Covariance active/passive control

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ABSTRACT

An integrated means for active controller design and structure redesign is presented. The tech-
niques of covariance control are used to parametrize all possible combinations of active con-
trollers/structure redesign parameters which can stabilize the plant, and achieve certain closed-loop
performance.

1. INTRODUCTION

To cope with the demanding task of structural control problems, one can improve performance
by combining both the design of active controllers and the design of the structure. The term
"Smart Structures" implies the use of feedback control sensors and actuators imbedded in a material
structure to improve the dynamic response. Until now, the design of the controller parameters and
the structural material parameters have not been integrated and combined to guarantee any specific
performance. Existing methods can choose the controller and structure parameters by trial and
error or by gradient approaches to nonlinear programming problems.

These computationally intensive approaches are devoid of physical insight, do not guarantee
stability, and make no attempt to produce all stabilizing solutions. Such is the goal of this paper,
to show the explicit relationship between all stabilizing state feedback control gains and the struc-
tural parameters. If an initial structure is given, and if a controller which satisfies performance
requirements (closed-loop stability, tracking accuracy... etc.) is given, the necessary and sufficient
condition is known \(^6^7\) for the existence of structure redesign parameters to duplicate the closed-
loop system performance while minimizing the active control effort. This condition derived from
the above setting is convex in the structural redesign parameters, and hence a global minimum is
guaranteed, as well as stability.

The drawback in this past approach \(^6^7\) is that we are given a controller before hand, hence the
space in which we search for the optimum is necessarily restricted by this fact. In other words, it
is possible that beginning with another controller, we can reduce the control effort even more. We
seek to simultaneously redesign the structure and the controller. The above algorithm \(^6^7\) does not
necessarily solve this problem, even if applied iteratively.

To obviate this difficulty we will use the covariance control technique \(^8^2^5\), which provides a way
to parametrize all stabilizing controllers in terms of a physically meaningful state covariance \(X\),
and the stabilizability conditions derived in the theory for active control forms a parametrization of
the set of all assignable covariances as function of structure parameters only (without the control
In this paper, we first present the problem formulation in section 2, and the main results are introduced in section 3. We give 2 examples in section 4, and in section 5, a short discussion of future directions is given. All proofs are given in appendices.

2. PROBLEM FORMULATION

We shall limit our attention to linear systems, although some advantages of our approach extend also to nonlinear systems. Assume that the equations of motion for the linear elastic structure have been put into the finite dimensional state form.

\[ \dot{z} = Ax + Bu + Dw \]  
\[ u = G_\alpha x \]

Where \( w(t) \) is a zero mean white noise (including actuator noise) with intensity \( W \), and \( G_\alpha \) is the state feedback control gain to be designed.

For a given state covariance \( X > 0 \), the necessary and sufficient condition for the existence of a \( G_\alpha \) which assigns this \( X \) was derived by Yasuda and Skelton\(^8\) as

\[
(I - BB^+)Q(I - BB^+) = 0
\]

\[ Q \triangleq AX + XA^* + DWD^* \]

and the set of \( G_\alpha \) that satisfies the requirement is parametrized as:

\[
G_\alpha = -\frac{1}{2}B^+Q(2I - BB^+)X^{-1} + B^+SB^+X^{-1} + (I - BB^+)Z
\]

Since \( w \) includes actuator noise, then \( D \) has the structure \( D = [B, D_2] \), and stabilizability (controllability) of \((A,B)\) implies stabilizability (controllability) of \((A,D)\). We assume controllability of \((A,B)\) to simplify the presentation. If \((A,B)\) is controllable, then \( X > 0 \) is equivalent to \((A + BG_\alpha)\) stable. Hence, since (3) parametrizes all \( X > 0 \) that can be assigned to the system. Condition (3), with \( X > 0 \), is also a necessary and sufficient condition for stability of the closed loop system.

