Module 4:
General Formulation of Electric Circuit Theory

4. General Formulation of Electric Circuit Theory

All electromagnetic phenomena are described at a fundamental level by Maxwell's equations and the associated auxiliary relationships. For certain classes of problems, such as representing the behavior of electric circuits driven at low frequency, application of these relationships may be cumbersome. As a result, approximate techniques for the analysis of low-frequency circuits have been developed. These specializations are used to describe the "ideal" behavior of common circuit elements such as wires, resistors, capacitors, and inductors. However, when devices are operated in a regime, or an environment, which lies outside the range of validity of such approximations, a more fundamental description of electrical systems is required. When viewed in this more general context, what may initially appear to be unexpected behavior of a circuit element often reveals itself to be normal operation under a more complex set of rules. An understanding of this lies at the core of electromagnetically compatible designs.

In this section, electric circuit theory will be presented in a general form, and the relationship between circuit theory and electromagnetic principles will be examined. The approximations associated with circuit theory and a discussion of the range of validity of these approximations will be included. It will be seen that effects due to radiation and induction are always present in systems immersed in time-varying fields, although under certain conditions these effects may be ignored.

4.1 Limitations of Kirchoff's laws

The behavior of electric circuits is typically described through Kirchoff's voltage and current laws. Kirchoff's voltage law states that the sum of the voltages around any closed circuit path is zero

$$\sum_n V_n = 0$$

and Kirchoff's current law states that the sum of the currents flowing out of a circuit node is zero

$$\sum_{n=1}^{N} I_n = 0$$

It is through application of these relationships that most descriptions of electric circuits proceed. However, both of these relationships are only valid under certain conditions:

- The structures under consideration must be electrically small . At 60 Hz, the wavelength of a wave propagating through free space is 5 million meters, while at 300 MHz a wavelength in free space is 1m long. Radiation and induction effects arise when the current amplitude and phase vary at points along the conductor.

- No variation exists along uninterrupted conductors.

- No delay time exists between sources and the rest of the circuit. Also all conductors are equipotential surfaces.

- The loss of energy from the circuit, other than dissipation, is neglected. In reality, losses due to radiation may become significant at high frequency.

In chapter 2, a time-dependent generalization of KVL was presented

$$v(t) = \left(R_{source} + R\right) i(t) + L \frac{d\,i(t)}{dt} .$$

Although this expression is valid for time-changing fields, it is assumed that the circuit elements are lumped, i.e, the resistance and inductance are concentrated in relatively small regions. This assumption begins to break down at frequencies where the circuit elements are a significant fraction of a wavelength long. In this regime, circuits must be described in terms of distributed parameters. Every part of the circuit has a certain impedance per unit length associated with it. This impedance may be both real (resistive) and imaginary (reactive). Also, interactions which occur in one part of the circuit may affect interactions which occur everywhere else in the circuit. In addition, the presence of other external circuits will affect interactions within a different circuit. Thus at high frequency, a circuit must be viewed as a single entity, not a collection of individual components, and multiple circuits must be viewed as composing a single, coupled system.

4.2 General formulation for a single RLC circuit

The general formulation of electric circuit theory will begin with an analysis of a single circuit constructed with a conducting wire of radius a that may include a coil (inductor), a capacitor, and a resistor. It will be assumed that the radius of the wire is much smaller than a wavelength at the frequency of operation

$$a/\lambda_o << 1$$

or

$$\beta_o a << 1$$

where

$$\beta_o = 2\pi/\lambda_o .$$

A current flows around the circuit. The tangential component of electric current

$$J_s = \hat{s}\cdot\vec{J} = \hat{s}\cdot(\sigma\vec{E}) = \sigma E_s$$

is driven by the tangential component of electric field E_s along the wire, which is produced by charge and current in the circuit.

Currents in the circuit are supported by a generator. The generator is a source region where a non-conservative (meaning that the potential rises in the direction of current flow) impressed electric field $\vec{E}^{\,e}$ maintains a charge separation. This impressed field is due to an electrochemical or other type of force, and is assumed to be independent of current and charge in the circuit. The charge separation supports an electric field $\vec{E}$ (Coulomb field) within and external to the source region which gives rise to the current flowing in the circuit.

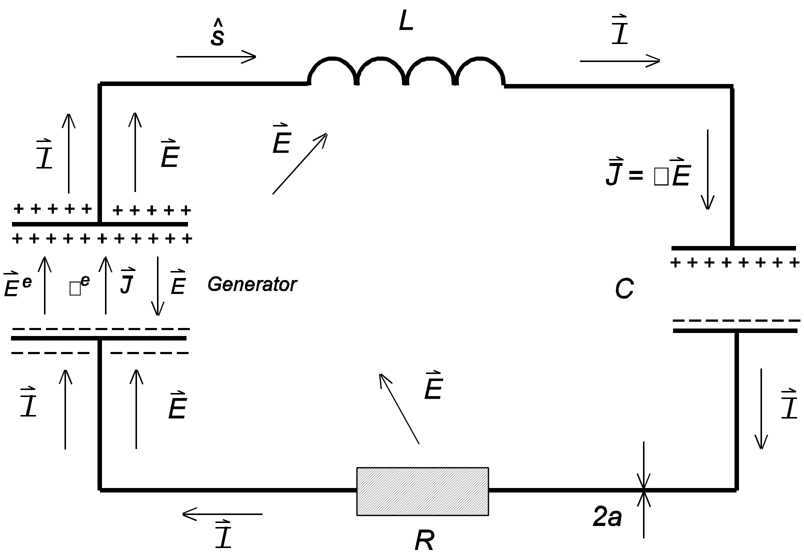

Figure 1. Generalized electric circuit.

