Nonlinear analysis of RC beams based on moment–curvature relation

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Abstract

Material nonlinear analyses of reinforced concrete (RC) beams considering the tension softening branch and bond slip have been conducted. Instead of adopting the sophisticated layer approach which has some limitations in application to large structures with many degrees of freedom, we have used the moment–curvature relationships of RC sections previously constructed through section analysis. To reduce numerical instability according to the finite element mesh size used, a relation simulating the tension softening branch has been taken into consideration. For the purpose of removing the imprecision in calculation of ultimate resisting capacity, we have included the plastic hinge length in finite element modeling. In addition, governing equations describing the bond-slip behavior in beams have been derived. Unlike the conventional bond elements using double nodes, the proposed model has used beam elements representing the structural response by two nodes at both ends, and has simplified the finite elements modeling and analytical process, besides effectively describing the bond-slip behavior. Moreover, the developed algorithm has been reflected in the moment–curvature relationship of RC section. Finally, correlation studies between analytical and experimental results have been conducted with the objective to establish the validity of the proposed algorithms. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Moment–curvature relation; Bond slip; Plastic hinge length; RC beam; RC section; Tension softening branch

1. Introduction

In accordance with the development of industrial society and the expansion of the magnitude of economies, structures have become larger and more complex. The safety and serviceability assessment of those complex structures necessitates the development of accurate and reliable methods and models for their analysis. To ensure the safety of structures in the case of earthquake, both analytical and experimental studies about structural behavior under over-load conditions and cyclic loads are carried out side by side [12], but experimental studies are expensive and time consuming and give us limited information. Especially for reinforced concrete (RC) structures which are brittle compared to steel structures, it is very important to describe the behavior of the RC structures under over-load conditions and estimate their ultimate strength accurately.

It is difficult in analytical studies to describe effectively the composite behavior of two wholly different materials, concrete and steel, and to consider the time dependent variation of material properties and effects between both materials. In recent years, due to the development of computers, improvement of analytical methods and explicit explanation of material properties, analytical studies are widely carried out. Because structural analysis requires great computational effort for iterations and numerical instability due to variation of structural appearance and material properties occurs as working
stress increases, we need to develop simpler analytical methods which effectively reduce the computational time and estimate the ultimate resisting capacity of structures [1].

In this study, material nonlinear analysis of RC beams which compose the primary members of RC structures have been conducted. Instead of taking the sophisticated layer approach which has some limitations in application to large structures with many degrees of freedom, we have used the moment–curvature relationships of RC sections previously constructed through section analysis. Especially, to reduce the numerical instability according to the finite element mesh size used in the stage of construction of moment–curvature relations, a relation simulating the tension softening branch has been taken into consideration. In addition, to simulate the concentric plastic deformation of RC beams after the yielding of the steel, we have included the plastic hinge length in finite element formulation. Governing equations describing the bond-slip behavior in beams according to cracking have been derived and applied to moment–curvature relations. Moreover, correlation studies between analytical and experimental results have been conducted to establish the validity of the proposed algorithms.

2. Material models

2.1. Concrete

The response of a structure under load depends to a large extent on the stress–strain relation of the constituent materials and the magnitude of stress. Since concrete is used mostly in compression, the stress–strain relation in compression is of primary interest. The concrete stress–strain relation exhibits a nearly linear elastic response up to about 30% of the compressive strength. This is followed by gradual softening up to the concrete compressive strength, when the material stiffness drops to zero. Beyond the compressive strength, the concrete stress–strain relation exhibits strain softening until failure takes place by crushing.

Many mathematical models of concrete are currently used in the analysis of RC structures. Among those models, the monotonic envelope curve introduced by Kent and Park and later extended by Scott et al. [14] was adopted in this study for its simplicity and computational efficiency. In this model, as shown in Fig. 1(a), the monotonic concrete stress–strain relation in compression is described by three regions:

\[
\sigma_c = Kf'_c \left[ 2 \left( \frac{\varepsilon_c}{\varepsilon_{c0}} \right) - \left( \frac{\varepsilon_c}{\varepsilon_{c0}} \right)^2 \right], \quad \varepsilon_c \leq \varepsilon_{c0}
\]

where

\[
\varepsilon_{c0} = Kf'_c \left[ 1 - Z(\varepsilon_c - \varepsilon_{c0}) \right], \quad \varepsilon_{c0} \leq \varepsilon_c \leq \varepsilon_u
\]

\[
\varepsilon_u = 0.002K, \quad K = 1 + \frac{\rho_h f_{yh}}{f'_c}
\]

\[
Z = \frac{3 + 0.0284f'_c}{14.21f'_c - 1000} + 0.75\rho \left( \frac{h'}{s_h} - 0.002K \right)
\]

\[
\varepsilon_{c0} \quad \text{is the concrete strain at maximum stress, } K \quad \text{is a factor which accounts for the strength increase due to confinement, } Z \quad \text{is the strain softening slope, } f'_c \quad \text{is the concrete compressive strength in kg/cm}^2 (1 \text{ kg/cm}^2 = 0.098 \text{ MPa}), f_{yh} \quad \text{is the yield strength of stirrups in kg/cm}^2, \quad \rho \quad \text{is the ratio of the volume of hoop reinforcement to the volume of concrete core measured to the outside of stirrups, } h' \quad \text{is the width of the concrete core measured to the outside of stirrups, and } s_h \quad \text{is the center to center spacing of stirrups or hoop sets.}
\]

