Mathematics Ph.D Qualifier Exam

Department of Mechanical Engineering
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Directions: Closed book, but one sheet of your own notes permitted.

All problems carry equal weight. In order to receive full credit for a solution, you must show all work clearly.

Exam prepared by Profs. G. Brereton and S. Baek
Problem 1.

Determine the centroid $C(\bar{x}, \bar{y})$ and the polar moment of inertia $J$ with respect to the centroid for a quarter semi-circle of radius $r$ in the figure below, where

$$
\bar{x} = \frac{1}{A} \int x \, dA \quad ; \quad \bar{y} = \frac{1}{A} \int y \, dA \quad ; \quad J = \int (x - \bar{x})^2 + (y - \bar{y})^2 \, dA
$$
Problem 2.

Let $r, \theta, \phi$ be spherical coordinates defined as

\[
x = r \cos \theta \sin \phi; \\
y = r \sin \theta \sin \phi; \\
z = r \cos \phi.
\]

When $u(x, y, z)$ is a function of $r$ only, use the chain rule to rewrite the Laplace equation $\nabla^2 = 0$ in spherical coordinates. Note that

\[
r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

in Cartesian coordinates.
Problem 3.

Consider a counter-clockwise plane rigid-body rotation about the z-axis with angular velocity $\Omega = \omega \mathbf{k}$ where $\omega$ is constant. Determine the curl of the velocity of a plane rigid rotation.

\[
\text{curl } \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} - \left( \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) \mathbf{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}
\]
Problem 4.

A carrier of an infectious disease joins a herd of 500 initially uninfected cattle. At any instant in time, the rate at which the disease spreads \( \frac{dx}{dt} \) is known to be proportional to the product of:

(i) the number of infected cattle \( x(t) \); and

(ii) the number of uninfected cattle.

If the number of cattle infected after 4 days is 50, how many will have been infected after 6 days?
Problem 5.

The small-amplitude motion of a taut string of length $L$ is described by the wave equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

where $u$ is the transverse deflection of the string, $t$ is time, and $\alpha^2$ is a physical constant describing the thickness, elasticity, density and tension in the string. The ends of the string are fixed so that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

(i) Find the general solution of this PDE, choosing the separation variable as $-\omega^2$ (where $\omega = \alpha \lambda$).

(ii) Find the eigenvalues for which the boundary conditions are satisfied and give the solution that satisfies these boundary conditions.

(iii) Suppose a guitarist could pluck the string with sufficient dexterity that he released the string at $t = 0$ with $\partial u / \partial t = 0$, with an amplitude $u(x, 0) = A \sin(\lambda_n x)$ (with the guitarist choosing $n$). Give the equation that describes the subsequent deflection of the string.
Problem 6.

The ordinary differential equation:

\[
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0
\]

has initial conditions: \( y = 1 \) at \( x = 0 \) and \( \frac{dy}{dx} = 2 \) at \( x = 0 \).

(i) Use Euler's method to find the approximate value of \( y \) at \( x = 0.2 \) using a step size of \( \Delta x = 0.1 \).

(ii) Compare your answer with the exact solution.