Module 5:
Non-Ideal Behavior of Circuit Components

5.0 Introduction

Most engineers are introduced to common circuit elements such as resistors, capacitors, and inductors through a course in basic circuit analysis. In this context, these common devices are often presented in the ideal sense, i.e. as being purely resistive, capacitive, or inductive, respectively. The effects due to wires, leads, and connectors are also usually neglected in circuit analysis. While this approach is clearly necessary to impart the fundamental concepts of circuit behavior, this ideal presentation often leads to misconceptions about how actual devices function. In reality, each type of circuit element exhibits some combination of resistive, capacitive, and inductive behaviors when operated at any frequency other than zero. The types of materials and construction techniques employed may also affect the performance of circuit elements. These factors can cause the impedance, capacitance, or inductance of these devices to differ greatly from the expected ideal values. As was demonstrated in previous chapters, even simple devices and circuits may require relatively complex models in order to correctly predict behavior over a wide frequency range. Through the application of certain approximations, these complex models may be simplified somewhat, while still providing the needed physical insight into device behavior. In this chapter, equivalent circuit models of basic circuit elements will be developed and analyzed. The response of these circuit elements to a broad range of operational inputs will be discussed and related to the topics presented in the earlier sections. From this it will become clear that what is usually referred to as "non-ideal" behavior of a circuit element is, in actuality, perfectly natural behavior in a regime that lies outside the range of validity of commonly accepted approximations.

5.1 Internal Impedance of Electric Circuit Elements

- **circular wires**

Effects due to wires are often overlooked when considering the behavior of electric circuits. For most low-frequency applications, wires are modeled as having no effect on circuit performance. However, the impedance of wires can become significant under certain conditions. In this section a general expression for the internal impedance of a long cylindrical conductor (i.e., a wire) will be developed.

Consider an infinitely-long, thin conductor, with cylindrical cross-section of radius a, having permittivity ε, permeability μ, and conductivity σ. It will be assumed that a current with density $\vec{J}$ flows through the conductor, supported by electric and magnetic fields $\vec{E}$ and $\vec{H}$. Because the conductor is infinite in length, and has rotational symmetry, longitudinal and axial invariance will be assumed, therefore

$$\frac{\partial \vec{E}}{\partial z} = \frac{\partial \vec{H}}{\partial z} = 0$$

and

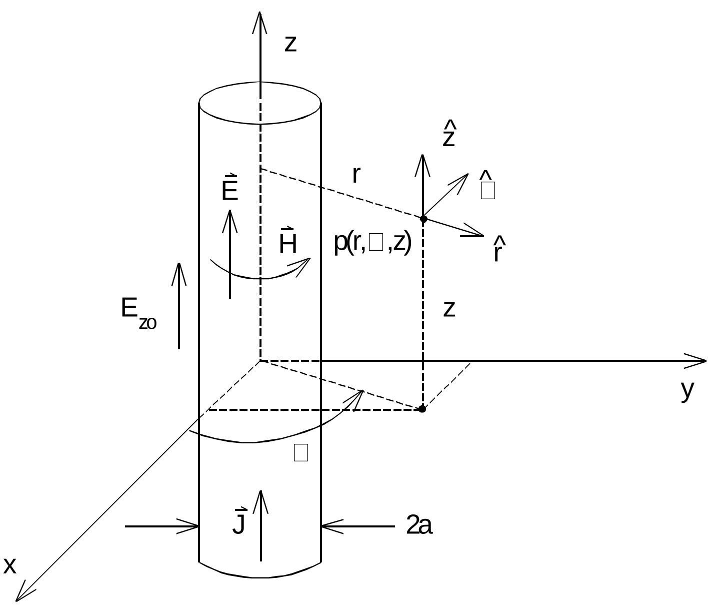

Figure 1. Infinitely-long cylindrical conductor.

$$\frac{\partial \vec{E}}{\partial \varphi} = \frac{\partial \vec{H}}{\partial \varphi} = 0 \, .$$

It will also be assumed that only an axially directed electric field exists within the conductor

$$\vec{E} = \hat{z} E_z(r)$$

which corresponds to the current density within the conductor. Also let the electric field maintained at the conductor surface by the external sources which excite the system be represented by

$$E_{zo} = E_z(r = a) \, .$$

A system of time-harmonic Maxwell's equations appropriately specialized for this structure is

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} = 0$$

$$\nabla \times \vec{E} = -j\omega \vec{B}$$

$$\nabla \times \vec{B} = \mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla \cdot \vec{B} = 0 .$$

Expanding the curl equations yields

$$-\hat{\varphi}\frac{\partial E_z}{\partial r} = -j\omega(\hat{r}B_r + \hat{\varphi}B_\varphi + \hat{z}B_z)$$

and

$$-\hat{\varphi}\frac{\partial B_z}{\partial r} + \hat{z}\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \mu(\sigma + j\omega\varepsilon)\hat{z}E_z .$$

From these it is apparent that $B_r = B_z = 0$ with

$$\frac{\partial E_z}{\partial r} = j\omega B_\varphi$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \mu(\sigma + j\omega\varepsilon)E_z . \quad (*)$$

Substitution of the expression

$$B_\varphi = \frac{1}{j\omega}\frac{\partial E_z}{\partial r}$$

into (*) yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{j\omega}\frac{\partial E_z}{\partial r}\right) = \mu(\sigma + j\omega\varepsilon)E_z .$$

which may be rewritten

$$\frac{\partial^2 E_z}{\partial^2 r} + \frac{1}{r}\frac{\partial E_z}{\partial r} + k^2 E_z = 0 \qquad (**)$$

where

$$k^2 = \omega^2 \mu\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon} \right) .$$

Equation (**) is known as *Bessel's equation* of order zero, with parameter k. This second order partial differential equation has solution

$$E_z(r) = A J_o(kr) + B N_o(kr)$$

where $J_o(kr)$ is a *Bessel function of the first kind* of order zero, and $N_o(kr)$ is a *Bessel function of the second kind* of order zero. A review of Bessel functions appears at the end of this chapter.

The region under consideration is that lying inside the conductor. This region contains the point $r = 0$, therefore the behavior of the Bessel functions for small arguments must be examined. As r approaches zero, the Bessel function of the second kind becomes infinitely large. In order for E_z to remain finite, the constant B must be equal to zero. The general expression for the electric field at any point inside the conductor is thus

$$E_z(r) = A J_o(kr) .$$

The unknown constant A is determined through application of boundary conditions. At the surface of the cylindrical conductor

$$E_z(r = a) = E_{zo} = A J_o(ka)$$

thus

$$A = \frac{E_{zo}}{J_o(ka)} .$$

Substitution into the general expression for electric field within the conductor yields the electric field distribution over the cross-section of the wire.

$$E_z(r) = E_{zo} \frac{J_o(kr)}{J_o(ka)} .$$

Application of Ohm's law gives the distribution of current density in the cross-section of the conductor

$$J_z(r) \; = \; \sigma E_{zo} \frac{J_o(kr)}{J_o(ka)} \, .$$

The total axial current is found by integration of the current density over the cross-section of the conductor

$$I_z \; = \; \int_{c.s.} J_z(r)\, ds \; = \; \int_0^a J_z(r)\, 2\pi r\, dr \; = \; \frac{2\pi\sigma E_{zo}}{J_o(ka)} \int_0^a r J_o(kr) dr \, .$$

A change of variables is now made by letting $kr = x$. Then when $r = 0,\ x = 0$, and when $r = a,\ x = ka$, and $dr = 1/k\, dx$. This results in

$$I_z \; = \; \frac{2\pi\sigma E_{zo}}{J_o(ka)} \int_0^{ka} \frac{1}{k^2} x J_o(x)\, dx \, .$$

Application of a look-up integration formula for Bessel functions (see the Bessel function review section) leads to

$$I_z \; = \; \frac{2\pi\sigma E_{zo}}{J_o(ka)} \frac{1}{k^2} \left[x\, J_1(x) \Big|_{x=0}^{x=ka} \right]$$

or

$$I_z \; = \; \frac{2\pi\, a\, \sigma E_{zo}}{k} \frac{J_1(ka)}{J_o(ka)} \, .$$

From this expression for the total axial current, the internal impedance per unit length of a cylindrical conductor is determined

$$z^{\,i} \; \equiv \; \frac{E_{zo}}{I_z} \; = \; \frac{k}{2\pi\, a\, \sigma} \frac{J_o(ka)}{J_1(ka)} \, .$$

This general expression is valid for all frequencies. Approximations to this general formula can be made to obtain expressions which are valid in the limit that the frequency of operation is either very low, or very high.