On the other hand, this model assumes that concrete is linearly elastic in the tension region. Beyond the tensile strength, the tensile stress decreases linearly with increasing principal tensile strain (see Fig. 1(b)). Ultimate failure is assumed to take place by cracking, when the principal tensile strain exceeds the value \( \varepsilon_0 \) in Fig. 1(b). The value of \( \varepsilon_0 \) is derived from the fracture mechanics concept by equating the crack energy release with the fracture toughness of concrete \( G_t \) [7].

\[
\varepsilon_0 = \frac{2G_t \ln \left( \frac{h'}{b} \right)}{f_t(3 - b)}
\]

where \( f_t \) is the tensile strength of concrete, \( \varepsilon_0 \) is the fracture tensile strain which characterizes the end of the strain softening process when the microcracks coalesce into a continuous crack, \( b \) is the element length, and \( G_t \) is the fracture energy which is dissipated in the formation of a crack of unit length per unit thickness and is considered a material property. The experimental study by Welch and Haisman [16] indicates that for normal strength concrete, the value of \( G_t/f_t \) is in the range of 0.005–0.01 mm. If \( G_t \) and \( f_t \) are known from measure-
ments, $e_0$ can be determined. Also, as shown in Eq. (6), $e_0$ depends on the finite element mesh size. Through previous numerical analyses, it was verified this approach of defining $e_0$ renders the analytical solution insensitive to the mesh size used, and guarantees the objectivity of the results.

2.2. Steel

Reinforcing steel is modeled as a linear elastic, linear strain hardening material with yield stress $\sigma_y$ as shown in Fig. 2. The reasons for this approximation are (1) the computational convenience of the model; (2) the behavior of R/C members is greatly affected by the yielding of reinforcing steel when the structure is subjected to monotonic bending moments [9,15].

3. Moment–curvature relation

Since a structure is composed of many structural members, and a member is formed by the integration of each section, the nonlinear behavior of a section causes nonlinear behavior in the structure. Especially, in the case of beams and columns which are the primary members of a frame structure, internal member forces are concentrated on the sections located at both ends and the center of a member. Using this structural characteristic, plastic hinge analysis is broadly adopted in the nonlinear analysis of frame structures. Accordingly, the nonlinear analysis of RC beams can be conducted with the moment–curvature relation constructed by section analysis because most of the deformations in beams arise from the strains associated with flexure. The curvature $\phi$ representing the gradient of the strain profile at a section can be calculated by classical beam theory [11] if the strains are measured from the increase of bending moment to failure, permitting the moment–curvature relation for a section to be obtained.

To ensure ductile behavior in practice, steel contents less than the balanced design value are always used for beams. The typical moment–curvature relation for a lightly RC section (under-RC section) can be idealized to the trilinear relation as shown in Fig. 3. The first stage is to cracking, the second to yield of the tension steel, and the third to the limit of useful strain in the concrete. The behavior of the section after cracking is dependant mainly on the steel content. Lightly reinforced sections result in a practically linear moment–curvature curve up to the point of steel yielding. When the steel yields, a large increase in curvature occurs, following the moment rising slowly to a maximum due to an increase in the internal lever arm, then decreasing.

The moment–curvature relation of a section is uniquely defined according to the dimensions of the concrete section and the material properties of concrete and steel. Also, the gradient of the moment–curvature relation means the elastic bending stiffness $EI$ which includes all the section properties in a typical loading condition. Therefore, using the moment–curvature relation instead of taking the layer approach abbreviates the accompanying sophisticated calculations in the nonlinear analysis such as the determination of neutral axis and change of elastic stiffness. This is why the nonlinear analysis of RC beams based on the moment–curvature relation is used in this study. In determining the theoretical moment–curvature curve for the RC section with flexure, we use the basic assumption that plane sections remain plain so that the longitudinal strain is directly proportional to the distance from the neutral axis of zero strain.

3.1. Cracking point

Up to the first cracking at the extreme tension fiber, the entire cross section effective for the applied internal moment, and the stress–strain relations for concrete and steel maintain linearly elastic. Fig. 4 depicts a doubly RC rectangular beam section in the elastic stage before
cracking. The dimensions of the concrete section and the steel area and positions are considered known quantities. The cracking moment \( M_{cr} \) and the corresponding curvature \( \phi_{cr} \) (point \( \Phi \) in Fig. 3) can be calculated using the requirements of strain compatibility and equilibrium of forces as follows:

\[
\phi_{cr} = \frac{\varepsilon_{cc}}{c} = \frac{\varepsilon_{cr}}{H - c} \quad (7)
\]

\[
M_{cr} = \frac{1}{3} \varepsilon_{cc} E_c c^2 B + \frac{(c - d')^2}{c} E_s A_{st} + \frac{1}{3} \frac{(H - c)^3}{c} E_s E_c B + \frac{(d - c)^2}{c} E_s A_{st} \quad (8)
\]

where \( A_{st} \) and \( A_{sc} \) are the areas of tension and compression steels, the corresponding steel ratios are \( q = A_{st}/Bd \) and \( q_0 = A_{sc}/Bd \), the modular ratio \( n = E_s/E_c \), and the distance from the extreme compression fiber to the neutral axis \( c = H + 2n(d\rho + d'\rho)/\{(1 + n(\rho + \rho'))\} \).

