^T]_{jj}, \quad j = 1, 2,
\]

where \([\cdot]_{jj}\) is the \(j\)th diagonal entry. Moreover, [23] and [24] show that the above bounds are the least upper bounds that hold for any arbitrary signal \(w \in W\).

In other word, if the control weighting matrix is defined as

\[
R := \text{diag}[r_1, r_2, \ldots, r_n_u],
\]

where \(n_u\) is the dimension of \(u\) (in this paper, \(n_u = 2\)), the OCC problem minimizes the sum of the worst-case \(\ell_\infty\) norms on the control signals given by

\[
J_{OCC} = \sum_{j=1}^{n_u} r_j \left\{ \sup_{w \in W} \|u_j\|_{\ell_\infty}^2 \right\}
\]

subject to the constraints on the worst-case \(\ell_\infty\) norms on the performance variables of the form

\[
\sup_{w \in W} \|y_j\|_{\ell_\infty}^2 \leq \bar{Y}_j, \quad j = 1, 2.
\]

Note that the corresponding cost function of the LQ controller is defined as

\[
J = \sum_{k=0}^{\infty} [y_i(k)^T Q y_i(k) + u_i(k)^T R u_i(k)]
\]

### C. Baseline Controller

To verify the performance of designed controllers, several different baseline controllers were also designed. The first kind of baseline controller is the proportional and integral (PI) controller.

\[
G_{PI}(z) = K_P + \frac{K_I}{z - 1}
\]

The proportional and integral gains ("P" and "I") of the PI controller are determined by three different methods: Ziegler-Nichols (ZN), Modified Ziegler-Nichols (MZN) and Tyreus-Luyben (TL), see references [25] and [26]. Their values are shown in Table II. PI controllers use only spark timing, \(\theta_{SPK}\), to control the CA50. An OCC baseline controller is also developed only using \(\theta_{SPk}\) as the controller output. The pre-chamber fuel and air injection amounts are determined by the feedforward controller.

### V. Experimental Validation

#### A. Model Validation

Experiments were conducted using the single cylinder DM-TJI engine described in Section II. A TMAP (temperature and manifold air pressure) sensor is mounted in the intake manifold to measure the pressure and temperature. Two Kistler pressure sensors are used to measure the main-chamber and exhaust manifold pressures. Another Kistler pressure sensor, integrated with the spark plug, is used to measure the pre-chamber pressure; see Fig. 1. The exhaust \(\lambda\) (AFR) is measured by an ECM lambda meter, and the CA50 is calculated by the combustion analysis system from A&D technology in real time. The experimental data used to calibrate and validate the model are shown in Fig. 4. The engine speed ranges from 1200 to 2000 rpm; and IMEP (indicated mean value pressure) ranges from 4.5 to 7.2 bar. The pre-chamber fuel injection amount varies from 0.7 to 1.35 mg.

Direct measurement of the four states in the state-space model is not possible on the TJI engine. In the authors’ previous research (see [18] and [15]) a crank-angle-resolved (CAR) TJI engine model was developed. It is able to calculate all the relevant parameters every crank angle degree. Since the CAR engine model was calibrated and validated by the experimental data under different steady-state operational conditions, the state-space model can be validated using the CAR engine model. The simulation results are shown in Fig. 5.

Fig. 5 compares the values of the four states calculated from the state-space engine model and those from the CAR engine model at different operational conditions. The two dash-dot lines on each side of the data points show the 5% error bound. All of the four states calculated by the state-space model match the CAR model very well.

The calculated CA50 values are compared with the experiment results in Fig. 6. The sensitivity of some control variables can also be found in Fig. 6, where the main chamber air-to-fuel ratio is related to \(m_{\text{fuel}}\max(k)\) and \(m_{\text{tur}}\) is related to both \(\theta_{SPK}(k)\) and \(m_{\text{pre},\text{min}}(k)\). Since \(m_{\text{fuel}}\) depends on engine load, both \(\theta_{SPK}(k)\) and \(m_{\text{pre},\text{min}}(k)\) are used as control inputs. The upper-left plot compares the experimental and

<table>
<thead>
<tr>
<th>PI controller parameters</th>
<th>P Gain</th>
<th>I Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ziegler-Nichols</td>
<td>0.427</td>
<td>0.128</td>
</tr>
<tr>
<td>Modified Ziegler-Nichols</td>
<td>0.188</td>
<td>0.090</td>
</tr>
<tr>
<td>Tyreus-Luyben</td>
<td>0.293</td>
<td>0.033</td>
</tr>
</tbody>
</table>

Fig. 4. Experimental matrix.
Fig. 5. State-space engine model validation.