- **internal impedance in the low frequency limit**

The frequency dependence of the impedance expression above occurs through the wavenumber term

$$k^2 = \omega^2 \mu\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon} \right) .$$

From this, it can be seen that the wavenumber k approaches zero as ω approaches zero in the low frequency limit. This would suggest performing a power series expansion to investigate the behavior of the Bessel function terms for small arguments. The Bessel function of the first kind may be expressed as a power series

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{n+2m}}{2^{n+2m}\, m!\, (m+n)!} .$$

For small values of z

$$J_o(z) = 1 - \frac{z^2}{4} + \ldots \approx 1 - \frac{z^2}{4}$$

and

$$J_1(z) = \frac{z}{2} - \frac{z^3}{16} + \ldots \approx \frac{z}{2}\left(1 - \frac{z^2}{8} \right) .$$

Substitution of these approximations into the general impedance expression results in

$$z^i \approx \frac{k}{2\pi a\sigma} \frac{1 - k^2a^2/4}{ka/2(1 - k^2a^2/8)} \approx \frac{1}{\pi a^2\sigma}\left(1 - \frac{k^2a^2}{4} \right) \left(1 + \frac{k^2a^2}{8} \right)$$

or

$$z^i \approx \frac{1}{\pi a^2 \sigma} - \frac{k^2}{8\pi\sigma}$$

but

$$k^2 = \omega^2 \mu\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon} \right) \approx -j\omega\mu\sigma$$

because $\sigma/(\omega\varepsilon) \gg 1$ at low frequencies. Substitution leads to the expression for the impedance per unit length of a cylindrical conductor in the low frequency limit

$$z^i = r^i + jx^i = r^i + j\omega l^i = \frac{1}{\pi a^2 \sigma} + j\omega\frac{\mu}{8\pi}$$

where

$$r^i = \frac{1}{\pi a^2 \sigma}$$

is the low frequency (dc) internal resistance per unit length, and

$$l^i = \frac{\mu}{8\pi}$$

is the low frequency (dc) internal inductance per unit length.

- **internal impedance in the high frequency limit**

Once again, we begin with an examination of the wavenumber term

$$k^2 = \omega^2 \mu\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon} \right) .$$

For a good conductor, conductivity σ is large, therefore it is assumed that $\sigma/\omega\varepsilon \gg 1$. It is apparent that k approaches infinity as the frequency ω approaches infinity, thus in the high frequency limit

$$k^2 \approx -j\omega\mu\sigma .$$

or

$$k = j\sqrt{j\omega\mu\sigma} = j\sqrt{\omega\mu\sigma}\left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}} = j e^{j\frac{\pi}{4}}\sqrt{\omega\mu\sigma}$$

$$= j\,\frac{\left(\sqrt{2} + j\sqrt{2}\right)}{2}\sqrt{\omega\mu\sigma} = (-1 + j)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{-1 + j}{\delta}$$

where

$$\delta \equiv \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}$$

is the skin depth or the depth that fields penetrate into a conductor, as discussed in Chapter 2.

In this case, the behavior of the Bessel function terms for large arguments must be investigated. The asymptotic formula for large arguments for the Bessel function of the first kind is

$$\lim_{x \to \infty} J_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(\pi x - \frac{n\pi}{2} - \frac{\pi}{4}\right) .$$

The internal impedance per unit length of the cylindrical conductor can now be expressed as

$$z^i = \frac{k}{2\pi a\sigma}\,\frac{J_o(ka)}{J_1(ka)} \approx \frac{k}{2\pi a\sigma}\,\frac{\cos\pi\left(ka - 1/4\right)}{\cos\pi\left(ka - 3/4\right)}$$

or

$$z^i = \frac{k}{2\pi a\sigma}\,\frac{e^{j\pi(ka - 1/4)} + e^{-j\pi(ka - 1/4)}}{e^{j\pi(ka - 3/4)} + e^{-j\pi(ka - 3/4)}} .$$

Now

$$e^{j\pi ka} = e^{j\pi(-1+j)(a/\delta)} = e^{-\pi a/\delta} e^{-j\pi a/\delta} \approx 0$$

when $a/\delta \gg 1$ at high frequency, therefore

$$z^i \approx \frac{k}{2\pi a\sigma} \frac{e^{-j\pi ka} e^{j\pi/4}}{e^{-j\pi ka} e^{j3\pi/4}} = \frac{k}{2\pi a\sigma} e^{-j\pi/2} = \frac{-jk}{2\pi a\sigma}$$

$$= \frac{-j(-1+j)}{2\pi a\sigma} \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{(1+j)}{2\pi a} \sqrt{\frac{\omega\mu}{2\sigma}} .$$

Thus the internal impedance per unit length of a good cylindrical conductor at high frequencies is

$$z^i = r^i + jx^i = \frac{(1+j)}{2\pi a} \sqrt{\frac{\omega\mu}{2\sigma}}$$

where

$$r^i = x^i = \frac{1}{2\pi a} \sqrt{\frac{\omega\mu}{2\sigma}} .$$

- **simplified derivation of wire resistance**

In the sections above, a rigorous development for the impedance of circular conductors was presented. Here a simplified derivation of the resistance of a wire will be developed. It is this model that is most often presented in textbooks and other EMC related publications.

The dc resistance of a wire with a circular cross-section having radius a, conductivity σ, and length l is given by

$$R = \frac{l}{\sigma\pi a^2} .$$

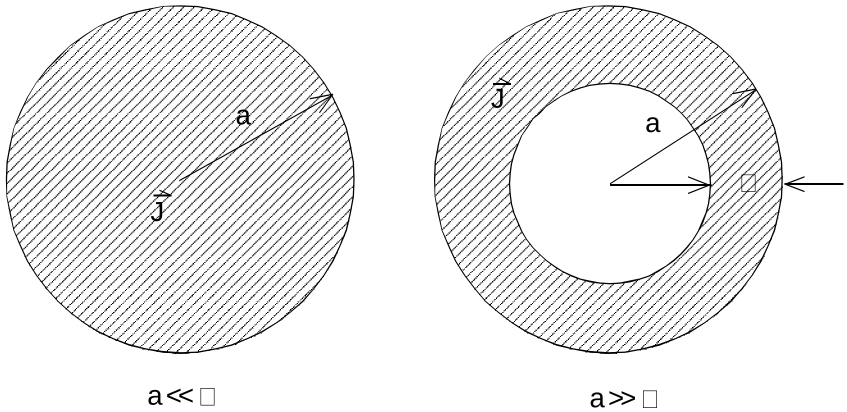

Figure 2. Current distribution in wire cross-section.

At low frequency, current is distributed nearly evenly throughout the cross-section of the wire. As the frequency of operation increases, current begins to accumulate at the periphery of the wire. This current will reside in a region that extends from the surface of the wire inward to a point equal to the skin depth, which is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu_o \sigma}}$$

where the skin depth is assumed to be smaller than the wire radius. Thus, as the frequency increases, the current becomes concentrated in an ever smaller region of the wire cross-section. The wire resistance per unit length is then simply

$$R_{low} \cong \frac{1}{\sigma \pi a^2} .$$

It can be seen that this value is the same as the real part of the low frequency impedance determined above.

At high frequency, as the current begins to accumulate near the wire surface, the resistance per-unit length can be approximated as

$$R_{high} = \frac{1}{\sigma\left[\pi a^2 - \pi(a-\delta)^2\right]} = \frac{1}{\sigma\pi\left[2a\delta - \delta^2\right]} .$$

Because at high frequency the skin depth is typically much smaller than the wire radius, the term δ^2 may be neglected, therefore

$$R_{high} \cong \frac{1}{\sigma 2\pi a\delta} = \frac{a}{2\delta} R_{low} = \frac{1}{2a}\sqrt{\frac{\mu f}{\pi\sigma}} = \frac{1}{2a}\sqrt{\frac{\mu\omega}{2\pi^2\sigma}}$$

or

$$R_{high} \cong \frac{1}{2\pi a}\sqrt{\frac{\omega\mu}{2\sigma}}.$$

Again, it is seen that this expression is the same as that obtained for the real part of the high frequency impedance per unit length.