The significance of state covariance is well known. A system must be stable to have a bounded covariance, and almost all robustness properties of linear systems (disturbance rejection, structured and unstructured parameter robustness) can be related directly to properties of the state covariance\(^4,1\).

Note that the control gain \( G \) in (4) is an explicit function of the covariance \( X \) and the plant (structure) data \((A,B)\). If we can relate a set of closed-loop performance requirements to an \( X > 0 \), then all control gains that assign this \( X \) to the system are given by (4). Let \( A_0 \) denote the original structure, and \( G_p \) denote the changes in the structural parameters that are allowed. Now if we write \( A \) (system matrix) as

\[ \mathbf{A} \]
\[ A = A_0 + B_p G_p M_p \]  

(5)

The structure of the connectivity matrices \( B_p, M_p \) allow changes in the system matrix to be accomplished in a physically achievable way.

We call \( G_p \) the “passive controller.” The closed-loop system matrix looks like so,

\[ A_{cl} = A_0 + B_p G_p M_p + B G_a . \]  

(6)

The question is, given a desired state covariance \( X > 0 \), is there a \((G_p, G_a)\) pair to assign this \( X \)? This is equivalent to asking the stabilizability question, since every stable system has a finite positive \( X > 0 \). If it is stabilizable, we desire the set of all \((G_p, G_a)\) which stabilize the system.

3. MAIN RESULT

Theorem 1 A state feedback system

\[ \dot{x} = (A + B_p G_p M_p)x + Bu \]

\[ u = G_a x \]

is stabilizable by some \( G_p, G_a \) iff there exists \( X > 0 \) satisfying

\[
\begin{align*}
P_\beta P Q P P_\beta &= 0 \\
P_M P Q P P_M &= 0 \\
[(I - P_M \beta \beta^+)(P_M \beta \beta^+)^+] P_M P Q P &= 0
\end{align*}
\]  

(7)

where

\[
\begin{align*}
P &\triangleq (I - BB^+), & Q &\triangleq (X A_0^* + A_0 X + DWD^*) \\
\beta &\triangleq P B_p, & M &\triangleq M_p X P \\
P_\beta &\triangleq (I - \beta \beta^+), & P_M &\triangleq (I - M^+ M)
\end{align*}
\]

Proof. See Appendix A.

The conditions shown in (7) are similar to the covariance assignability conditions derived in for the measurement feedback system. This similarity is due to the fact that the plant redesign part \( B_p G_p M_p \) is mathematically equivalent to measurement feedback, and we take advantage of this fact in our proof for Theorem 1. Next, compare (3) (interpret with \( A_0 \) in lieu of \( A \)) with (5)-(7), we can see that the set of assignable \( X \) is enlarged because of the added flexibility of plant redesign. (Conditions (5)-(7) are less restrictive than (3). Any \( X \) which satisfies (3) will automatically satisfy (5)-(7), but the reverse is not true.)

The next theorem gives the parametrization of active and passive covariance controllers.
Theorem 2 Suppose $X$ is assignable. Then all $(G, G_a)$ that assign $X$ to the system are given by:

$$G_a = -\frac{1}{2} B^+ [Q + B_p G_p M_p X + X M_p^* G_p^* B_p^* + L_a^+ \Phi - (L_a^+ \Phi)^* - L_a^+ L_a^+ L_a] X^{-1} + B^+ (I - L_a^+ L_a) S_a (I - L_a^+ L_a)^+ + L_a Z_a$$

$$G_p = -\frac{1}{2} \beta^+ [P Q P + L_p^+ \Phi_p - (L_p^+ \Phi_p)^* - L_p^+ \Phi_p L_p^+ L_p] M^+ + \beta^+ (I - L_p^+ L_p) S_p (I - L_p^+ L_p)^+ M^+ + Z_p - \beta^+ Z_p M M^+$$

(8)

Where

$$L_a \triangleq P = (I - BB^+) \quad \Phi \triangleq -P [Q + B_p G_p M_p X + X M_p^* G_p^* B_p^*]$$

$$L_p \triangleq \begin{bmatrix} I & -\beta \beta^+ \\ -\beta \beta^+ & I - M^+ M \end{bmatrix} \quad \Phi_p \triangleq \begin{bmatrix} -I + \beta \beta^+ \\ -I + M^+ M \end{bmatrix} P Q P$$

(9)

$P, Q, M$ are defined as in Theorem 1. $S_a, S_p$ are arbitrary skew-symmetric matrices, $Z_a, Z_p$ are arbitrary matrices of proper dimension.

