A Method for Predicting Drawdown at the Radius of a Pumping Well for Large Complex Systems

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Abstract

One of the challenges in groundwater modeling is the prediction of hydraulic head in close proximity to a pumping well using a regional-scale model. Typical applications of numerical models to field-scale problems generally require large grids that can seldom accommodate cells as small as the actual well diameter [Anderson and Woessner 1992]. Several methods have been used to simulate a more accurate head at the well scale. The two primary methods for head prediction are: (1) an analytical method, and (2) grid refinement using a numerical model. However, these two methods have limitations, particularly for applications that involve the development of numerical models for large-scale hydrogeologic systems with multiple pumping wells. Traditional methods may become computationally problematic and increase the time and resources needed for storing and processing simulation results.

A method for predicting head at the well scale that is more general, computationally efficient, and accurate was presented. Interactive Ground Water (IGW) [Li et al. 2002; Li and Liu 2003; Li and Liu in review], which adopts a hierarchical modeling paradigm, was used to illustrate this approach and determine hydraulic head at the well scale. The performance of the hierarchical-modeling approach against the analytical solution for a single well and against a superposition of analytical solutions for a well field was presented. Our results demonstrate that the hierarchical-modeling approach is capable of matching an exact solution of well drawdown and providing an accurate representation of head and groundwater velocities in a well field in large-scale hydrogeologic systems.

Introduction

In groundwater modeling, it is usually impractical to employ grids that are comparable in size and dimension to a pumping well. However, predicting head, or drawdown, in close proximity to a pumping well (well scale) may be important for certain groundwater-flow and solute-transport modeling applications, especially in complex environments. In groundwater-flow modeling, long-term drawdown prediction at the well is critical for proper well design [Beljin 1987]. It is also important to predict drawdown at the well scale where models are used for evaluation of groundwater-management and sustainability strategies. In solute-transport modeling, the estimated velocity field is derived from head values at the model nodes; this is needed to predict the advection component of the transport of solutes. It is also necessary for the proper design of remedial actions.

In finite-difference models, numerous finite-difference cells, that typically have relatively-large spatial dimensions, are used to represent the aquifer system. A point source or sink of water is injected, or extracted, over the volume of aquifer represented by the cell that contains the point source or sink [Anderson and Woessner 1992]. However, the diameter of the well is typically much smaller than the dimensions of the cell. Field-scale problems generally cover large geographic areas that require grids having cells with large spatial dimensions and can seldom accommodate cells as small as the actual well diameter. The resulting simulated heads
are generally not a good approximation of heads or hydraulic gradients in close proximity to the pumping well, or any other source or sink; numerous small cells are needed to simulate the relatively-steep hydraulic gradients near a point source or sink accurately. However, regional-model-derived heads may be correct at nodes located away from the point source or sink [Anderson and Woessner 1992].

Several methods have been adopted in an attempt to improve the accuracy of simulated heads at the well scale. Two methods used in finite-difference models are: (1) an analytical method (corrected drawdown) [Prickett 1967, Peaceman 1978, Pritchett and Garg 1980], and (2) grid refinement using a numerical model local-grid-refinement (LGR) [e.g. Mehl 2002], or telescopic-mesh-refinement (TMR) [Ward et al. 1987]. However, these two methods may have severe limitations, particularly for applications that involve the development of numerical models for large-scale, complex hydrogeological systems with multiple pumping wells, or other sources or sinks. In particular, the application of grid-refinement techniques to these complex problems, may lead to slow convergence, numerical oscillations, computational inefficiencies, and solution failure [Li et al. in review]. In addition, the time and computational resources needed to store and process simulation results may be great.

The objectives of this paper are to address the limitations and drawbacks of the traditional methods used to predict heads at the well-scale for large complex groundwater systems, and to present an innovative methodology for determining well-scale heads for such systems. Interactive Ground Water (IGW) [Li and Liu 2003; Li and Liu in review], which uses the hierarchical-modeling approach, overcomes the difficulties encountered using traditional finite-difference methods in calculating well-scale heads. Furthermore, IGW is capable of modeling large complex groundwater systems in a flexible and computationally-efficient framework on typical desktop computers.

In the following sections, we review the two traditional methods for calculating hydraulic heads at the well scale, and point out the limitations with the assumptions and implementation of these methods. Next, we introduce the hierarchical-modeling approach and explain the concept behind this approach. Finally, we present illustrative examples to verify and show the capabilities of the hierarchical-modeling method.