5.2 External Impedance of Electric Circuits

- **self and mutual inductances of two coaxial circular loops**

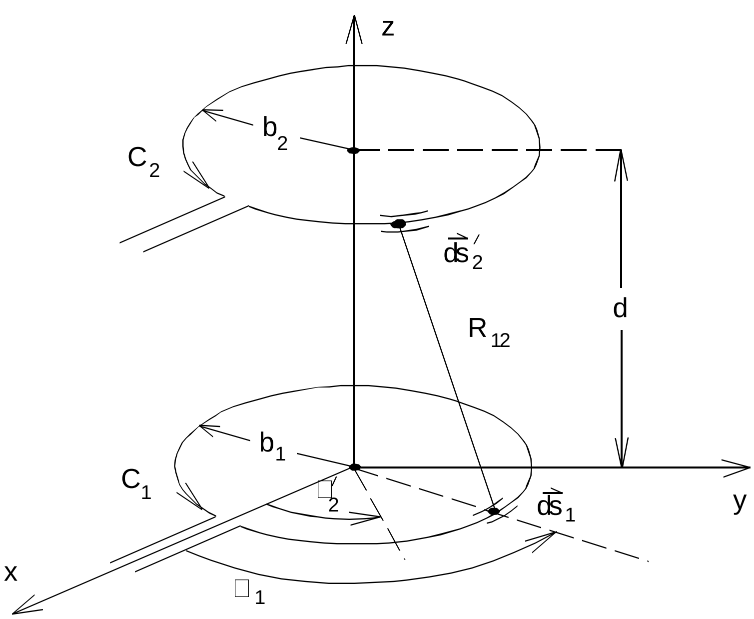

Figure 3. Coaxial wire loops.

Consider a pair of coaxial conducting loops, centered about the z-axis, and separated by a distance *d*. The conductors which compose each loop are assumed to have radii *a*. Let the primary loop have radius b_1 and the secondary loop have radius b_2. Also let $c=b_1-a$ represent the radius of the inner periphery of the primary loop.

As developed in Chapter 4, the mutual inductance of the two conducting loops is given by

$$L_{12} = \frac{\mu_o}{4\pi}\oint_{C_1} \vec{ds}_1 \cdot \oint_{C_2'} \frac{\vec{ds}_2'}{R_{12}(s_1,s_2')} = \frac{\mu_o}{4\pi}\oint_{C_1}\oint_{C_2'} \frac{\vec{ds}_1\cdot\vec{ds}_2'}{R_{12}(s_1,s_2')}$$

where $s_1 = c\varphi_1$, $\vec{ds}_1 = \hat{\varphi}_1 c d\varphi_1$, $s_2' = b_2\varphi_2'$, $\vec{ds}_2' = \hat{\varphi}_2' b_2 d\varphi_2'$, therefore

$$\vec{ds}_1\cdot\vec{ds}_2' = cb_2(\hat{\varphi}_1\cdot\hat{\varphi}_2')d\varphi_1 d\varphi_2' = cb_2\cos(\varphi_1 - \varphi_2')d\varphi_1 d\varphi_2' .$$

Now

$$R_{12}^2(s_1,s_2') = d^2 + l^2 = d^2 + c^2 + b_2^2 - 2cb_2\cos(\varphi_1 - \varphi_2')$$

by the cosine law, thus

$$L_{12} = \frac{\mu_o c b_2}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos(\varphi_1 - \varphi_2')d\varphi_2' d\varphi_1}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2\cos(\varphi_1 - \varphi_2')}} .$$

A change of integration variable for φ_2' is made by letting $\varphi_1 - \varphi_2' = \theta$. It is noted that φ_1 is a constant during the integration over φ_2', therefore $d\varphi_2' = -d\theta$. When $\varphi_2' = 0$, $\theta = \varphi_1$, and when $\varphi_2' = 2\pi$, $\theta = \varphi_1 - 2\pi$, and

$$L_{12} = \frac{\mu_o c b_2}{4\pi}\int_0^{2\pi}\int_{\varphi_1 - 2\pi}^{\varphi_1} \frac{\cos\theta\, d\theta\, d\varphi_1}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2\cos\theta}} .$$

Note that the integrand of this new integral is independent of φ_1 so that the outside integration is trivial. Furthermore since the inner integral is evaluated for the closed loop such that θ varies over a total of 2π, then the origin for θ is unimportant so the mutual inductance between the two loops becomes

$$L_{12} = \frac{\mu_o c b_2}{2}\int_0^{2\pi} \frac{\cos\theta\, d\theta}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2\cos\theta}} .$$

This expression for mutual inductance can be expressed in terms of *complete elliptical integrals* of the first and second kinds which are tabulated in most handbooks. First, a change of variables is made where $\theta = \pi - 2\beta$, and $d\theta = -2d\beta$. When $\theta = 0$, $\beta = \pi/2$, and when $\theta = 2\pi$, $\beta = -\pi/2$. As a result

$$\cos\theta = \cos(\pi - 2\beta) = \cos\pi\cos 2\beta + \sin\pi\sin 2\beta = -\cos 2\beta$$
$$= 2\sin^2\beta - 1$$

therefore

$$L_{12} = \mu_o c b_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\sin^2\beta - 1)\,d\beta}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2(2\sin^2\beta - 1)}}\,.$$

Because the integrand is even in β, this may be written

$$L_{12} = 2\mu_o c b_2 \int_{0}^{\frac{\pi}{2}} \frac{(2\sin^2\beta - 1)\,d\beta}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2(2\sin^2\beta - 1)}}\,.$$

The denominator of this expression may be rewritten

$$\begin{aligned} d^2 + c^2 + b_2^2 - 2cb_2(2\sin^2\beta - 1) &= d^2 + c^2 + 2cb_2 + b_2^2 - 4cb_2\sin^2\beta \\ &= d^2 + (c + b_2)^2 - 4cb_2\sin^2\beta \\ &= \frac{4cb_2}{k^2}(1 - k^2\sin^2\beta) \end{aligned}$$

where

$$k^2 = \frac{4cb_2}{d^2 + (c + b_2)^2}\,.$$

From the expression above it is apparent that

$$2\sin^2\beta - 1 = \frac{2}{k^2} - 1 - \frac{2}{k^2}(1 - k^2\sin^2\beta)$$

and

$$\frac{2\sin^2\beta - 1}{\sqrt{d^2 + c^2 + b_2^2 - 2cb_2(2\sin^2\beta - 1)}} = \frac{k}{2\sqrt{cb_2}} \frac{\left[\frac{2}{k^2} - 1 - \frac{2}{k^2}(1 - k^2\sin^2\beta)\right]}{\sqrt{1 - k^2\sin^2\beta}}.$$

This leads to

$$\begin{aligned} L_{12} &= 2\mu_o c b_2 \frac{1}{2\sqrt{cb_2}}\left[\left(\frac{2}{k} - k\right)\int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2\sin^2\beta}} - \frac{2}{k}\int_0^{\frac{\pi}{2}}\sqrt{1 - k^2\sin^2\beta}\, d\beta\right] \\ &= \mu_o\sqrt{cb_2}\left[\left(\frac{2}{k} - k\right)F\left(k, \frac{\pi}{2}\right) - \frac{2}{k}E\left(k, \frac{\pi}{2}\right)\right] \end{aligned}$$

where

$$F(k,\psi) = \int_0^{\psi} \frac{d\beta}{\sqrt{1 - k^2\sin^2\beta}}$$

is known as a *complete elliptic integral of the first kind*, and

$$E(k,\psi) = \int_0^{\psi}\sqrt{1 - k^2\sin^2\beta}\, d\beta$$

is known as a *complete elliptic integral of the second kind*. These elliptic integrals are tabulated in mathematical tables such as the C.R.C Handbook.