Fig. 6. Comparison between experimental and calculated CA50 under different operational conditions.

calculated CA50 under five different operational conditions. These conditions differ from each other mainly by their main-chamber λ. The other parameters, P_{int}, Ω, \( x_{CSF} \), and \( m_{tur} \) in Eqn. (20) remain unchanged or within small variations. The upper-right and bottom-left plots are obtained similarly by mainly changing \( P_{int} \) and \( m_{tur} \), respectively. However, varying one parameter while maintaining other parameters unchanged is challenging during the experiment. For example, all the parameters, except Ω, will be changed when changing \( P_{int} \). This is why the simulation points cannot form a straight line in the upper-right plot. However, the trend of the curve is evident: increasing \( P_{int} \) leads to reduced burn duration. The last plot compares CA50 values for all the experimental conditions. The dash-dot line shows the 10% error bound.

Fig. 7 shows the CA50 values when the pre-chamber fuel injection amount changes from 1.9 to 1.1 mg. The experimental CA50 data have a lot of noise, so it is hard to see the transient response of CA50 from a single set of experimental data. Therefore, the experimental data in Fig. 7 are obtained by averaging the results of 14 identical experiments. It can be seen that the CA50 does not reach its steady-state value immediately after reducing the pre-chamber fuel injection. The CA50 right after the engine cycle of fuel injection change is about 1 crank angle degree lower than the steady-state value and it takes a few engine cycles to reach the steady state. Both are caused by prechamber fuel vaporization (from fuel film on the prechamber wall), residual gas and thermal dynamics.

B. Controller Validation

In the experiment, the controller is implemented into the MotoTron controller shown in Fig. 8. The pressure and other signals from the engine are captured by the Phoenix AM module from A&D Technology. The Phoenix RT module calculates the CA50 in real-time and sends it to MotoTron controller through the CAN bus.

Fig. 9 compares the experimental results when the controller is turned on and off. When the OCC controller is turned on, CA50 is adjusted close to the optimal value. The standard deviation (S.D.) of CA50 is reduced from 1.58 to 1.12 deg after the controller is turned on. The S.D. of IMEP is also reduced accordingly. This indicates that the combustion phasing controller is able to improve the combustion stability by regulating the CA50 close to its optimal value.

The OCC controller is compared with the PI baseline controllers in Fig. 10. The settling times for the MZN and TL PI controllers are very long compared with the OCC controller. Since the ZN controller has higher PI gains than these of MZN and TL controllers, the settling time is improved. However,
The high gain results in large steady state CA50 variations. This is because PI controllers do not use the model-based feedforward control developed for the OCC controller; they are not model-based and do not consider the engine cycle-to-cycle dynamics, disturbances and measurement noises. However, the OCC controller utilizes all the plant information for feedforward and feedback control, and thus, has better performance.

In Fig. 11, the OCC controller is compared with the OCC baseline controller, which does not use pre-chamber fuel injection amount, $m_{\text{inj, fuel}}^{\text{pre}}$, as the control input. The CA50 takes longer time to track the desired value. Although it takes a similar number of engine cycles to track the desired CA50 for both cases, the combustion is less stable when only using spark timing as the control input, which can be observed from the standard deviation of CA50 in Fig. 11. The reason is quite obvious. Since $m_{\text{inj, fuel}}^{\text{pre}}$ is not used for closed loop control, the controller relies only on adjusting spark timing to track the desired CA50. This is why the variation of spark timing of the baseline OCC controller is much larger than the proposed OCC controller; see the bottom plot in Fig. 11. To achieve similar settling time, the baseline OCC controller needs a larger controller gain than the proposed OCC controller, resulting in significant CA50 variations. It should also be noted that the OCC baseline controller has better performance than that of other PI controllers. This further demonstrates the benefit of using the model-based OCC control scheme.

Finally, Fig. 12 shows the experiment results under different operational conditions (engine speed: 1200 and 2000 rpm; IMEP: 6.5 and 4.55 bar) with the same controller parameters. Overall, the OCC controller is able to track the desired CA50 within 10 engine cycles. Fig. 13 compared the standard deviations of the CA50 with different controllers discussed in this paper. The proposed and baseline OCC controllers have overall lower S.D. values than the PID controllers. The proposed OCC controller has much lower value than the baseline OCC controller because of utilization of pre-chamber fuel as a control input.