**Traditional methodologies**

Two traditional methodologies employed in finite-difference models for predicting the drawdown or hydraulic head at the radius of, or in close proximity to, a pumping well (well scale) using regional-scale models are (1) an analytical method (corrected drawdown), and (2) grid refinement using a numerical model (LGR or TMR).

**Analytical method: the corrected drawdown**

The model-calculated head can be thought to represent the head at some distance \(( r_e )\) from the well node. An estimation of the head in the well can be obtained from formulas based on the steady-state Thiem equation [Thiem 1906], which can be applied to quasi-steady-state conditions when the rate of removal of water from storage near the pumping well is zero. The head in the well is calculated from the following equation [Anderson and Woessner 1992]:

\[
h_w = h_{i,j} - \frac{Q_{WT}}{2\pi T} \ln \left( \frac{r_e}{r_w} \right)
\]  

(1)

Where \( h_w \) is the head in the well; \( h_{i,j} \) is the head computed by the finite-difference model for the well node \((i,j)\); \( Q_{WT} \) is the total pumping or injection rate from the well; \( T \) is the transmissivity; \( r_e \) is the radial distance measured from the node at which head is equal to \( h_{i,j} \); and \( r_w \) is the radius of the actual well. The radius \( r_e \) is the effective well-block radius. Prickett [Prickett 1967], Peaceman [Peaceman 1978, 1983] and Trescott [Trescott et al. 1976] provided an equation, which approximates \( r_e \) based on different model grid sizes. Their assumptions in applying this approximation are: (1) flow to the well is within a
The square finite-difference cell (well block) and can be described by a steady-state equation with no source term except for the well discharge, (2) the aquifer is isotropic and homogeneous in the well block, (3) only one well, located at the cell center, is in the well block, (4) the well fully-penetrates the aquifer, (5) flow to the well is laminar, and (6) well loss is negligible. Trescott [Trescott et al. 1976] examined the performance of the corrected drawdown against the analytical solution. Results revealed a reasonable improvement for predicting the drawdown in the pumping well. Pritchett [Pritchett and Garg 1980], and Beljin [Beljin 1987] presented a more general approximation for the Thiem equation. They investigated the cases where: (1) a well is not positioned at the center of the well block, (2) the well block is rectangular having different aspect ratios, (3) the aquifer media is anisotropic, and (4) there is more than one well within the block.

The application of the analytical correction method results in a drawdown prediction that is comparable to the exact solution [Trescott et al. 1976]. However, this method suffers from several limiting factors; one of which is that it cannot easily be applied for multiple sources and sinks within the well block. In addition, the corrected drawdown method is not valid for other general cases, such as variable hydraulic conductivity, variable recharge, anisotropic media with specified orientation (e.g., fractured rock), partial well penetration, and transient flow or pumping conditions.

The second traditional methodology, grid refinement, overcomes the limiting assumptions of the corrected-drawdown method. However, for large grids and complex hydrogeologic systems, this method suffers from other conceptual, structure, and computational difficulties.

### Numerical method: grid refinement

A model covering a relatively-large domain and having grids with large spatial dimensions relative to the well scale is often referred to as the “regional model.” With the grid-refinement method, the relatively-large finite-difference grid cell from a regional model is subdivided into multiple cells with progressively-smaller spatial dimensions. This results in a more accurate estimation of hydraulic head or drawdown at the well scale. In other words, there will be less approximation error involved with averaging the predicted head for that particular well node.

Two different approaches are used to implement the grid-refinement method. In the first approach, LGR, the grid spacing is subdivided so that it is finer in the area of interest (e.g., surrounding of the well node). With LGR, there is one model with the area of grid refinement that is part of the regional model. This often results in a very large number of grid cells. In second approach, TMR, grid-dependent information from the regional model is used to construct a separate model with finer grid spacing to obtain more information around the area of interest. The procedure is called “telescopic-mesh-refinement” because it can be applied repeatedly to construct multiple, successively-smaller, embedded models [Ward et al. 1987].

With TMR, the model with the finer grid is called a “submodel” or “local model.” In order to obtain detailed information for the local model, it is necessary to interpolate grid-dependent information from the regional model at the finer grid spacing of the local model. The transformation of information from regional model to local model and the selection of boundary conditions and starting conditions for the local model are the most important issues using this numerical methodology [Townley and Wilson 1980; Ward et al. 1987; Buxton and Reilly 1986].