The mutual inductance between two coaxial wire loops is therefore

$$L_{12} = 4\pi \times 10^{-7}\sqrt{(b_1 - a)b_2}\left[\left(\frac{2}{k} - k\right)F\left(k, \frac{\pi}{2}\right) - \frac{2}{k}E\left(k, \frac{\pi}{2}\right)\right]$$

where

$$k^2 = \frac{4cb_2}{d^2 + (c + b_2)^2} = \frac{4(b_1 - a)b_2}{d^2 + (b_1 + b_2 - a)^2} .$$

The external self-inductance of a circular loop can be obtained from the above mutual inductance between two loops by a neat trick. The external self-inductance of the primary loop is defined by

$$L_1^e = \frac{\mu_o}{4\pi} \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_2^/} \frac{d\vec{s}_1^{\,/}}{R_{11}(s_1, s_1^/)} .$$

Can L_1^e be deduced from L_{12} in a simple manner? The answer is yes if $C_2^/ \rightarrow C_1^/$, $d\vec{s}_2^{\,/} \rightarrow d\vec{s}_1^{\,/}$, and $R_{12} \rightarrow R_{11}$, i.e. if

$$\left. \begin{array}{l} b_2 \rightarrow b_1 \\ d \rightarrow 0 \end{array} \right\} \; \ldots \text{ then } L_{12} \rightarrow L_1^e .$$

Then

$$L_1^e = 4\pi \times 10^{-7} \sqrt{b_1(b_1 - a)} \left[\left(\frac{2}{k} - k \right) F\left(k, \frac{\pi}{2}\right) - \frac{2}{k} E\left(k, \frac{\pi}{2}\right) \right]$$

where

$$k^2 = \frac{4(b_1 - a)b_1}{(2b_1 - a)^2}$$

but

$$\sqrt{b_1(b_1 - a)} = \frac{k}{2}(2b_1 - a)$$

therefore

$$L_1^e = 4\pi \times 10^{-7} (2b_1 - a) \left[\left(1 - \frac{k^2}{2} \right) F\left(k, \frac{\pi}{2}\right) - E\left(k, \frac{\pi}{2}\right) \right]$$

where

$$k^2 = \frac{4b_1(b_1 - a)}{(2b_1 - a)^2} .$$

For the special case of a thin loop having $b_1/a \gg 1$, (i.e. $k \to 1$)

$$F\left(k \to 1, \pi/2\right) \cong \ln\left(\frac{4}{\sqrt{1 - k^2}} \right)$$

and

$$E\left(k \to 1, \pi/2\right) \cong 1$$

from the asymptotic forms of the elliptic integrals for $k \to 1$. This leads to

$$L_1^e = 4\pi \times 10^{-7} b_1 \left[\ln\left(\frac{4}{\sqrt{(a/2b_1)^2}} \right) - 2 \right]$$

as $b_1/a \gg 1$ and $k \to 1$. Finally

$$L_1^e = 4\pi \times 10^{-7} b_1 \left[\ln\left(\frac{8b_1}{a} \right) - 2 \right]$$

for a thin-wire loop with $b_1/a \gg 1$.

- **external radiation resistance of arbitrarily-shaped planar electric circuit**

It was established in Module 4 that the external radiation resistance of a closed electric circuit is given by

$$R_1^e = -5\beta_0^4 \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_1^{\prime}} R_{11}^2(s_1, s_1^{\prime})\, d\vec{s}_1^{\,\prime}$$

which can be expressed as

$$R_1^e \; = \; -5\beta_0^4 \oint_{C_1}\oint_{C'_1} R_{11}^2(s_1,s_1^{/})\,(\vec{ds}_1 \cdot \vec{ds}_1^{/})$$

with

$$R_{11}^2(s_1,s_1^{/}) \; = \; (x-x^{/})^2 + (y-y^{/})^2 \; = \; (x^2+x^{/2}+y^2+y^{/2}) - 2\,(xx^{/}+yy^{/})$$

$$\vec{ds}_1 \cdot \vec{ds}_1^{/} \; = \; (\vec{dx}+\vec{dy})\cdot(\vec{dx}^{/}+\vec{dy}^{/}) \; = \; dx\,dx^{/} + dy\,dy^{/}$$

leading to

$$\oint_{C_1}\oint_{C_1^{/}} R_{11}^2(s_1,s_1^{/})\,(\vec{ds}_1 \cdot \vec{ds}_1^{/}) \; = \; \oint_{C_1}\oint_{C_1^{/}} \left[(x^2+x^{/2}+y^2+y^{/2}) - 2\,(xx^{/}+yy^{/})\right](dx\,dx^{/}+dy\,dy^{/}) \; .$$

For a closed circuit, however

$$\oint_{C_1} dx \; = \; \oint_{C_1} x\,dx \; = \; \oint_{C_1} x^2\,dx \; = \; \oint_{C_1} dy \; = \; \oint_{C_1} y\,dy \; = \; \oint_{C_1} y^2\,dy \; = 0$$

leading to

$$\begin{aligned} \oint_{C_1}\oint_{C_1^{/}} R_{11}^2(s_1,s_1^{/})\,(\vec{ds}_1 \cdot \vec{ds}_1^{/}) \; &= \; -2\oint_{C_!}\oint_{C_1^{/}} (xx^{/}dy\,dy^{/} + yy^{/}dx\,dx^{/}) \\ &= \; -2\left[\oint_{C_1} x\,dy \oint_{C_1^{/}} x^{/}dy^{/} + \oint_{C_1} y\,dx \oint_{C_1^{/}} y^{/}dx^{/}\right] . \end{aligned}$$

It is observed that the integrals in the above expression are simply the areas bounded by the inner periphery and centerline circuit paths such that

$$\oint_{C_1} x\,dy \; = \; \oint_{C_1} y\,dx \; = \; S_1$$

$$\oint_{C_1^{/}} x^{/}dy^{/} \; = \; \oint_{C_1^{/}} y^{/}dx^{/} \; = \; S_1^{/}$$

and the external radiation resistance finally becomes

$$R_1^e \; = \; 20\beta_0^4 S_1 S_1^{/} \; \cong \; 20\beta_0^4 S_1^2$$

where the latter approximation follows because the two areas are nearly equal for thin-wire circuits. The latter expression yields a measure of when an electric circuit can be expected to radiate significantly. When the radiation resistance becomes non-negligible compared to other circuit impedances then the radiated power becomes significant.

5.3 Frequency Dependence and Equivalent Circuits of Common Circuit Elements

- **resistors**

Perhaps the most common circuit element, resistors usually belong to one of three basic classes:

- **carbon composition**

The most common type of resistor consists of finely divided carbon particles (usually graphite) which are mixed with a non-conductive material. This short cylinder of carbon is then connected to two wire leads.

The carbon used in this type of device has a high resistivity, therefore a relatively small carbon resistor will have a resistance much greater than a very long wire (the resistance of carbon is about 2200 times greater than the resistance of copper).

Carbon resistors tend to be the most common due to low cost and ease of fabrication.

Carbon resistors usually are not designed to carry large currents. If too much current passes through this type of resistor, it will heat to the point that permanent damage results. Even currents that are slightly too large may cause changes in the resistivity of the carbon material.

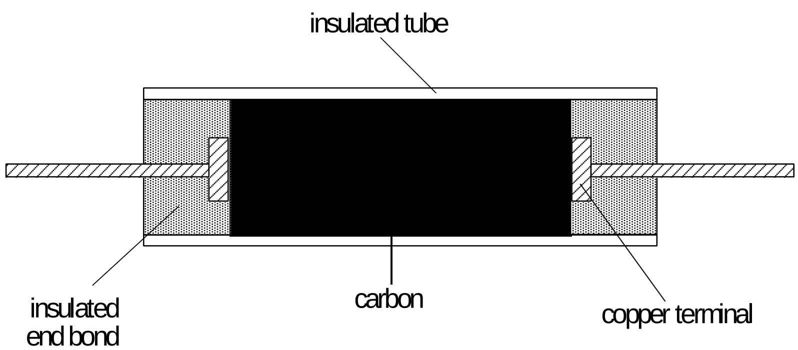

Figure 4. Carbon resistor.

- **wire wound**

Before the invention of radio, nearly all resistors were of the *wire-wound* type. This device consists of a resistive wire which is wound tightly around a hollow tube made of a non-conductive, heat-dissipative material (usually porcelain). This assembly is coated with an enamel-like substance which protects the wire, and prevents oxidation and changes due to temperature and atmospheric humidity.

Although expensive and more difficult to fabricate than carbon resistors, wire-wound resistors are capable of withstanding large current loads which are required for applications such as powerful radio transmitters.

Wire-wound resistors can be fabricated to much tighter tolerances than carbon composition resistors, which typically have tolerances of 5-10%.

Due to the amount of tightly coiled wire present in a wire-wound resistor, this type of resistor typically has a large inductance.

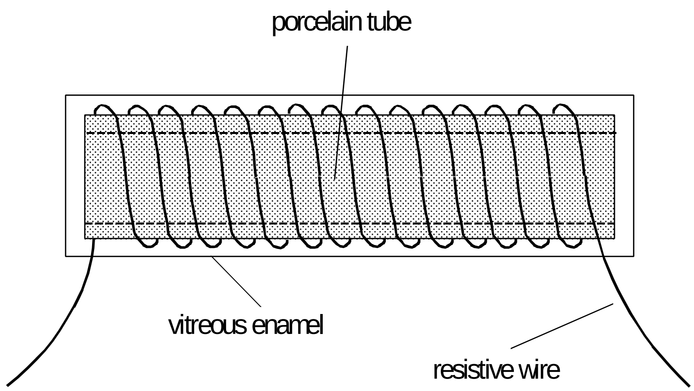

Figure 5. Vitreous enamel resistor.