3.2. Yielding point

The moment and curvature at first yield of the tensile steel (point \( \Phi \) in Fig. 4) should be calculated using the defined stress–strain relations for the concrete. Based on the normal force equilibrium, the section analysis is carried out by assuming first that the tension steel reaches the yielding point. With the assumed neutral axis depth, the internal tension \( T \) and compression \( C \) can be calculated by

\[
T = \sigma_{st} A_s + \int_{A_{tc}} \sigma_{tc} dA = f_y A_s + \int_{A_{tc}} \sigma_{tc} dA \quad (9)
\]

\[
C = \sigma_{sc} A_{sc} + \int_{A_{cc}} \sigma_{cc} dA \quad (10)
\]

where \( f_y \) is the yield strength of steel, \( A_{tc} \) and \( A_{cc} \) represent the areas of concrete acted on by the tensile and compressive stresses, respectively.

After iterations based on the reassumed neutral axis depth until the difference between the tensile force and compressive force is less than the given tolerance, the moment and curvature are finally determined.

3.3. Cracked stage between two points

After dividing the curvature range between \( \phi_{cr} \) and \( \phi_{y} \) into four equal parts, first the bending moment corresponding to each boundary curvature can be calculated, following the same iteration procedure described in the preceding section. Then the connection of each point with straight lines gives the simplified moment–curvature relation used in this study (see Fig. 5(b)). Also, the moment–curvature relation to the post-yielding stage

![Fig. 4. Doubly RC section before cracking.](image)

![Fig. 5. Moment–curvature relations of RC section. (a) Classical layered section approach and (b) simplified approach.](image)
is approximated as a straight line with the bending stiffness \( E_s t^2 \), where \( E_s \) is the elastic modulus of steel after yielding, because the moment capacity depends wholly on the structural behavior of steel in this stage.

Fig. 5(a) shows the theoretical moment–curvature relations for a typical rectangular cross-section of width 15.24 cm and overall depth 30.48 cm. The tension steel of 2.88 cm\(^2\) was placed at \( d = 27.33 \) cm. Previously defined stress–strain relations of concrete and steel were used, and the material properties of \( f_c = 323.6 \) kg/cm\(^2\), \( f_y = 2236.4 \) kg/cm\(^2\), and \( E_s = 1.98 \times 10^6 \) kg/cm\(^2\) were assumed. The layer approach was accompanied by changing \( \varepsilon_0 \) in Fig. 1(b). As shown in this figure, the tension softening branch must be considered to simulate exactly the bending behavior for a lightly RC section. On the other hand, Fig. 5(b) represents the simplified moment–curvature relations for the same section, where \( b \) is the element length used to calculate \( \varepsilon_0 \) by Eq. (6).

4. Bond-slip effect

Since bond stresses in RC structures arise from the change in the steel force along the length, the effect of bond becomes more pronounced in the cracked region. In the simplified analysis of RC structures, complete compatibility of strains between concrete and steel is usually assumed, which implies a prefect bond. This assumption is realistic only in regions where negligible stress transfer between the two components takes place. In regions of high transfer stresses along the interface between reinforcing steel and surrounding concrete, such as near cracks, the bond stress is related to the relative displacement between reinforcing steel and concrete. Therefore, the bond-slip effect must also be considered to simulate the structural behavior more exactly.

Two basically different elements, namely, the bond-link element and bond-zone element, have been proposed to date for inclusion of the bond-slip effect in the finite element analysis of RC structures [5,9]. However, the use of those elements requires a double node to represent the relative slip between reinforcing steel and concrete. In a beam structure defined by both end nodes along the direction, it is impossible to use the double node at each end node. To address this limitation in adopting the bond model, a numerical algorithm which includes the bond-slip effect into the moment–curvature relation is proposed in this study.

4.1. Bond-slip behavior

When a RC beam is subjected to a concentrated lateral load \( Q \) at midspan, the force equilibrium in the axial direction requires the summation of the axial force component. Fig. 6 shows an infinitesimal beam element of length \( dx \). If the linear bond stress–slip relation is assumed, the variations of the axial force components can be represented in terms of bond stress and slip

\[
dP_s = f_b \Sigma_0 dx = E_b A_x dx \Sigma_0 = A_y E_s d\varepsilon_s \quad (11)
\]

\[
dP_c = \int E_c d\delta_c dA_c = -f_b \Sigma_0 \quad (12)
\]

where \( f_b \) is the bond stress at the steel interface, \( E_b \) is the slope of the bond stress–slip curve, \( \Sigma_0 \) is the perimeter of the steel bar, \( A \) is the bond slip, \( A_c \) is the uncracked area of the cross-section, and \( E \) is Young's modulus.

Since the longitudinal strains are directly proportional to the distance from the neutral axis of zero strain, the variation of curvature \( \phi \), representing the gradient of the strain profile at the section means the variation of strain. In addition, the actual curvature distribution up to the yielding of reinforcing steel can be idealized into a linear curvature distribution along the member as shown in Fig. 7.

By ignoring the tensile force in the concrete which makes a negligibly small contribution after cracking, the variation of axial force in the concrete can be determined as

\[
\begin{align*}
\end{align*}
\]
\[ dP_c = -f_b \Sigma_0 \, dx = \int E_c \, d\varepsilon_c \, dA_c \]

\[ = - \int_0^c E_c \, d\phi, y \, dy = -E_c b \, d\phi, \frac{1}{2} e^2 \]  

(13)

where \( c \) is the distance from the extreme compression fiber to the neutral axis.