Our objective here is to identify advantages and limitations in using either approach to calculate head or drawdown at the well scale. In using LGR for simple or small-scale problems, the solution is obtained quickly, and consistency between the regional and local model is maximized. Furthermore, there will not be a significant increase in effort for processing the results of the model simulations for the regional model or local model [Leake et al. 1998]. However, for large-scale, regional groundwater models where there may be a significant increase in the number of nodes (e.g., millions rather than thousands), the cost of computation increases exponentially and the process becomes computationally intensive. Additionally, for large problems that involve multiple sources and sinks, multiple scales of interest, transient flow conditions, complex aquifer
structure, and strong anisotropy and heterogeneity, the solution process using LGR can become problematic. In these cases, the structure of the matrices may be highly “ill-conditioned,” which oftentimes leads to lack of convergence or numerical oscillations [Li et al. in review]. Selecting a proper numerical algorithm or a matrix solver can improve convergence or dampen oscillations, to some extent; however, the major problem, which often results in solution failure, is the large number of grids in a very large, non-uniformly-structured matrix.

TMR avoids the potential difficulties encountered with LGR by separating the regional model from the local model, and solving each model individually. Therefore, instead of solving very large, complex matrices, a problem is solved using multiple, smaller-scale local models. This procedure can be implemented (in principle) successively in a continuous manner until the desired resolution for the well scale is obtained. The local model will derive its boundary and starting conditions from the parent model. Once the local model is created, it performs as an independent model. The primary advantage of TMR is that it can reduce a large and complex matrix system into multiple smaller and better-conditioned matrices.

However, the major drawback to implementing TMR is that the interaction between the parent and local models (of which there can be several) depends on the offline analysis and processing of model modifications or simulation results from the parent model to obtain the boundary and starting conditions for the local model. For example, once a new simulation is completed using the regional model, new boundary conditions and starting flow and solute-transport conditions, if applicable, are determined for each local model at the local-model grid spacing. Making modifications to models or processing simulation results for use in different scales of models can be very time consuming, especially when the problem is a transient-flow or transport condition, or there is uncertainty in selecting the boundary conditions. The effort involved may become impractical when the offline conceptual changes must be made iteratively or in more than one model. Because of this, applications of the TMR are limited, in most cases, to very small number of submodels (e.g., 1 or 2), and are implemented with little flexibility.

We present an innovative method that is more computationally efficient than LGR or TMR while maintaining the capability of accurately predicting detailed head or drawdown at the well scale.

**Hierarchical modeling approach**

Our approach for predicting head or drawdown at the well scale employs the concept of hierarchical modeling in a unique, general purpose, and object-oriented computational environment, Interactive Ground Water (IGW) [Li and Liu 2003; Li and Liu in review]. With this approach we are able to reduce a large, very-complex problem into a number of small, less complex problems that are solved individually in a dynamically-coupled and fully-integrated environment “patch-dynamic framework” [Li et al. in review]. The hierarchical modeling approach overcomes problems encountered when attempting to solve a very large set of complex, ill-conditioned matrices often encountered with LGR. The approach is more advantageous than the traditional, loosely coupled TMR in that the interaction between the regional model and all submodels is seamless, dynamic, and fully-integrated. In other words, regional-model-simulation results (e.g., boundary conditions), and any changes to the regional model propagate automatically to all submodels, without the need for offline post-processing of data or simulation results. This transfer of information between the regional model and submodels is accomplished for each time step in near real-time. IGW hierarchical modeling achieves these distinct capabilities through the adoption of the following new, integrated computational paradigm [Li and Liu 2003; Li and Liu in review](showing in the following text box).

The dynamic paradigm eliminates the traditional TMR disconnect and makes generalized hierarchical modeling practical. In particular, this paradigm provides an efficient means of routing data between models and also provides visualization controls at each time step for all models during the simulation. This gives the modelers the perception of using a single model.
that provides high resolution dynamics at the speed that is very similar to that of a low-resolution coarse-grid model. The object-oriented implementation enables flexible, interactive creation of hierarchical models. This “hierarchical modeling process” is an interactive and recursive process by which a modeler can create a hierarchy of submodels that are embedded in a parent model in order to provide greater detail where it is required. IGW provides an intelligent and integrated modeling environment in which to conduct hierarchical model [Li and Liu 2003; Li and Liu in review].

Our experience with hierarchical modeling and submodel development has shown that to maintain computational efficiency and robustness in modeling detailed flow dynamics around wells in a very large complex system it is best to limit the number of grid cells and keep grid spacing relatively uniform. This may result in the development of multiple submodels with successively-smaller grid spacing until it is possible to estimate head accurately at the well scale. However, with hierarchical modeling and IGW, this process of creating an appropriate number of submodels with successively-smaller grid spacing is accomplished easily and in a way that is naturally intuitive [Li and Liu 2003; Li and Liu in review].