- **thin film**

This type of resistor is constructed by depositing a thin metallic film on an insulating substrate. Leads are attached to the ends of the metallic film.

Thin film resistors often tend to meander over the surface of the substrate, and therefore have inductances that are typically greater than carbon composition resistors, but less than wire wound resistors.

Thin film resistors can be fabricated to very precise values of resistance.

The behavior of all real resistors begins to depart from the ideal response as the frequency of operation increases. The degree to which this occurs at any given frequency depends greatly upon the type of resistor under consideration. For example, because a wire wound resistor contains a long, tightly-wound wire element, it is expected that the inductance of this type of resistor would be more dominant at high frequencies than for carbon resistors. Most resistors share certain non-ideal behaviors, however. At higher frequencies, charge tends to leak around the resistor body, giving rise to a stray capacitance, although this effect is usually not significant. A more pronounced effect arises from the inductance and capacitance associated with the leads which are connected to the resistor.

– equivalent model of resistor

Despite differences in device construction, a general equivalent circuit for resistors may be constructed.

- A lumped lead inductance L_{lead} is considered to be in series with a parallel combination of the lead capacitance C_{lead} , the ideal bulk resistance of the device itself R, and the stray leakage capacitance $C_{leakage}$.

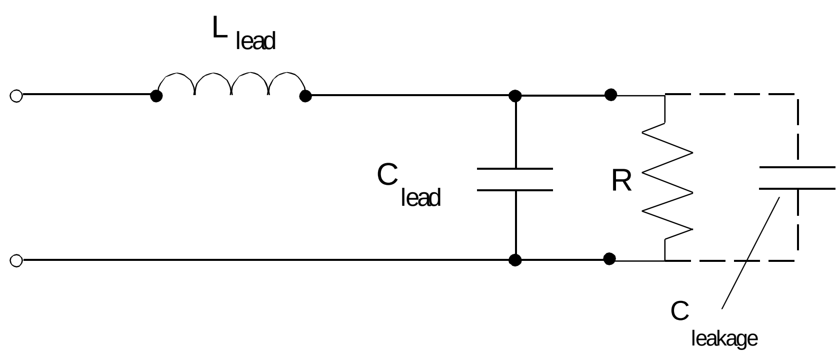

Figure 6. Equivalent circuit for resistor.

- This equivalent circuit may be simplified by combining the lead and leakage capacitances, so that

$$C_{parasitic} = C_{lead} + C_{leakage} .$$

- The impedance of the equivalent circuit is determined by first finding the impedance of the parallel combination of the parasitic capacitance $C_{parasitic}$ and the bulk resistance R

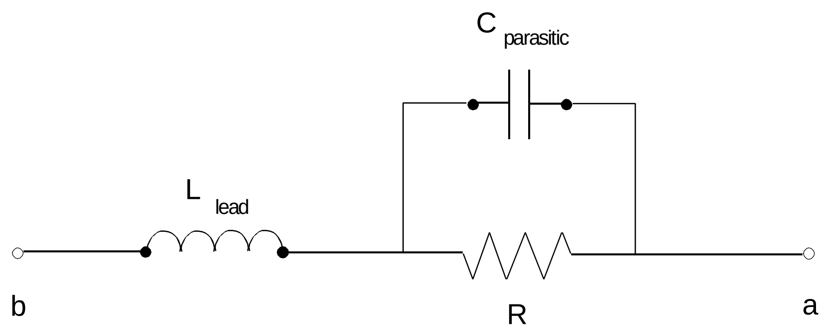

Figure 7. Simplified equivalent circuit for a resistor.

$$\frac{1}{Z_{RC}} = \frac{1}{[1/(j\omega C_{par.})]} + \frac{1}{R} = j\omega C_{par.} + \frac{1}{R} = \frac{j\omega RC_{par.} + 1}{R}$$

or

$$Z_{RC} = \frac{R}{j\omega RC_{par.} + 1} .$$

- The total impedance of the equivalent circuit is then

$$Z_{circuit} = j\omega L_{lead} + Z_{RC} = j\omega L_{lead} + \frac{R}{j\omega RC_{par.} + 1} = \frac{(j\omega RC_{par.} + 1)j\omega L_{lead} + R}{j\omega RC_{par.} + 1}$$

which becomes

$$Z_{circuit} = \frac{j\omega L_{lead} + R(1 - \omega^2 L_{lead} C_{par.})}{j\omega RC_{par.} + 1} .$$

The behavior of the generalized equivalent circuit for the resistor is determined by examining this impedance expression for a wide range of frequencies.

- For dc operation (ω=0), the impedance of the equivalent circuit is simply equal to the bulk resistance *R*, as expected. This is physically due to the fact that at dc the parasitic

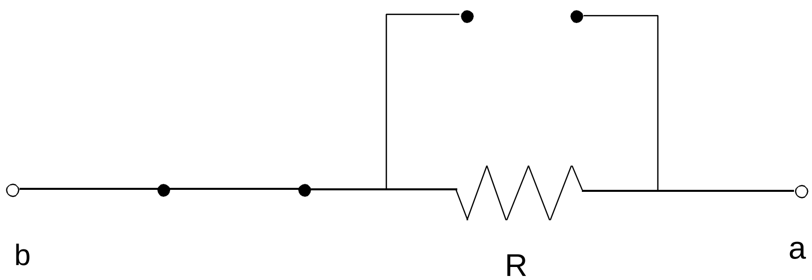

Figure 8. Equivalent circuit for resistor operated at dc.

capacitance behaves like an open circuit, and the lead inductance behaves like a short

- As the frequency of operation increases, the impedance associated with the parasitic capacitance begins to decrease. At some point the impedance of this capacitance is equal to the bulk resistance, or

$$R = \frac{1}{j\omega C_{parasitic}} .$$

As the frequency increases beyond this point, more current begins to flow through the conducting path provided by the parasitic capacitance than flows through the bulk resistance. In this regime, the lead inductance remains small (i.e., nearly a short circuit).

- As the frequency increases further, the impedance of the equivalent circuit decreases until the lead inductance and the parasitic capacitance cause the resistor to resonate. The equivalent circuit impedance is a minimum at the *self resonant frequency* of the resistor

$$\omega_o = \frac{1}{\sqrt{L_{lead} C_{par.}}} .$$

Above this frequency, the impedance of the lead inductance begins to dominate.

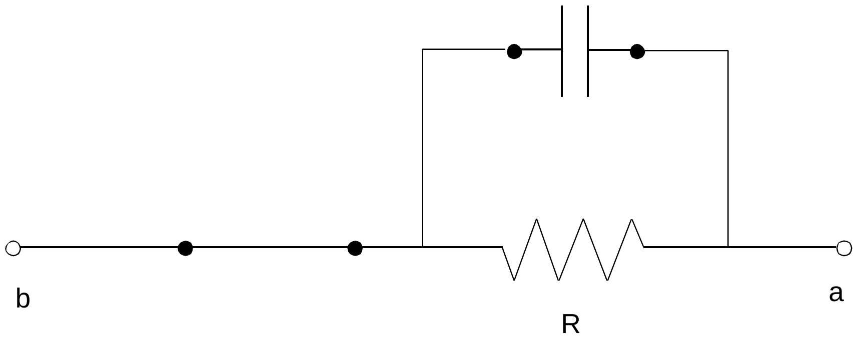

Figure 9. Equivalent circuit for resistor below self-resonance.

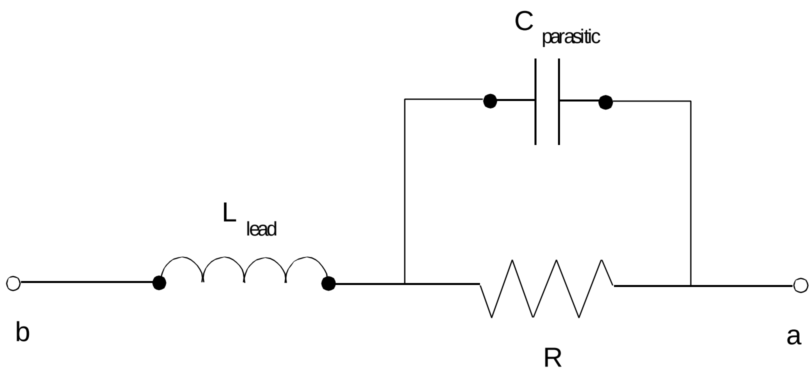

Figure 10. Equivalent circuit for resistor near self-resonance.