Namely, the variation of curvature along the length can be expressed by \( d\phi, y / dx = 2f_b \Sigma_0 / (E_c b c^2) \). In addition, the neutral axis depth \( c \) maintains an almost constant value of \( c = 0.4d \) from the initial cracking up to the yielding of reinforcing steel. The variation of concrete strain at the steel location, therefore, can be simplified as

\[ \frac{dc}{dx} = (d - c) \frac{d\phi, y}{dx} = (d - c) \frac{2E_b \Sigma_0}{E_c b c^2} \]

\[ = 3.75 \frac{2E_b \Sigma_0}{(E_c A_s / n\nu)} \, d \]

(14)

where \( n = E_s / E_c \), the steel ratio \( \rho = A_s / b d \).

When the bond slip \( \Delta \) at the steel–concrete interface is defined by the relative displacement between reinforcing steel and concrete (\( \Delta = u_s - u_c \)), the first order and second order differential equation of bond slip lead to

\[ \frac{dd_A}{dx} = \frac{du_s}{dx} - \frac{du_c}{dx} = \varepsilon_s - \varepsilon_c \]  

(15)

\[ \frac{d^2 A_s}{dx^2} = \frac{d\varepsilon_s}{dx} - \frac{d\varepsilon_c}{dx} = E_b \Sigma_0 (1 - 7.5\nu) \Delta_s \]

(16)

Accordingly, the very well known following governing differential equation for the bond slip is obtained.

\[ \frac{d^2 A}{dx^2} - k^2 A = 0 \]

(17)

where \( k^2 = (E_b \Sigma_0 / A_s E_c)(1 - 7.5\nu) \).

As shown in this equation, the governing equation has the same form as that of the axial member in uniaxial tension except a minor difference in \( k^2 \), that is, the proportional coefficient has \((1 - 7.5\nu)\) instead of \((1 + \nu)\). The general solution to Eq. (17) is given by \( A = C_1 \sinh kx + C_2 \cosh kx \), in which \( C_1 \) and \( C_2 \) are constants that have to be determined from the boundary conditions.

5. Solution algorithm

The region ranged from the end to the loaded point where the maximum bending moment occurs is assumed to be subdivided into \( n \) elements, as shown in Fig. 8. The general solution of Eq. (17) represents the force equilibrium and the compatibility condition at the steel interface along the length. Focusing attention on the \( i \)th element (see Fig. 8), the following boundary condition at both end points must be satisfied: \( \Delta A / dx = -(\varepsilon_s - \varepsilon_c) \) at \( x = l_i / 2 \), \( \Delta A / dx = +(\varepsilon_s - \varepsilon_c) \) at \( x = l_i / 2 \), and the bond slip at each end of the \( i \)th element is determined as

\[ \Delta_i \left( \frac{l_i}{2} \right) = \Delta_i = \frac{\cosh k l_i}{k \sinh k l_i} P_i + \frac{1}{k \sinh k l_i} Q_i \]

\[ = A_i P_i + B_i Q_i \]  

(18)

\[ \Delta_i \left( -\frac{l_i}{2} \right) = \Delta_i = \frac{1}{k \sinh k l_i} P_i + \frac{\cosh k l_i}{k \sinh k l_i} Q_i \]

\[ = B_i P_i + A_i Q_i \]  

(19)

where \( P_i = \varepsilon_s - \varepsilon_c \), \( Q_i = \varepsilon_s - \varepsilon_c \), \( A_i = \cosh k l_i / (k \sinh k l_i) \) and \( B_i = 1 / (k \sinh k l_i) \).

Solving the above equations for the bond slip \( \Delta_i \) and strain difference \( (\varepsilon_s - \varepsilon_c) \) at \( x = l_i / 2 \) yields

\[ \{D_i\}^{\dagger} = \left[ \begin{array}{c} A_i \\ P_i \\
\end{array} \right] = \left[ \begin{array}{cc}
A_i & B_i \\
B_i & A_i - B_i \\
\end{array} \right] \left[ \begin{array}{c}
\Delta_i \\
Q_i \\
\end{array} \right] \]

\[ = [C]\{D_0\}^{\dagger} \]  

(20)

For the transition from the \((i - 1)\)th to the \(i\)th element using the force equilibrium and the compatibility condition, the following equation can be determined

\[ \{D_0\}^{\dagger} = \left[ \begin{array}{c}
A_i \\
Q_i \\
\end{array} \right] = \left[ \begin{array}{cc}
1 & 0 \\
0 & -1 \\
\end{array} \right] \left[ \begin{array}{c}
\Delta_i^{i-1} \\
P_{i-1} \\
\end{array} \right] = [S]\{D_i\}^{i-1} \]  

(21)

and the substitution of Eq. (20) into Eq. (21) yields

\[ \{D_i\}^{\dagger} = [C]\{D_0\}^{\dagger} = [C_i][S]\{D_i\}^{i-1} = [C_i][D_i]^{i-1} \]  

(22)

This equation relates the bond-slip and strain difference at the end of the \(i\)th element with those of \((i - 1)\)th elements. By applying Eq. (22) successively to elements \( i = 2, 3, \ldots , 2, 1 \) and summing up the results, and extending to the 0th element, the following relation between the bond slip \( \Delta \) and strain difference \( (\varepsilon_s - \varepsilon_c) \) at the two ends of the steel interface can be obtained.

\[ \{D_0\} = [C_0][C_0-1][C_0-2]\ldots[C_0][D_0] = [R_0][D_0] \]  

(23)

where \([C_i] = [C][S] \).