**Illustrative examples**

In this paper, we apply IGW to illustrate and verify the hierarchical-modeling-approach for predicting the drawdown at the radius of a pumping well by performing a systematic comparison with an analytical solution (Theis equation). Following the first example, we extend our examination to a more general and complex well-field system and the simulation of detailed flow dynamics around wells in a large regional system.

**Single well example**

The simulation results obtained using IGW are compared to the analytical solution given by Theis [Theis 1935] to demonstrate the ability of the hierarchical-modeling-approach to predict drawdown at the radius of a pumping well. The analytical solution assumes that the confined aquifer is homogenous and isotropic, with impermeable boundaries bounding the top and bottom of the aquifer. It further assumes that the aquifer has infinite lateral extent. The objectives of this example are to (1) verify the use of hierarchical modeling in predicting the drawdown at the radius of a pumping well, and (2) to illustrate the process and concept of hierarchical modeling in assessing the solution, as it is implemented in IGW.

In the finite-difference model developed to solve this problem, the aquifer is represented by a rectangular layer whose extent is 10,000 meters by 10,000 meters. No-flow boundary conditions are imposed along each face of the model. The aquifer parameters are as follows: the transmissivity and storage coefficient of the aquifer are 17.3 m²/day and 0.001, respectively; the discharge rate of the pumping well is 518.4 m³/day; the well radius is 0.1 meters (0.1m) and the duration of pumping is 50 days.

Figure 1 presents the hierarchical-network and solutions to the single well example problem. The regional model and the six submodels have grid resolutions of 500m, 100m, 30m, 9m, 2.7m, and 0.81m respectively.

Each model has a grid that is 21 rows by 21 columns in size. Figure 2 illustrates the drawdown comparison between the numerical model and analytical solution. Drawdown data from every other node were used to prepare this graph. It is apparent, in examining the results obtained from hierarchical-modeling-approach, that
the numerical model closely-approximates the analytical solution. Also, it is important to point out that the results from each submodel were obtained and visualized instantaneously. And, with hierarchical modeling, different drawdown values may be recomputed, displayed, and analyzed very quickly whenever the model stresses or other model parameters are changed.

Figure 1 - Presents the hierarchical-network and simulated heads after 50 days pumping for the single well example problem. Each model has a grid that is 21 rows by 21 columns in size.
Figure 2 - Illustrates the drawdown comparison at the end of 50 days between the numerical model and analytical solution. Each model has a grid that is 21 rows by 21 columns in size.

An alternative hierarchical model applied to the same single well problem is shown in Figure 3. In this case, each model has a larger grid that is 51 rows by 51 columns in size, requiring only three nested submodels to accurately predict the drawdown at the well scale. Figure 4 illustrates the drawdown comparison between the analytical solution and numerical hierarchical models. On this graph, only the simulated drawdown results from every other model node are plotted. The alternative hierarchical solution show that there is more than one way to subdivide a larger regional model into several smaller submodels. The decision on how to subdivide a regional model depends primarily on the computational ability of the computer available to perform the analysis. One may choose to divide the regional model into larger numbers of smaller submodels when using a less powerful computer.

Likewise, an approach using a smaller number of larger submodels would be taken when using a more powerful computer. This flexibility eliminates the longstanding, infamous “curse of dimensionality” in large-scale groundwater...
modeling and allows the modeler to simulate detailed flow dynamics, even around low capacity wells, in a large, regional groundwater system.

**Well field example**

The objective of this example is to illustrate the capabilities of the hierarchical-modeling-approach in calculating hydraulic heads for a large grid containing a well field with multiple wells each pumping at different rates. This example applies for groundwater-management problems, delineating the wellhead protection area (WHPA) for a well field, or prediction of groundwater-flow directions and velocities for solute-transport modeling.
Computational difficulties encountered in modeling well fields center around calculating heads and groundwater flow rates where there are clusters of wells that pump at different rates and with different pumping schedules. In this case, the system may be very dynamic with spatially- and temporally-variable heads and groundwater flow. In areas far removed from the well field, predictions using a regional model are usually accurate; however, closer to the well node, a finer grid is needed to model the system. Model grids of different extent and cell size are needed to model the system accurately so that the flow dynamics in a well field with multiple well clusters and variable pumping rates are resolved.

Another difficulty in modeling a well field is representing the exact location of the individual pumping wells, especially those in close proximity to each other. With a regional model, or coarse grid submodel, clusters of closely spaced wells are
grouped together in a single well node. A locally refined grid is needed to simulate the impact of individual wells on heads and flow rates.