Proof. See Appendix B.

Note that we have 4 free parameters, i.e., $(S_a, S_p, Z_a, Z_p)$ in the characterization of $(G_p, G_a)$. These free parameters provide us with additional freedom for further optimization of a secondary objective (for example, searching for lowest fuel consumption or highest precision etc.), without changing the closed-loop state covariance.

In the following examples, we will show that with the additional freedom of plant redesign, we can achieve closed-loop performance which is not feasible with active control alone.

4. EXAMPLE

Example 1. Single Mass, Spring and Damper

The original system with active control only is

$$\dot{x} = Ax + Bu + Dw$$
$$u = G_a M a x$$
$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad M_a^T = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad M_p^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From (3), we can derive the assignable set of covariances as

$$X_a = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{3} \end{bmatrix}; \quad a > 0$$

where $X_a$ satisfies
\[ 0 = X_a(A + B G_a M_a)^T + (A + B G_a M_a)X_a + DD^T. \]

With simultaneous plant/control design
\[ \dot{x} = (A + B_p G_p M_p)x + B_a u + Dw, \]
the set of assignable closed loop covariances is
\[ X = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad a_1 > 0, \; a_2 > 0. \]

where \( X \) satisfies
\[ 0 = X(A + B_p G_p M_p + B_a G_a M_a)^T + (A + B_p G_p M_p + B_a G_a M_a)X + DD^T. \]

It is obvious that the set of all assignable covariances by active control gain \( G_a \) is included in the set by the simultaneous plant/control design, i.e., the addition of plant redesign has enlarged the set of assignable covariances, and consequently, the closed-loop performance capability.

**Example 2.** Euler Bernoulli Beam, 2 modes.

Consider
\[ \dot{x} = (A + B_p G_p M_p)x + B_a u + Dw \]
where
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -0.01 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -16 & -0.04
\end{bmatrix}
\quad B_a = D = \begin{bmatrix} 0 \\ 0.5878 \\ 0 \\ 0.955 \end{bmatrix}
\]
\[
M_p = B_p^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

The choice of \( M, B \) corresponds to a damping mechanism to change the damping of each mode.

Given an assignable closed-loop covariance matrix:
\[
X = \begin{bmatrix}
0.1183 & 0 & 0.0013 & -0.0296 \\
0 & 0.1538 & 0.0296 & 0.047 \\
0.0013 & 0.0296 & 0.0264 & 0 \\
-0.0296 & 0.047 & 0 & 0.4484
\end{bmatrix}
\]

From Theorem 2, eq. (9), we compute
\[
G_p = \begin{bmatrix}
-0.7236 & 0.4472 \\
0.4472 & -0.2764
\end{bmatrix} + \begin{bmatrix} 0.2764 & 0.4472 \\ 0.4472 & 0.7236 \end{bmatrix} Z_p.
\]
Substitute $G_p$ into $G_a$, then, from (8),

$$G_a = [-0.5 \ -0.4702 \ -1 \ -0.7608] + G_{a2}S_aG_{a3} + G_{a4}Z_pG_{a5} + G_{a6}Z_p^TG_{a7}$$

Where $S_a$ is arbitrary skew symmetric and $Z_p$ arbitrary.