If two boundary values among the four unknowns are determined, it is possible to solve Eq. (23) for the remaining two unknowns. For a beam structure, it can be
assumed that the strain difference at the end face of a structure $Q_i = (\varepsilon_{si}^0 - \varepsilon_{ei}^0)$ is zero because the bending moment by the applied lateral load is zero. Another boundary condition is derived from the relation between the crack width and steel stress. Based on the experimental study for many cracked beams, Oh [10] found that the crack width has almost a linear relation with the crack width at the steel interface, $\varepsilon_s$ is the steel strain, and $T$ is the proportional constant.

As the concentrated lateral load $Q$ increases, the cracking develops first at the location where the maximum bending moment occurs, i.e., the midspan in Fig. 7. Since it can be assumed that the crack width $w$ is equivalent to two times of the bond slip $d$ at the cracked location, the bond slip $A_s^i$ can be expressed in terms of the corresponding steel strain from Eq. (23). If the terms of matrix $[R_0]$ in Eq. (23) are $r_{11}$, $r_{12}$, $r_{21}$, and $r_{22}$, the relations of $A_s^i = r_{11}A_s^0 + r_{12}Q_i = r_{11}A_s^0 = 4w = 4T\varepsilon_{si}^0$ and $\varepsilon_s^0 - \varepsilon_{si}^0 = r_{21}A_s^0$ can be constructed. Therefore, the steel strain at the right end of the $n$th element can be expressed by

$$
\varepsilon_s^n = \varepsilon_{si}^0 \frac{1 - r_{21} \frac{\varepsilon_{si}^0}{r_{11}}}{1 - r_{21} \frac{\varepsilon_s}{2}}
$$

where it should be noted that the concrete strain $\varepsilon_{si}^0$ is known from the global nonlinear finite element analysis of RC beams based on the perfect bond assumption.

Moreover, from the relation of $r_{11}A_s^0 = 4T\varepsilon_{si}^0$, the bond slip at the left end of the first element $A_s^0$ is finally calculated. After obtaining the bond-slip and strain difference at one end of the structure, those components at each node can be found through the successive application of Eqs. (20) and (21).

With the determined deformations at both ends of each element, the revised curvature considering the bond-slip effect can be calculated by using the mean curvature concept as follows:

$$
\phi' = \frac{\varepsilon_s - \varepsilon_{cc}}{d} = \phi + \frac{1}{2} \left( \frac{\varepsilon_s^0 + \varepsilon_{si}^0}{2} - \frac{\varepsilon_s^0 - \varepsilon_{si}^0}{2} \right)
$$

where $\phi$ is the curvature corresponding to a moment $M$ under the perfect bond, $\phi'$ is the revised curvature considering the bond slip, $\varepsilon_{cc}$ is the concrete strain at the extreme compression fiber, and $\varepsilon_s$ and $\varepsilon_{si}$ are the steel and concrete strains at the steel interface in the case of considering the bond slip, as shown in Fig. 10.

On the other hand, if the pure bending region is widely distributed in the center of a member, it is assumed that the bond slip in this region is not varied but constant, and the bond slip and strain difference have the same values with those at the end of the adjacent element ($(a - 1)$th element), that is, the revised moment–curvature relation of the adjacent element can be used to the $n$th element without any modification.

### 6. Finite element idealization

For the analysis of RC beams, Timoshenko beam theory was used in this study [11]. Since this theory is well established and widely used in the analysis of beams, attention is focused below on some theoretical aspects of the bending problem followed by the finite element implementation of the moment–curvature relation and bond-slip effect in the analysis of RC beams. In a typical Timoshenko beam, it is usual to assume that normals to the neutral axis before deformation remain straight but not necessarily normal to the neutral axis after deformation. In addition, the effects of shear deformation are not taken into consideration in simulating nonlinear behavior since the normal bending stresses reach a maximum at the extreme fibers, where the transverse shear stresses are at their lowest value, and reach a minimum at mid-depth of the beam.
transverse shear stresses are highest. Thus, the iteration between transverse shear stresses and normal bending stresses is relatively small and can be ignored.

Using a virtual work approach, the governing equilibrium equations can be expressed as

\[
[K_f + K_s] \phi^T - f = \left[ \int_l [B_f]^T (EI) [B_f] \right] \, dx \\
+ \int_l [B_s]^T (GA) [B_s] \, dx \] \phi^T - f = 0
\]

where the nodal displacement vector \( \phi = [w_1, \theta_1, w_2, \theta_2] \), the subscripts 1 and 2 mean both end nodes of an element with length \( l \). If \( K_f \) and \( K_s \) are evaluated using a one-point Gauss–Legendre rule, those take the following forms.