- Finally, as the frequency begins to approach infinity, the impedance of the lead inductor becomes very large, and the impedance of the parasitic capacitance approaches zero. Thus resistor behaves as an open circuit.

A plot of the response of the equivalent circuit for a 1000 ohm resistor is shown in Figure 12. Here the lead inductance is 15 nH, and the parasitic capacitance is 1 pF. The resonant frequency is seen to occur at 1.299 GHz.

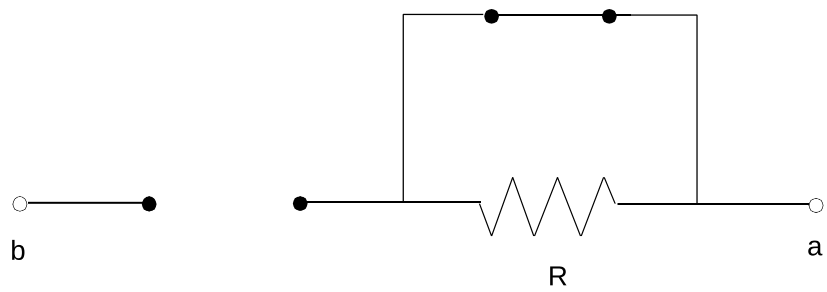

Figure 11. Equivalent circuit for resistor as frequency approaches infinity.

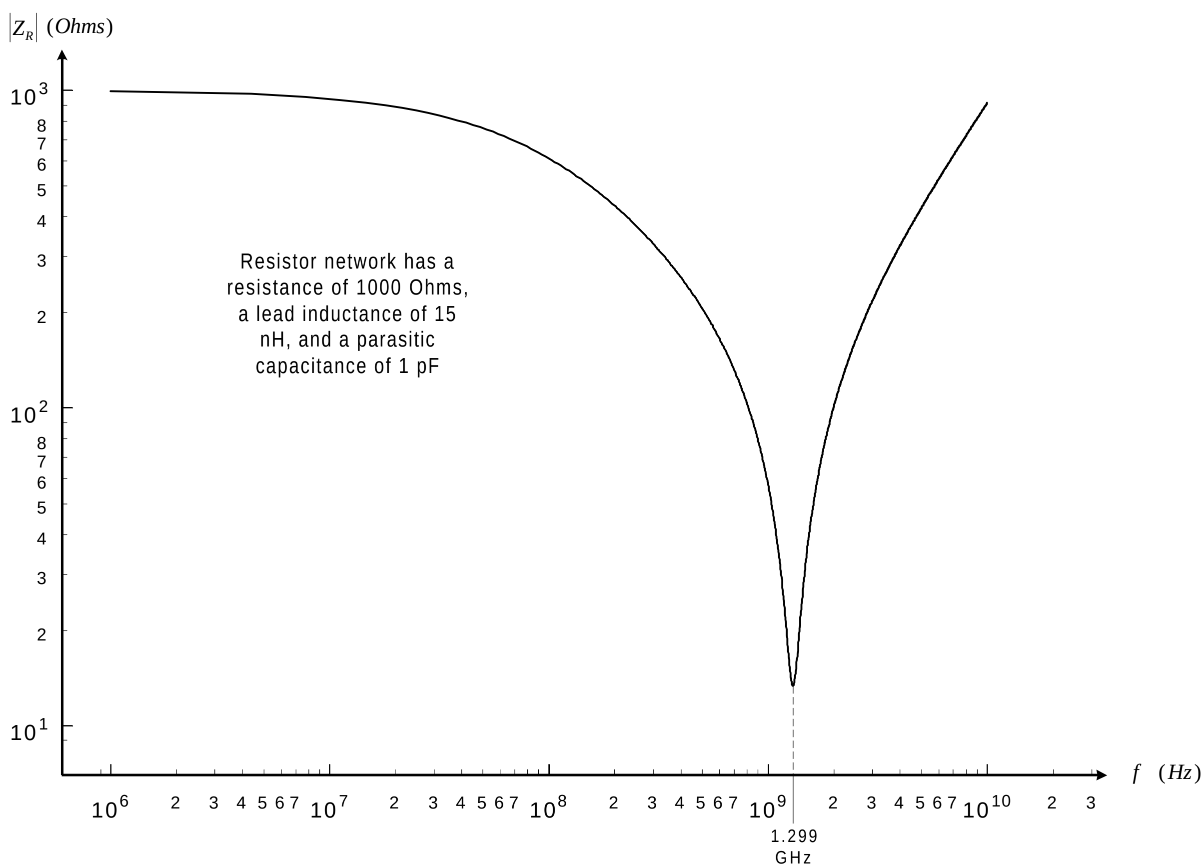

Figure 12. Frequency dependent behavior of equivalent circuit for 1000 ohm resistor.

- **capacitors**

  While many types of capacitors exist, two are widely used in EMC design:

  – **tantalum electrolytic**

  Small tantalum electrolytic capacitors can have large capacitances (1 - 1000 μF).

  – **ceramic**

  Ceramic capacitors typically have smaller capacitances (1 μF - 5 pF), but tend to exhibit ideal behavior over a much broader range of frequencies.

Although typically thought of in the context of dc operation, operating frequency is one of primary consideration in choosing capacitors. A table of various capacitors and their typical operating ranges is presented below.

Type	Approximate operating range
Tantalum electrolytic	1 - 1000 Hz
Large value aluminum electrolytic	1 - 1000 Hz
Ceramic	10 kHz - 1 GHz
Mica	10 kHz - 1 GHz

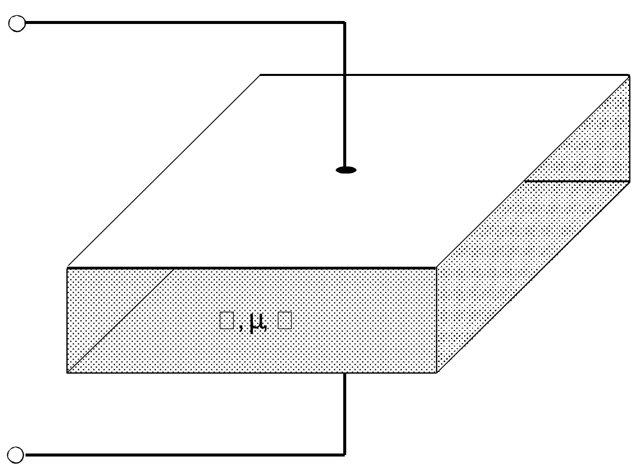

Figure 13. Generalized capacitor.

– **equivalent circuit of capacitor**

Both types of capacitors discussed above share the same basic configuration. Wire leads are connected to a pair of parallel plates, each having area *A*, which are usually separated by some type of dielectric material. A generalized equivalent circuit for capacitors can be constructed, however, specific component values may differ for different types of capacitors.

- The component leads introduce an associated inductance (L_{lead}) and capacitance (C_{lead}).

- A large resistance ($R_{dielectric}$) associated with the dielectric layer between the capacitor plates exists in parallel with the ideal bulk capacitance (C).

- The plates of the capacitor themselves introduce a resistance (R_{plates}).

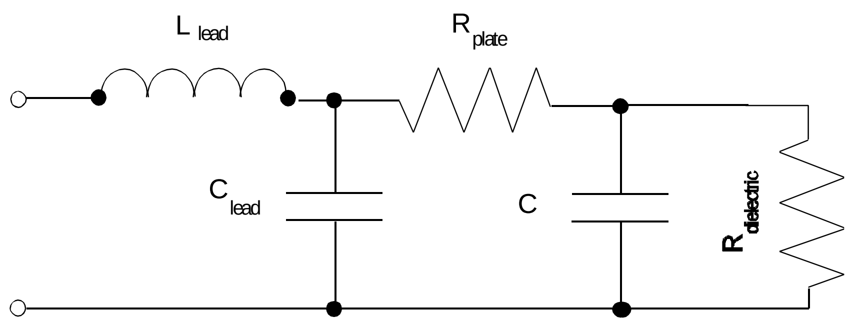

Figure 14. Equivalent circuit for capacitor.