In the regions external to the source, the impressed field $\vec{E}^{\,e}$ does not exist, however within the source both fields exist, therefore by Ohm's law the current density present at any point in the circuit is given by

$$\vec{J} = \sigma(\vec{E} + \vec{E}^{\,e})$$

where σ varies from point to point. From this it is apparent that in order to drive the current density $\vec{J}$ against the electric field $\vec{E}$ which opposes the charge separation in the source region, the impressed electric field must be such that

$$|\vec{E}^{\,e}| > |\vec{E}| \,.$$

Positions along the circuit are measured using a displacement variable s, having it's origin at the center of the generator. The unit vector parallel to the wire axis at any point along the circuit is $\hat{s}$. The tangential component of electric field along the conductor is therefore

$$E_s = \hat{s} \cdot \vec{E}$$

and the associated axial component of current density is

$$J_s = \hat{s} \cdot \vec{J} = \hat{s} \cdot (\sigma \vec{E}) = \sigma E_s \,.$$

The total current flowing through the conductor cross-section is then

$$I_s = \int_{c.s.} J_s \, ds \,.$$

- **boundary conditions at the surface of the wire circuit**

According to the boundary condition

$$\hat{t} \cdot (\vec{E}_2 - \vec{E}_1) = 0$$

the tangential component of electric field is continuous across an interface between materials. Application of this boundary condition at the surface of the wire conductor leads to

$$E_s(r = a^-) = E_s(r = a^+) \,,$$

where

$$E_s^{inside}(s) = E_s(r = a^-)$$

is the field just inside the conductor at position s along the circuit, and

$$E_s^{outside}(s) = E_s(r = a^+)$$

represents the field maintained at position s just outside the surface of the conductor by the current and charge in the circuit. Therefore the fundamental boundary condition employed in an electromagnetic description of circuit theory is

$$E_s^{inside}(r=a^-,s) \;=\; E_s^{outside}(r=a^+,s)\;.$$

- **determination of** $E_s^{inside}(s)$

In order to apply the boundary condition above, the tangential electric field that exits at points just inside the surface of the circuit must be determined. This is not an easy task, because the impedance may differ in the various regions of the circuit. At any point along the conducting wire, including coils and resistors, the electric field is in general

$$E_s^{inside}(s) \;=\; z^{\,i}\, I_s(s)$$

where

$$z^{\,i} \;\equiv\; \frac{E_s^{inside}(s)}{I_s(s)}$$

is the internal impedance per unit length of the region.

- **source region**

In the source region, both the impressed and induced electric fields exist, therefore the current density is

$$J_s \;=\; \sigma^e (E_s + E_s^{\,e})$$

where σ^e is the conductivity of the material in the source region, E_s is the tangential component of electric field in the source region maintained by charge and current in the circuit, and $E_s^{\,e}$ is the tangential component of impressed electric field. From this, it is apparent that

$$E_s = \frac{J_s}{\sigma^e} - E_s^e = \frac{J_s S^e}{\sigma^e S^e} - E_S^e = \frac{I_s}{\sigma^e S^e} - E_s^e$$

where S^e is the cross-sectional area of the source region. In an ideal generator, the material in the source region is perfectly conducting ($\sigma^e \rightarrow \infty$), and has zero internal impedance. Thus $E_s \rightarrow -E_s^e$ in a good source generator. In general, in the source region

$$E_s(s) = z_e^i I_s(s) - E_s^e(s)$$

where

$$z_e^i = \frac{1}{\sigma^e S^e}$$

is the internal impedance per unit length of the source region.

- capacitor

The tangential component of electric field at the edge of the capacitor E_s and the current flowing to the capacitor lie in the same direction. Therefore, within the capacitor

$$E_s(s) = z_c^i I_s(s)$$

where z_c^i is the internal impedance per unit thickness of the dielectric material contained in the capacitor. The total potential difference across the capacitor is the line integral of the normal component of electric field which exists between the capacitor plates

$$V_{AB} = V_B - V_a = -\int_A^B E_s\, ds = \int_B^A E_s\, ds = \int_B^A z_c^i I_s(s)\, ds$$

or

$$V_{AB} = I_s \int_B^A z_c^i\, ds$$

if it is assumed that a constant current I_s flows to the capacitor.