\[
k_f = \frac{(EI)}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},
\]

\[
k_s = \frac{(GA)}{l} \begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{l}{2} & \frac{\alpha}{2} & -\frac{l}{2} & \frac{\alpha}{2} \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{l}{2} & \frac{\alpha}{2} & -\frac{l}{2} & \frac{\alpha}{2} \end{bmatrix}
\]

where \( EI \) is the flexure rigidity, \( GA \) is the shear rigidity (\( A = \alpha A \)), and the parameter \( \alpha \) is a correction factor to allow for cross-sectional warping (\( \alpha \approx 1.5 \) for a rectangular section). Moreover, the internal member forces \( M \) and \( S \) can be represented in terms of nodal displacements.

\[
M = (EI) [B_f] \phi^T = (EI) \begin{bmatrix} 0 & 1/l, 0, -1/l \end{bmatrix} [w_1, \theta_1, w_2, \theta_2]^T = \frac{(EI)}{l} (\theta_1 - \theta_2) \]

\[
S = (GA) [B_s] \phi^T = (GA) \left[ -\frac{1}{7} \begin{bmatrix} 1 & -1/2 & 1 & -1/2 \end{bmatrix} \right] \phi^T = (GA) \left\{ \left( \frac{w_2 - w_1}{l} \right) - \left( \frac{\theta_1 + \theta_2}{2} \right) \right\}
\]

where the curvature–displacement matrix \( [B_f] = [0, -dN_1/dx, 0, -dN_2/dx] \), the shear strain–displacement matrix \( [B_s] = [dN_1/dx, -N_1, dN_2/dx, -N_2] \), and the shape functions \( N_1 = (x_2 - x)/l \) and \( N_2 = (x - x_1)/l \). The shear force varies linearly over each element but it is assumed to be constant over the element and calculated at the midpoint of the element, \( x = (x_1 + x_2)/2 \).

The internal force vector \( P \) caused by the internal moment \( M \) and shear force \( S \) can be calculated, and \( P - f \) becomes a residual force to be applied for the iteration. More details can be found elsewhere [11].

\[
P = \int_l \begin{bmatrix} 0 \\ 1/l \\ -1/l \end{bmatrix} M \, dx + \int_l \begin{bmatrix} -1/l \\ (x-x_2)/l \\ 1/l \\ (x_1-x)/l \end{bmatrix} S \, dx
\]

\[
= [ -S, \ M - (1/2), \ S, \ -M - (1/2)]^T
\]

As mentioned in the assumptions, the effect of shear deformation, that is, the dowel action and aggregate interlocking in the R/C structure, is neglected in the simulation of nonlinear behavior. It means that the flexural rigidity \( EI \) is replaced by that corresponding to the calculated curvature \( \phi = (\theta_1 - \theta_2)/l \) whereas the shear rigidity \( GA \) is assumed to be unchanged.

7. Solution algorithm

Every nonlinear analysis algorithm consists of four basic steps: the formation of the current stiffness matrix, the solution of the equilibrium equations for the displacement increments, the state determination of all elements in the model and the convergence check. These steps are presented in some detail in the flow diagram of Fig. 11. Since the global stiffness matrix of the structure depends on the displacement increments, the solution of equilibrium equations is typically accomplished with an iterative method through the convergence check. The nonlinear solution scheme selected in this study uses the tangent stiffness matrix at the beginning of each load step in combination with a constant stiffness matrix during the subsequent correction phase, that is, the incremental-iterative method.

The criterion for measuring the convergence of the iterative solution is based on the accuracy of satisfying the global equilibrium equations or on the accuracy of determining the total displacements. The accuracy of satisfying the global equilibrium equation is controlled by the magnitude of the unbalanced nodal forces. The accuracy of the node displacements depends on the magnitude of the additional displacement increment after each iteration. The latter convergence criterion is used in this study. This can be expressed as

\[
E_d = \left[ \frac{\sum (\Delta d_j)^2}{\sum (d_j)^2} \right]^{1/2} \leq \text{TOLER}
\]

where the summation extends over all degrees of freedom \( j \), \( d_j \) is the displacement of degree of freedom \( j \), \( \Delta d_j \) is the corresponding increment after iteration \( i \) and \( \text{TOLER} \) is the specified tolerance.

In the nonlinear analysis of RC structures, the load step size must be small enough so that unrealistic
“numerical cracking” does not take place. These spurious cracks can artificially alter the load transfer path within the structure and result in incorrect modes of failure. Crisfield [4] has shown that such numerical disturbance of the load transfer path after initiation of cracking can give rise to alternative equilibrium states and, hence, lead to false ultimate strength predictions. In order to avoid such problems after crack initiation, the load is increased in steps of 2.5–5.0% of the ultimate load of the member.

8. Numerical applications

Three simply supported RC beams were investigated with the objective of establishing the ability of the proposed model in simulating the response of RC beams. These beams are specimen T1MA tested by Gaston et al. [6], specimen B5 tested in NJIT [8], and J4 tested by Burns and Siess [2]. The material properties and geometries of the three test specimens are summarized in Tables 1 and 2, respectively. Since the mean crack spacing at the ultimate loading condition is equal to about three times the concrete cover [3], an improvement in numerical results can not be expected for the smaller mesh size. Accordingly, the minimum length of elements is determined as three times the concrete cover.

For beams T1MA and B5 where the concentrated loads are applied at one-third points of the structure (see Fig. 12), five elements for half of the structure are used because of the symmetry in geometry and loading while considering the bond-slip effect. Especially, the range from the one-third point of the span to the midspan is modeled with only one element since the plastic deformation is uniformly distributed within this range. In the case of beam J4, subjected to one concentrated load at the midspan (see Fig. 12), the plastic deformation is concentrated at the midspan with narrow width, where the occurrence of plastic rotation is initiated and concentrated.