$$G_{a2} = \begin{bmatrix} 0 & 0.4702 & 0 & 0.7608 \ 0 & 0 & 0 & 0 \ 0.2248 & 1.9775 & -2.2228 & 0.8048 \ 0 & 0 & 0 & 0 \ \end{bmatrix}$$

$$G_{a3} = \begin{bmatrix} 0 & 0 \ 0 & 0 \ 0.3637 & 3.1998 \ \end{bmatrix}$$

$$G_{a4} = \begin{bmatrix} -0.2351 & -0.3804 \ -0.0517 & 1.5453 & 0.5111 & -0.1850 \ -0.1737 & -1.5278 & 1.7173 & 1.3782 \ \end{bmatrix}$$

$$G_{a5} = \begin{bmatrix} -0.0547 & -0.1816 \ -0.2248 & 1.9775 & -2.2228 & 0.8048 \ 0.3637 & 3.1998 \ \end{bmatrix}$$

Hence $G_p(Z_p)$ and $G_a(Z_p, S_a)$ form all possible combinations of active and passive controllers that are stabilizing.

5. CONCLUSION

All controllers (passive and state feedback active) that assign a specified covariance are parametrized. These parametrizations are expressed in terms of system matrices only. The next step will be to expand the theory to measurement feedback system and systems with dynamic controllers, and to apply the theory to specific smart structure problems.

6. APPENDIXES

Appendix A. Proof of Theorem 1.

For state feedback system (1)-(2) if $X > 0$ is assignable, we have

$$(I - B B^+)(X A^* + A X + D W D^*)(I - B B^+) = 0 \quad (10)$$

Now let $A = A_o + B_p G_p M_p$.

From (10). $\Rightarrow$

$$(I - B B^+)(X(A_o + B_p G_p M_p)^* + (A_o + B_p G_p M_p)X + D W D^*)(I - B B^+) = 0 \quad (11)$$

Let $P \triangleq (I - B B^+)$  $Q \triangleq (X A_o^* + A_o X + D W D^*)$.

From (11). $\Rightarrow$

$$P (X M_p^* G_p^* B_p^* + B_p G_p M_p X) P = -P Q P$$
\[ PB_p G_p M_p X P = -\frac{1}{2} (PQP + S_k) \quad S_k^* = -S_k \]  \hfill (12)

Let \( \beta \triangleq P B_p, \mathcal{M} \triangleq M_p X P \).

\[ \Rightarrow \beta G_p \mathcal{M} = -\frac{1}{2} (PQP + S_k) \]  \hfill (13)

For \( G_p \mathcal{M} \) to have solution:

\[ (I - \beta \beta^+) (PQP + S_k) = 0 \]  \hfill (14)

For \( \beta G_p \) to have solution

\[ (PQP + S_k)(I - \mathcal{M}^+ \mathcal{M}) = 0 \]  \hfill (15)

It is easy to prove that (14), (15) are necessary and sufficient conditions for the existence of \( G_p \).

Next, solve \( S_k \), because \( S_k \) is skew-symmetric.

\[ \Rightarrow jS_k \text{ is Hermitian.} \]

\[ From \ (14) \Rightarrow (I - \beta \beta^+)(jS_k) = -j(I - \beta \beta^+)PQP \]  \hfill (16)

\[ From \ (15) \Rightarrow (jS_k)(I - \mathcal{M}^+ \mathcal{M}) = -jPQP[I - \mathcal{M}^+ \mathcal{M}] \]  \hfill (17)

We need the following lemma:

**Lemma 1** \( AX = C, \ XB = D \) has common Hermitian solution \( X \) if and only if these two equations have common solution and

\[ H^* = H; \quad H = \begin{bmatrix} CA^* & CB \\ D^*A^* & D^*B \end{bmatrix}. \]

Now, if we set \( A = (I - \beta \beta^+) \), \( B = (I - \mathcal{M}^+ \mathcal{M}) \)

\[ C = -j(I - \beta \beta^+)PQP, \quad D = -JPQP[I - \mathcal{M}^+ \mathcal{M}] \]

then, from lemma 1, we can see that if \( jS_k \) has Hermitian solution

\[ CA^* = AC^* \Rightarrow [I - \beta \beta^+]PQP[I - \beta \beta^+] = 0 \]  \hfill (18)

\[ D^*B = B^*D \Rightarrow [I - \mathcal{M}^+ \mathcal{M}]PQP[I - \mathcal{M}^+ \mathcal{M}] = 0. \]  \hfill (19)