The model simulation results are compared to an analytical solution for drawdown in a well field obtained by superimposing Theis solutions for each pumping well to verify the hierarchical-modeling-approach for a well field in a regional model. The distribution of the pumping wells and their pumping rates for the well-field example are shown in Figure 5. The regional-model domain is 100 km by 100 km with a uniform grid size of 4000 meters. As with the single-well example, the outer boundaries are represented by no-flow boundaries. The no-flow boundaries in the regional model have been placed at this distance so they do not affect the simulated drawdown in the well field, satisfying the assumption of an infinite aquifer. The aquifer parameters are as follows: the transmissivity and storage coefficient of the aquifer are 200 m²/day and 0.001, respectively; the well radius is 0.1 meter (0.1m); and the duration of pumping is 64 days. The discharge rates of the pumping wells are variable and are shown in Figure 5.

**Figure 5** - The distribution of the pumping wells and their pumping rates for the well-field example.
Figure 6 - Hierarchical modeling layout which illustrates the relationship between submodels and their parent models. [In label of Submodel $M^{a}_{bc}$, $a$ represents the generation level (grandparent), $b$ is the parent index, and $c$ is the kid index which is related to parent index].
Figure 7 - The modeling results are compared to the analytical solution on a cross-section profile (A-A) for different submodel levels and grid resolutions (a)- large scale (b)- local scale.
One possible example of the application of hierarchical modeling is shown in Figure 6. In this example, different levels of submodels are developed for each pumping well. Starting from a coarse model we can successively approach the area of interest and obtain detailed information as close as the effective well radius. The diagram shown in the upper right quadrant of this figure illustrates the hierarchical relationship between parent models and subsequent submodels. The remainder of the figure shows the different model domains, pumping wells and simulated drawdowns. Lines showing the relationship between parent model and dependent submodel domains were not included for figure clarity reasons. However, an examination of the hierarchical tree and the different model domains should be sufficient to understand the relationship between models. The grid resolution from the regional model to finest well-scale model varies between $4000 \text{ m}$ to $0.49 \text{ m}$. The number of rows or columns in these models is typically 30 to 100, depending on the size of the model area and the uniform grid spacing for that model.

Figures 7a and 7b show the comparison between modeling results for different submodel levels and grid resolutions and the analytical solution obtained by superimposing Theis solutions for the different pumping wells. The results are plotted along a cross-section profile (A-A) drawn through the regional model whose location is shown in Figure 6. In the drawdown comparison we show the results from the regional model ($M_{11}^{11}$) through the finest-grid submodel ($M_{11}^{8}$). The area of interest in this problem is the pumping well shown in the center of model $M_{11}^{8}$ (pumping well of interest). Figure 7a shows the simulated heads along a profile through the entire well field from model $M_{11}^{2}$ including the pumping well of interest. Figure 7b shows simulated heads from model $M_{11}^{8}$ along the same profile through the pumping well of interest. However, in this figure, the lateral extent of the profile is limited to the extent of the domain of model $M_{11}^{8}$ in the X-direction. As with the single well example, these results show that, as the grid spacing is reduced, the numerical solution approaches the analytical solution. In addition, the results from each submodel were obtained and visualized instantaneously.

With IGW and hierarchical modeling, different drawdown values may be recomputed, displayed, and analyzed very quickly whenever the model stresses or other model parameters are changed.

**Summary**

With numerical models, the simulated head at the well scale is the average of the head over the well block and does not reflect the head value at the well or the hydraulic gradients in close proximity to the well. Since the applications of numerical models to field-scale problems typically cover relatively-large geographic areas, a great number of cells having large size, relative to a pumping well, are used. Analytical and numerical methods have been adopted, to better evaluate the drawdown at the radius of a pumping well. However, these traditional methodologies have limitations. The analytical method has limiting assumptions that cannot be applied for general groundwater modeling. With numerical methods, LGR creates computational difficulties because of the large number of nodes, and TMR suffers from the discontinuity between parent and submodels and the time and effort required to process data between parent and submodels.

In this paper we presented an innovative methodology to predict the head at the radius of a pumping well for large-scale and complex groundwater-flow systems. Utilizing IGW, our method employs the dynamically integrated, object-oriented, hierarchical-modeling concept resulting in a highly efficient and flexible way to simulate heads at the well scale. When modeling large-scale complex groundwater systems, the IGW hierarchical-modeling-approach allows the modeler to obtain an accurate solution at the desired grid resolution quickly and with little difficulty. Convergence to a solution using this method is computationally efficient and can be obtained using typical desktop computers, even for large complex field applications.
References