This range is called the plastic hinge length. Various empirical expressions have been proposed by investigators for the equivalent length of the plastic hinge $l_p$ [12]. Since the structure is modeled with beam elements whose displacement field is defined by the average deformation of both end nodes, the ultimate capacity can be overestimated if the plastic hinge length is not precisely taken into consideration. In this study, the relatively simple equation proposed by Sawyer [13] was used. Assuming that the ratio of yielding moment to ultimate moment is $M_y/M_u = 0.85$, the calculated plastic hinge length $l_p$ is 25 cm from $l_p = 0.25d + 0.075z$, where $z$ means the distance of critical section to the point of contraflexure ($z = l/2$ in this example). Namely, a small element of $l = 25$ cm must be located at the midspan, to predict the ultimate strength of a structure exactly.

Table 1
Material properties used in applications

<table>
<thead>
<tr>
<th>Beam</th>
<th>$E_c$ (kg/cm²)</th>
<th>$E_s$ (kg/cm²)</th>
<th>$f_c$ (kg/cm²)</th>
<th>$f_y$ (kg/cm²)</th>
<th>$\rho = A_s/Bd$</th>
<th>$\rho' = A_{cc}/Bd$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1MA</td>
<td>2.71 × 10⁵</td>
<td>1.98 × 10⁶</td>
<td>323</td>
<td>3236</td>
<td>0.0062</td>
<td>0.0</td>
</tr>
<tr>
<td>B5</td>
<td>1.10 × 10⁵</td>
<td>2.04 × 10⁶</td>
<td>334</td>
<td>5623</td>
<td>0.015</td>
<td>0.0075</td>
</tr>
<tr>
<td>J4</td>
<td>2.67 × 10⁵</td>
<td>2.07 × 10⁶</td>
<td>340</td>
<td>3157</td>
<td>0.0099</td>
<td>0.0</td>
</tr>
</tbody>
</table>

$v_c = 0.167$, $G_c = E_c/(1 + v_c)$, $G_c/f_t = 0.0075$ mm.  

---

Fig. 11. Outline of solution algorithm.
8.1. Beam T1MA

Fig. 13(a) and (b) shows the contribution of the tension softening branch and bond-slip to the moment–curvature relation of the example structure, respectively. The moment–curvature relation of this under-reinforced specimen is notably affected by the tension softening branch at the cracked stage, as shown in Fig. 13(a), because the contribution by the concrete to the structural behavior increases as the structure is under-reinforced. By using the tension softening branch introduced in Eq. (6), the increase of flexural rigidity according to the decrease of finite element mesh size is reflected. The modified moment–curvature relation leads to response predictions which are essentially independent from the finite element mesh size. On the other hand, bond slip represents the opposite effect to tension softening on the structure response, that is, the bond slip reduces the flexural rigidity of the structure. Fig. 13(b) shows the difference between the moment–curvature relations considering bond slip and those based on the perfect bond assumption. As shown in this figure, the flexural rigidity has a marked reduction at the cracked stage, following the increase of yielding curvature more than two times. However, the difference according to the location of the considered element is relatively small. It means that the governing parameters which affect bond-slip behavior are the relative slip and steel stress corresponding to the crack width. Especially, the element located at support (fifth element in Fig. 13(b)) represents a slightly more flexible bending rigidity than the element at midspan (first element). This phenomenon can be explained by the increase of average slip as the decrease of specimen length in the pull-out test of the reinforcing bar.

Based on the moment–curvature relations defined in Fig. 13, the numerical analyses are conducted, and the results are compared with experimental data in Fig. 14. If the tension softening branch is not taken into account in the analysis \( (\varepsilon_c = \varepsilon_c^0) \), the load–deflection curves exhibit a more flexible response than the experiment regardless of the finite element mesh size. In spite of the flexible response, however, the central deflection near the ultimate loading condition is still underestimated because of not considering the bond-slip effect. In the case of considering the tension softening branch, the load–deflection behavior is a little stiffer than the experiments (see Fig. 14(b)).

However, the predicted responses show that the tension softening branch adopted in this study exhibit satisfactory behavior even in beam element and gives reasonable results which are essentially independent from the finite element mesh size up to the ultimate loading stage.

Fig. 15 shows the effect of bond slip and tension softening and the relative contribution of each source

<table>
<thead>
<tr>
<th>Beam</th>
<th>B (cm)</th>
<th>H (cm)</th>
<th>d (cm)</th>
<th>Loading</th>
<th>a or b (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1MA</td>
<td>15.24</td>
<td>30.48</td>
<td>27.23</td>
<td>B-type</td>
<td>90.00</td>
</tr>
<tr>
<td>B5</td>
<td>17.78</td>
<td>38.10</td>
<td>32.385</td>
<td>B-type</td>
<td>91.44</td>
</tr>
<tr>
<td>J4</td>
<td>15.24</td>
<td>50.80</td>
<td>45.72</td>
<td>A-type</td>
<td>180.0</td>
</tr>
</tbody>
</table>

Table 2: Details of RC beams
of deformation to the load–displacement response of the specimen. The initial discrepancy between analysis and experiment stems from the fact that the specimen was probably extensively cracked before loading. Otherwise, the proposed model shows very satisfactory agreement with the measured response. Through the numerical analysis of this example structure, it can be concluded that the inclusion of both effects yields a very satisfactory agreement with the experimental data.