Notice that

\[ (17) \Rightarrow (I - \mathcal{M}^+ \mathcal{M})(jS_k)^* = j(I - \mathcal{M}^+ \mathcal{M})PQP \]  \hfill (20)

\[ (16), (20) \Rightarrow \begin{cases} (I - \beta \beta^+)(jS_k) = -j(I - \beta \beta^+)PQP \\ (I - \mathcal{M}^+ \mathcal{M})(jS_k) = j(I - \mathcal{M}^+ \mathcal{M})PQP \end{cases} \]

We require (16), (20) to possess common solutions.

It is obvious that (16) always has solution:
Substitute (21) into (20)

\[ \Rightarrow -P_m(I - P_\beta)Z = jP_m(I - P_\beta)PQP \]  \hspace{1cm} (22)

(22) is solvable iff

\[ [I - (P_m\beta\beta^+)(P_m\beta\beta^+)^+]P_m(I + P_\beta)PQP = 0 \]

\[ \Rightarrow [I - (P_m\beta\beta^+)(P_m\beta\beta^+)^+]P_mPQP = 0 \] \hspace{1cm} (23)

Hence, we have proved that (14), (15) are equivalent to (18), (19), (23), i.e., for \( G_p \) to be solvable iff (18), (19), (23) hold.

Q.E.D.

Appendix B. Proof of Theorem 2

We need an extension of lemma 1.

Lemma 2

Suppose the equation \( AX = C, XB = D \) have a common Hermitian solution, the form of the solution is

\[
X = \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix} - \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix}^* \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix} - \begin{bmatrix} A^* \\ B^* \end{bmatrix}^* \begin{bmatrix} I - A^*B \end{bmatrix} U \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix}^* \] \hspace{1cm} (24)

Where \( U \) is an arbitrary Hermitian matrix and \([-]^-\) is the generalized inverse.

To generate all solutions, we must parameterize all pseudo inverse \([-]^-\) as follows

\[
\begin{bmatrix} A \\ B^* \end{bmatrix}^- = \begin{bmatrix} A \\ B^* \end{bmatrix}^- + Z - \begin{bmatrix} A \\ B^* \end{bmatrix}^+ \begin{bmatrix} A \\ B^* \end{bmatrix} Z \begin{bmatrix} A^* \\ B^* \end{bmatrix} \] \hspace{1cm} (25)

where \( Z \) is arbitrary, \([\cdot]^+\) denotes the Moore Penrose inverse.

Now, if we define

\[
L_p \triangleq \begin{bmatrix} I - \beta\beta^+ \\ I - M^+M \end{bmatrix}, \hspace{0.5cm} \Phi_p \triangleq \begin{bmatrix} -(I - \beta\beta^+) \\ -(I - M^+M) \end{bmatrix} PQP
\]

then according to (24), (25),

\[
S_k = L_p^+\Phi_p + (L_p^+\Phi_p)^* - L_p^+\Phi_p L_p^+L_p + (I - L_p^+L_p)S(Z - L_p^+L_p) \] \hspace{1cm} (26)
is an arbitrary skew-symmetric matrix. Next, plug (26) into standard form of solution for (13), we get:

\[ G_p = -\frac{1}{2} \beta^* [PQ^* + L_p^* \Phi_p - (L_p^* \Phi_p)^* - L_p^* \Phi_p L_p^* L_p] M^* + \beta^* (I - L_p^* L_p) S_p (I - L_p^* L_p) M^* + Z_p - \beta \beta Z M M^* \]

where \( Z_p \) is arbitrary.

The parametrization of \( G_a \) was derived in\(^7\).

Q.E.D.

7. REFERENCES