**VI. Conclusions**

In this paper, a cycle-by-cycle state-space TJI engine model is first developed and validated using both experimental data and high fidelity model. The validation results show that the CA50 (crank location when 50% fuel is burned) modeling error is less than 10%. An OCC (output covariance constraint) combustion phasing controller is developed based on the state-space engine model. The controller uses both spark timing and pre-chamber fuel quantity as the control inputs. Experimental results show that the proposed OCC controller has better performance than the baseline controllers with a minimal improvement of over 22% for CA50 standard deviation. This demonstrates that utilizing multiple pre-chamber control variables and OCC control scheme is able to improve the combustion phasing control performance. The future work is...
to study if further performance improvement can be achieved by utilizing linear-parameter-varying (LPV) control based on an LPV model that captures the engine dynamics accurately.

**APPENDIX A**

**MAIN-CHAMBER MODEL DEVELOPMENT**

From IVO to IVC, the volume occupied by residual gas expands due to the pressure change from $P_{\text{main}}^{\text{IVO}}$ to $P_{\text{main}}^{\text{IVC}}$.

$$V_{\text{r,IVC}}^{\text{main}}(k) = V_{\text{r,IVO}}^{\text{main}}(k) \left( \frac{P_{\text{main}}^{\text{IVO}}(k)}{P_{\text{main}}^{\text{IVC}}(k)} \right) \pi$$

(58)

where $V_{\text{r,IVO}}^{\text{main}}$ is the total volume of the residual gas at IVO that is approximated by the cylinder volume at IVO.

The volume of the fresh air-fuel mixture flowing into the main-chamber can be calculated. Then, the mass of the fresh air-fuel mixture and the average in-cylinder temperature can be calculated by ideal gas law:

$$m_{\text{fresh}}^{\text{main}}(k) = \frac{P_{\text{main}}^{\text{IVO}}(k)[V_{\text{main}}^{\text{IVO}}(k) - V_{\text{r,IVC}}^{\text{main}}(k)]}{R T_{\text{IVO}}(k)}$$

(59)

$$T_{\text{IVC}}^{\text{main}}(k) = \frac{P_{\text{main}}^{\text{IVO}}(k) V_{\text{main}}^{\text{IVO}}(k)}{m_{\text{main}}^{\text{IVO}}(k) R}$$

(60)

where $R$ is the gas constant and $m_{\text{main}}^{\text{IVO}}(k)$ is the total mass inside the main-chamber at IVC:

$$m_{\text{main}}^{\text{IVO}}(k) = m_{r}^{\text{IVO}}(k) + m_{\text{fresh}}^{\text{main}}(k)$$

(61)

**APPENDIX B**

**PRE-CHAMBER MODEL DEVELOPMENT**

Before IVC, the gas flow through the orifice connecting the two chambers mainly goes from the pre-chamber to the main-chamber. Therefore, the gas mixture in the pre-chamber is mainly the residual gas left from last cycle. At IVC, it is assumed that the pressures in the two chambers and the intake manifold are the same.

During the compression stroke, the residual gas in pre-chamber is compressed from $P_{\text{IVC}}$, $V_{\text{r,SPK}}$ to $P_{\text{SPK}}$, $V_{\text{r,SPK}}$. It is assumed that $P_{\text{SPK}}$ is the same as $P_{\text{main}}^{\text{IVO}}$. The mass of the residual gas $m_{r}^{\text{pre}}$ remains unchanged.

$$V_{\text{r,SPK}}^{\text{pre}}(k) = \left( \frac{P_{\text{IVO}}(k)}{P_{\text{SPK}}^{\text{pre}}(k)} \right)^{\frac{1}{\gamma}} V_{\text{pre}}^{\text{pre}}$$

(62)

Then $T_{\text{r,SPK}}^{\text{pre}}$ can be obtained from ideal gas law.

The volume and mass of the gas coming from the main-chamber at SPK can be then calculated by the following equations:

$$V_{\text{main,SPK}}^{\text{pre}}(k) = V_{\text{pre}}^{\text{pre}} - V_{\text{r,SPK}}^{\text{pre}}(k) - V_{\text{inj}}^{\text{pre}}(k)$$

(63)

where $V_{\text{inj}}^{\text{pre}}$ is the total volume of the injected air and vaporized fuel. And $m_{\text{main,SPK}}$ can be then calculated.