8.2. Beam B5

Differently from beam T1MA which is under-reinforced, the structural response of the over-reinforced beam B5 is predominantly affected by the reinforcing steel, as shown in the moment–curvature relations of Fig. 16(a). The tension softening is negligible, and the difference according to the location of the element in the case of considering the bond-slip effect also decreases. This result arises from the fact that the concrete reaches its maximum capacity before the steel yields, and that the widths of the flexural cracks in the tension zone of the concrete at the failure section are small, owing to the low steel stress.

To identify the relative contribution of tension softening and bond-slip effect, three different analyses were performed. The responses represented in Fig. 17(a) exclude both effects, while the responses represented in Fig. 17(b) include the tension softening branch only. It is clear from the comparison of these results with the experimental data that the tension softening is negligible in the over-RC beam, as predicted in the moment–curvature relations. Nevertheless, if the tension softening branch is not considered, the structural behavior is relatively underestimated at the cracked stage in spite of the strong brittle behavior.

Fig. 18 shows two analytical results which consider the tension softening branch. As shown in this figure, the inclusion of bond slip (dotted line) produces a slightly more flexible response even if the difference is very small. This result means that bond slip has a negligible effect, and it was also predicted since the increase of yielding curvature due to the consideration of bond-slip effect is about 50%, which is a relatively small increase in comparison with that of the under-RC beam T1MA (see Figs. 16(b) and 13(b)). However, any more improvement in the structural behavior is not achieved. It seems to be the limitation in analysis of over-RC beams using the beam element. Finally, it can be concluded that neither tension softening nor bond slip is important in the analysis of heavily over-RC beams.

8.3. Beam J4

Since this specimen is still under-reinforced, the increase of curvature near yielding according to bond slip is rather closer to that of beam B5 than to that of beam T1MA (see Fig. 19(b)). The only remarkable difference between beam B5 and beam J4 is that the under-reinforced concrete beam J4 sustains the post-yielding deformation to the load–displacement response of the specimen. The initial discrepancy between analysis and experiment stems from the fact that the specimen was probably extensively cracked before loading. Otherwise, the proposed model shows very satisfactory agreement with the measured response. Through the numerical analysis of this example structure, it can be concluded that the inclusion of both effects yields a very satisfactory agreement with the experimental data.

Fig. 14. Bond-slip effect of beam T1MA. (a) Exclusion of tension softening branch and (b) inclusion of tension softening branch.

Fig. 15. Tension softening and bond-slip effect of beam T1MA.
behavior to the large curvature, but the over-RC beam B5 fails immediately after the steel yields. This structure will be affected by bond slip, because the steel yielding around the midspan accompanies the large crack widths, and the structural response depends mainly on the structural behavior at midspan.

Fig. 20 compares the analytical results with the measured load–displacement response of beam J4. The effect of tension softening is included in both analytical results. The responses shown in Fig. 20(a) exclude the effect of plastic hinge length, while the responses shown in Fig. 20(b) include this effect. It is clear from the comparison of those results with the experimental data that the exclusion of plastic hinge length when the element size is greater than the expected plastic hinge length may yield an overestimated ultimate load. It arises from the fact that the change of flexural rigidity is calculated by averaging the deformations at both end nodes. Accordingly, the plastic hinge length must be taken into account to estimate the ultimate load exactly.

Fig. 21 shows that the inclusion of three effects, that is, the term tension softening, bond slip, and plastic hinge length, yields a very satisfactory agreement of the model with reality. Moreover, this figure shows that

![Fig. 16. Moment–curvature relations of beam B5. (a) Exclusion of tension softening branch and (b) inclusion of tension softening branch.](image)

![Fig. 17. Load–deflection relations of beam B5. (a) Exclusion of tension softening branch and (b) inclusion of tension softening branch.](image)

![Fig. 18. Tension softening and bond-slip effect of beam B5.](image)
contribution of bond slip to the load–displacement response of the specimen increases with the load. Near the ultimate strength of the beam, the magnitude of the bond slip contribution to the load–displacement response is remarkably increased.

9. Conclusions

Based on the moment–curvature relations of RC sections including the bond-slip and tension softening branch, an analytical model for the material nonlinear analyses of RC beams has been introduced. In addition, the plastic hinge length has been taken into consideration with the purpose of removing imprecision in calculations of ultimate resisting capacity. The introduced model has been verified by comparison between experimental results and numerical examples.

The representative RC beams were analyzed with the purpose of investigating the relative effects of bond slip, tension softening and plastic hinge length and the following conclusions were obtained: (1) the plastic hinge length must be considered to predict the ultimate
strength of RC beams where the plastic deformation is concentrated at any location with narrow range; (2) for under-RC beams, the tension softening and bond slip have the dominant influences at the cracked range, while the tension softening can be negligible for over-RC beams; (3) the numerical results for over-RC beams indicate that the bond-slip effect can also be ignored. However, this result seems to be caused by using the beam element. That is, there are some restrictions in modeling the over-RC structure with the beam element; (4) the simplified numerical analysis with the moment–curvature relation still can be effectively used in the nonlinear analysis of RC beams.

References