- The lead capacitance is typically so small compared to the bulk capacitance that it may be neglected. Likewise, the resistance of the dielectric layer is typically so large that it may be represented as being an open circuit. Thus a simplified equivalent circuit may be developed consisting of a series combination of the lead inductance, the plate resistance, and the ideal bulk capacitance.

- The impedance associated with this simplified equivalent circuit is clearly

$$Z_{circuit} = j\omega L_{lead} + R_{plates} + \frac{1}{j\omega C} = R_{plates} + j\left(\omega L_{lead} - \frac{1}{\omega C}\right) .$$

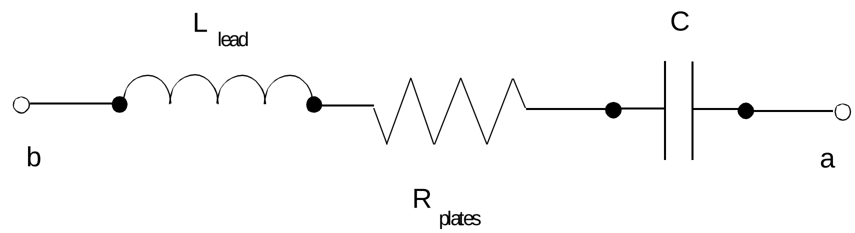

Figure 15. Simplified equivalent capacitor circuit.

Again the behavior of the generalized equivalent circuit is determined by examining this impedance expression over a wide range of frequencies:

- For dc operation, the lead inductance behaves as a short circuit, and the ideal bulk capacitance is an open circuit. Thus, the capacitor itself acts as an open circuit.

- As frequency is increased, the impedance, which is dominated by the ideal capacitance term, decreases linearly, until reaching a minimum when

$$\omega L_{lead} = \frac{1}{\omega C}.$$

At this point, the impedance is purely real, and the equivalent circuit is in resonance. This occurs at the self-resonant frequency of the capacitor, which is given by

$$\omega_o = \frac{1}{\sqrt{L_{lead} C}}.$$

- As frequency increases beyond self-resonance, the impedance increases linearly, with the inductive term dominating.

- As the frequency approaches infinity, the lead inductance begins to behave like an open circuit. Thus the maximum operating frequency of a capacitor is typically limited by the inductance of the capacitor and the device leads.

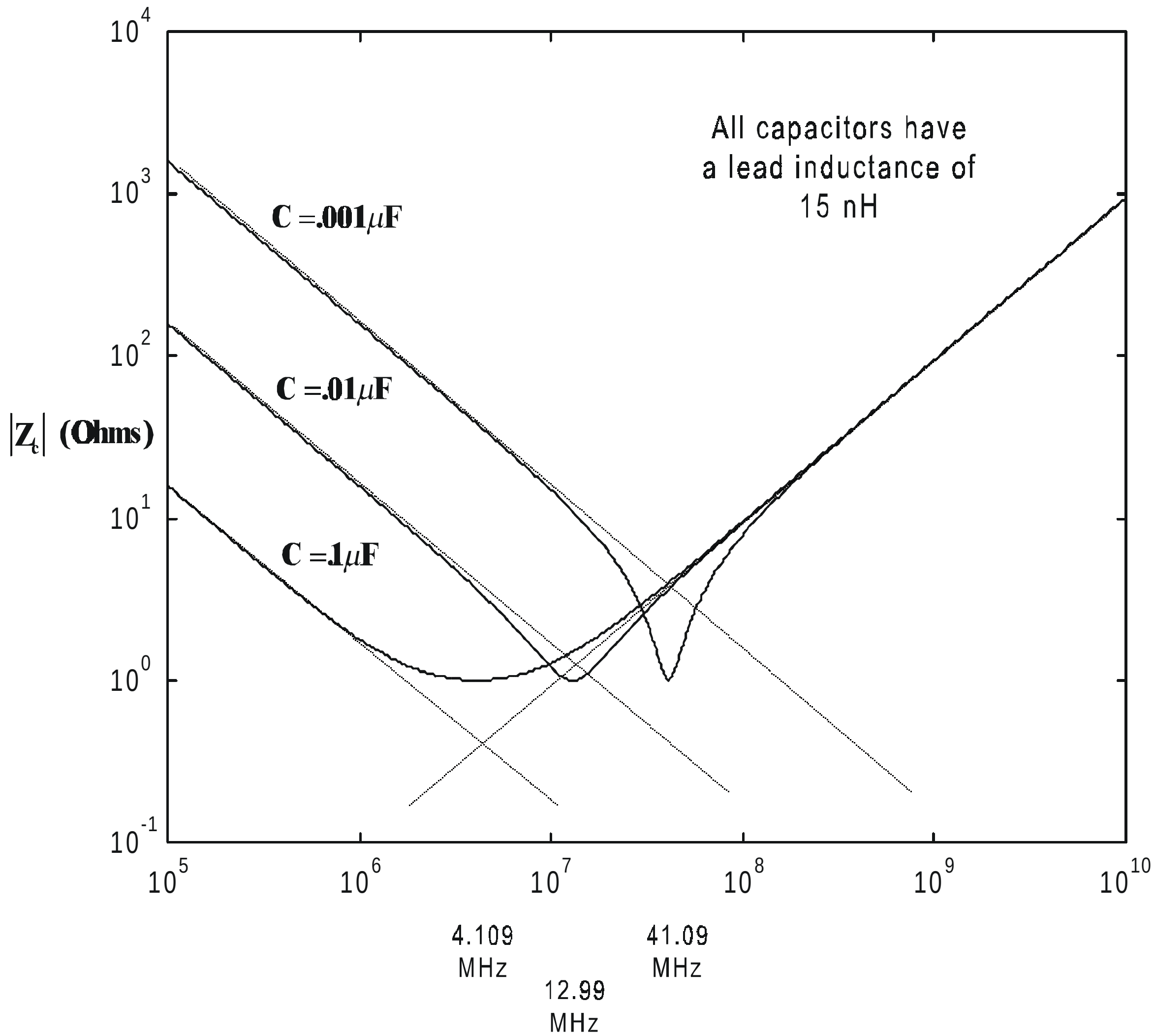

Figure 16: Plot of frequency dependent behavior of equivalent circuit for various capacitors.

A plot of the responses of the equivalent circuits for various capacitors is shown in Figure 16. The response of 0.001 μF, 0.01 μF, and 0.1 μF capacitors is shown. Here the lead inductances are all 15 nH, and the plate resistances are 1Ω. The resonant frequencies are seen to occur at 4.109 MHz, 12.99 MHz, and 41.09 MHz.

- **inductors**

Although all inductors consist of a coil of wire, many variations on the actual method of device construction exist. Inductors may be wound on cores made of non-magnetic material, or, more commonly, on materials with magnetic properties such as ferrite.

Due to their physical geometry, inductors, more than any other type of common circuit element, tend be sources of stray magnetic fields. Likewise, inductors are also more susceptible to effects due to external magnetic fields than other basic circuit elements. The type of inductor has a great deal to do with how susceptible it is to external fields. Air core and open magnetic core inductors tend to be the most susceptible to external fields, and also tend to generate fields which may interfere with other devices. It is often desirable to shield inductors in order to insure proper operation.

– **equivalent circuit of inductor**

As with the other circuit elements, an equivalent circuit for a generalized inductor may be constructed.

- The wire leads of the inductor introduce a series inductance L_{lead}, and a capacitance C_{lead} that is in parallel with the ideal inductance.

- Because a relatively large amount of wire is contained in the inductor coil, a parasitic resistance is modeled in series with the ideal inductance.

- Finally, a parasitic capacitance exists in parallel with the series combination of the parasitic resistance, and the ideal inductance. This capacitance is due mainly to the individual windings of the coil being in such close proximity to one another.

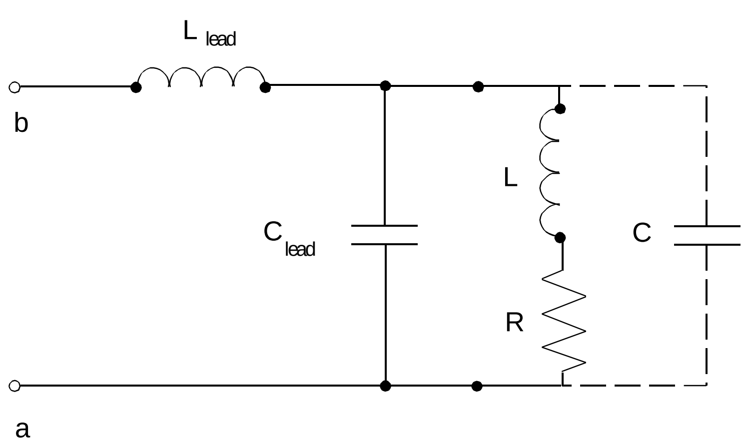

Figure 17. Equivalent circuit for inductor.