The average temperature in pre-chamber at SPK is:

$$T_{\text{SPK}}^{\text{pre}}(k) = \frac{P_{\text{SPK}}^{\text{pre}}(k) V_{\text{pre}}^{\text{pre}}}{m_{\text{SPK}}^{\text{pre}}(k) R}$$

(64)

$$m_{\text{SPK}}^{\text{pre}}(k) = m_{\text{inj}}^{\text{pre}}(k) + m_{r}^{\text{pre}}(k) + m_{\text{main,SPK}}^{\text{pre}}(k)$$

(65)

where $m_{\text{inj}}^{\text{pre}}$ is the total mass of the injected air and vaporized fuel in pre-chamber.

**APPENDIX C**

**NONLINEAR STATE-SPACE EQUATION**

$$m_{r}^{\text{pre}}(k + 1) = P_{\text{main}}^{\text{IVC}}(k + 1) \frac{2}{\gamma} V_{\text{pre}}^{\text{pre}}$$

$$\left( P_{\text{SPK}}^{\text{pre}}(k) V_{\text{pre}}^{\text{pre}} + Q_{\text{LHV}} R m_{\text{fuel,ap}}^{\text{main}} \right)^{\frac{1}{\gamma}} \{ m_{\text{inj}}^{\text{pre}}(k) + \}$$

$$\left[ m_{r}^{\text{pre}}(k) + \frac{m_{\text{main,SPK}}^{\text{pre}}(k)}{R} \left( V_{\text{pre}}^{\text{pre}} - \frac{m_{\text{inj}}^{\text{pre}}(k) P_{\text{SPK}}^{\text{pre}}}{R} \right) \right] \}$$

$$m_{r}^{\text{film}}(k + 1) = \left[ m_{r}^{\text{pre}}(k + 1) + m_{\text{inj,wall}}(k) \right] \exp \left\{ \Delta t \left[ - \frac{M_{f} P}{\delta_{f} P_{f}} \frac{P_{f}}{R_{f}} \exp \left\{ \frac{\Delta H}{R_{f}} \right\} \right] \right\}$$

$$m_{r}^{\text{main}}(k + 1) = P_{\text{IVC}}^{\text{IVO}}(k + 1) \frac{2}{\gamma} - 1 R^{\frac{1}{\gamma}} V_{\text{pre}}^{\text{EVC}}$$

$$\left[ \frac{m_{\text{main}}^{\text{IVO}}(k) V_{\text{IVO}}^{\text{main}}(k)}{R} \left( V_{\text{EVC}}^{\text{main}}(k) \right)^{\frac{1}{\gamma}} + Q_{\text{LHV}} m_{\text{fuel}}^{\text{main}}(k) \right]$$

(66)

where $\theta_{CA} = \theta_{\text{CA}} + \Delta \theta_{s} + \theta_{\text{int}} - \Delta \theta_{\text{int}} + b(\Omega - \Omega_{0}) + c(\lambda - \lambda_{s}) + d_{x_{\text{CSP}}}^{\text{main}} - x_{\text{CSP}}^{\text{main}} + e_{c_{\text{inj}}}^{\text{main}} - m_{\text{urb}}^{\text{main}}(k + 1) + m_{\text{inj}}^{\text{main}}(k + 1) R^{\frac{1}{\gamma}} V_{\text{EVC}}^{\text{main}}$$

(67)

$$m_{\text{SPK}}^{\text{pre}} = \frac{m_{r}^{\text{pre}} + m_{\text{inj}}^{\text{pre}}}{R} \left[ m_{r}^{\text{pre}} \frac{P_{\text{IVO}}^{\text{pre}}}{m_{\text{SPK}}^{\text{pre}}^{\text{main}}} \right] \}$$

(68)

**APPENDIX D**

**ENGINE MODEL PARAMETERS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Eqn.</th>
<th>Value</th>
<th>Symbol</th>
<th>Eqn.</th>
<th>Value</th>
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<tr>
<td>$C_{1}$</td>
<td>(9)</td>
<td>0.85</td>
<td>$a$</td>
<td>(20)</td>
<td>-60.9 deg/bar</td>
</tr>
<tr>
<td>$OF$</td>
<td>(9)</td>
<td>1.20 deg/m</td>
<td>$c$</td>
<td>(20)</td>
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<tr>
<td>$\delta_{f}$</td>
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<td>0.03 mm</td>
<td>$d$</td>
<td>(20)</td>
<td>-660 deg</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>(20)</td>
<td>-0.0105 s</td>
<td>$b$</td>
<td>(20)</td>
<td>0.0105 s</td>
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