As with the devices examined previously, the generalized equivalent circuit for the inductor shown above may be simplified.

- The lead inductance L_{lead} is typically much smaller than the ideal inductance L, and may therefore be neglected.

- Additionally, the lead capacitance C_{lead} is typically much smaller than the parasitic capacitance $C_{parasitic}$.

Thus the simplified equivalent circuit for the inductor consists of a series combination of $R_{parasitic}$ and L in parallel with $C_{parasitic}$.

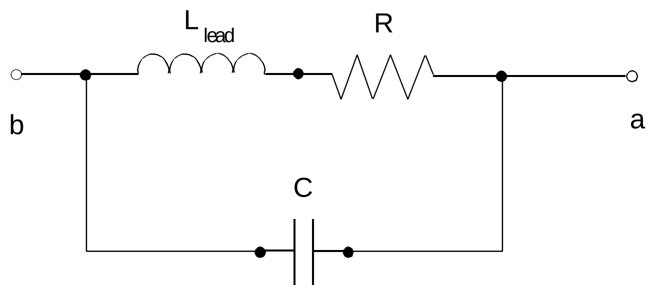

Figure 18. Simplified equivalent circuit for inductor.

The impedance of this simplified circuit is given by

$$\frac{1}{Z_{total}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

where

$$Z_1 = j\omega L + R_{parasitic}$$

and

$$Z_2 = \frac{1}{j\omega C_{parasitic}}$$

This leads to

$$\frac{1}{Z_{total}} = \frac{j\omega C_{par.}\left(j\omega L + R_{par.}\right) + 1}{j\omega L + R_{par.}}$$

or

$$Z_{total} = \frac{j\omega L + R_{par.}}{1 - \omega^2 LC_{par.} + j\omega C_{par.}R_{par.}} .$$

As before, the behavior of the impedance of the equivalent circuit for the inductor is examined over a wide range of frequencies:

- At low frequencies, the parasitic resistance term dominates and the impedance is approximately equal to $R_{parasitic}$.

- As the frequency of operation increases, the ideal inductance L begins to dominate the impedance of the equivalent circuit near the frequency

$$\omega = \frac{R_{parasitic}}{L} .$$

- As the frequency increases further, the impedance of the parasitic capacitance decreases until it's magnitude is equal to that of the ideal inductance. This occurs at the self resonant frequency

$$\omega_o = \frac{1}{\sqrt{LC_{par.}}} .$$

The impedance of the equivalent circuit is a maximum at this frequency.

- Above the self-resonant frequency, the parasitic capacitance begins to dominate the behavior of the equivalent circuit. In this range of operation the impedance decreases with increasing frequency.

A plot of the responses of the equivalent circuits for various inductors is shown in Figure 19. The responses of 0.1 mH, 10 µH, and 1 µH inductors are shown. Here the parasitic capacitances are all 1 pF, and the parasitic resistances are 1Ω. The resonant frequencies are seen to occur at 15.92 MHz, 50.33 MHz, and 159.2 MHz.

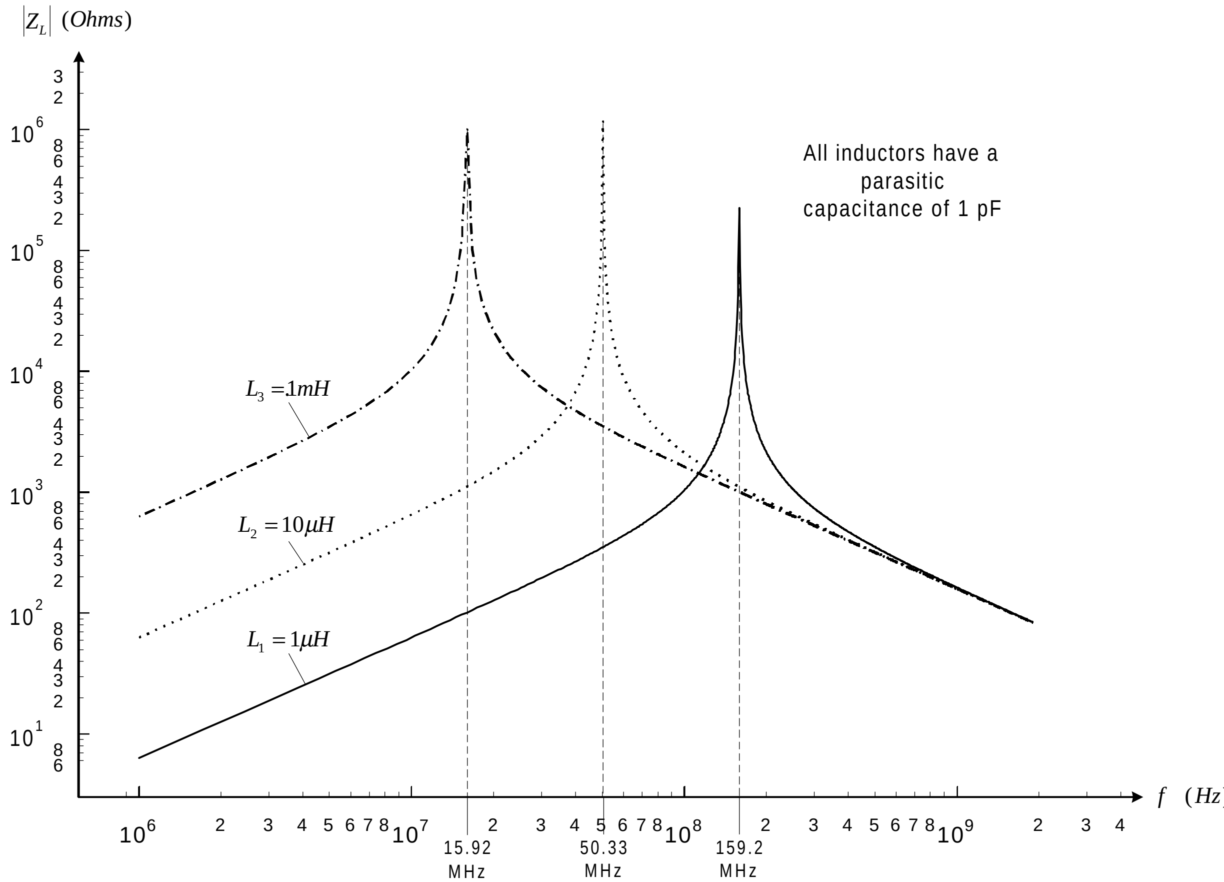

Figure 19. Plot of frequency dependent behavior of equivalent circuits for various inductors.

Appendix - Bessel Functions

The equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(k^2 - \frac{n^2}{r^2}\right)y = 0$$

is known as *Bessel's equation* of order n with parameter k. General solutions to Bessel' s differential equation are given by

$$y(x) = AJ_n(kx) + BN_n(kx)$$

$$y(x) = CH_n^{(1)}(kx) + DH_n^{(2)}(kx)$$

where

$$J_n(x) = \sum_{m=0}^{N} \frac{(-1)^m x^{n+2m}}{2^{n+2m} m!\,(m+n)!}$$

is known as a *Bessel function of the first kind* of order n,

$$N_n(x) = \frac{\cos(n\pi)\, J_n(x) - J_{-n}(x)}{\sin(n\pi)}$$

is known as a *Bessel function of the second kind (Neumann function)* of order n,

$$H_n^{(1)}(x) = J_n(x) + jN_n(x)$$

is known as a *Hankel function of the first kind* of order n, and

$$H_n^{(2)}(x) = J_n(x) - jN_n(x)$$

is known as a *Hankel function of the second kind* of order n.

The expression

$$\int x^{n+1} Z_n(x)\, dx = x^{n+1} Z_{n+1}(x)$$

is a commonly used integration formula, where $Z_n = J_n,\ N_n,\ H_n^{(1)},\ H_n^{(2)}$.

The behavior of the Bessel functions for large arguments are determined by the asymptotic forms

$$\lim_{x \to \infty} J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$\lim_{x \to \infty} N_n(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$\lim_{x \to \infty} H_n^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{j\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)}$$

and

$$\lim_{x \to \infty} H_n^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-j\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)}.$$