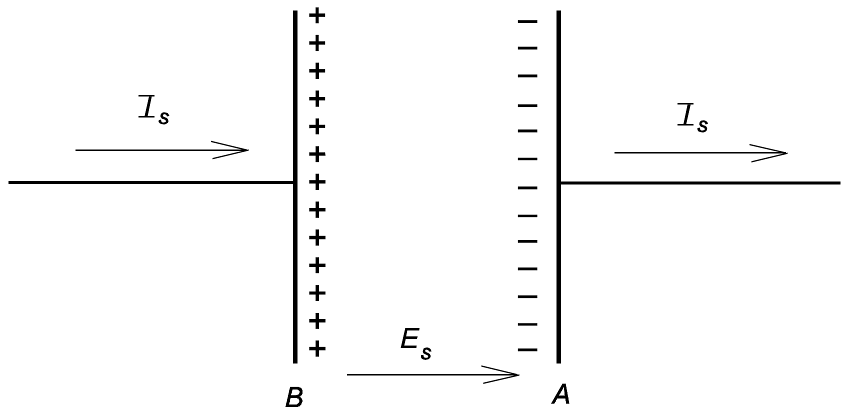

Figure 2. Capacitor.

The time-harmonic continuity equation states

$$\nabla \cdot \vec{J} = -j\omega\rho \, .$$

Volume integration of both sides of this expression, and application of the divergence theorem yields

$$\oint_S (\hat{n} \cdot \vec{J}) ds = -j\omega \int_V \rho \, dv$$

resulting in

$$-I_s = -j\omega Q$$

where Q is the total charge contained on the positive capacitor plate. The total potential difference between the capacitor plates is then

$$V_{AB} = j\omega Q \int_A^B z_c^i ds$$

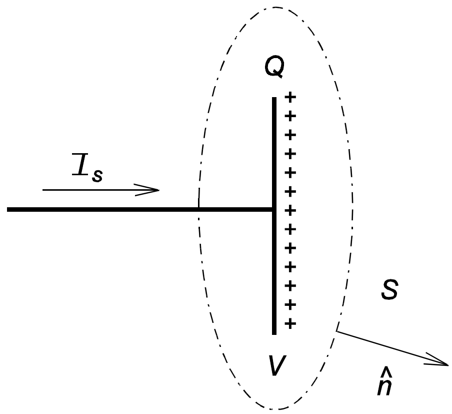

Figure 3. Single capacitor plate.

but by the definition of capacitance

$$C = \frac{Q}{V_{AB}}$$

therefore

$$\frac{Q}{C} = j\omega Q \int_A^B z_c^i \, ds \, .$$

From this comes the expected expression for the impedance of a capacitor

$$\int_B^A z_c^i ds = \frac{1}{j\omega C} = -jX_c \, .$$

- arbitrary point along the surface of the circuit

By combining the results for the three cases above, a general expression for the tangential component of electric field residing just inside the surface of the conductor at any point along the circuit is determined

$$E_s^{inside}(s) = z^i(s) I_s(s) - E_s^e(s)$$

where $z^i(s)$ is the internal impedance per unit length which is different for the various components of the circuit, and $E_s^e(s)$ is the impressed electric field which is zero everywhere outside of the source region.

- **determination of** $E_s^{outside}(s)$

In Chapter 2 it was shown that an electric field may be represented in terms of scalar and vector potentials. Thus at any point in space outside the electric circuit the electric field is

$$\vec{E} = -\nabla\Phi - j\omega\vec{A}$$

where Φ is the scalar potential maintained at the surface of the circuit by the charge present in the circuit, and $\vec{A}$ is the vector potential maintained at the surface of the circuit by the current flowing in the circuit. The well known *Lorentz condition* states that

$$\nabla\cdot\vec{A} + \frac{jk^2}{\omega}\Phi = 0$$

where

$$k^2 = \omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right).$$

In the free space outside the circuit $\mu = \mu_o$, $\varepsilon = \varepsilon_o$, and $\sigma = 0$ thus

$$k^2 = \beta_o^2 = \omega^2\mu_o\varepsilon_o .$$

Applying this to the Lorentz condition yields

$$\Phi = \frac{j\omega}{\beta_o^2}\nabla\cdot\vec{A}$$

which, upon substitution into the expression for electric field outside the circuit gives

$$\vec{E} = -\frac{j\omega}{\beta_o^2}\left[\nabla(\nabla\cdot\vec{A}) + \beta_o^2\vec{A}\right].$$

The component of electric field tangent to the surface of the circuit is then given by

$$E_s^{outside}(s) = \hat{s}\cdot\vec{E} = -\hat{s}\cdot\nabla\Phi - j\omega(\hat{s}\cdot\vec{A}) = -\frac{j\omega}{\beta_o^2}\hat{s}\cdot\left[\nabla(\nabla\cdot\vec{A}) + \beta_o^2\vec{A}\right]$$

or

$$E_s^{outside}(s) = -\frac{\partial\Phi}{\partial s} - j\omega A_s = -\frac{j\omega}{\beta_o^2}\left[\frac{\partial}{\partial s}(\nabla\cdot\vec{A}) + \beta_o^2 A_s\right].$$

- **satisfaction of the fundamental boundary condition**

The boundary condition at the surface of the circuit states that the tangential component of electric field must be continuous, or

$$E_s^{inside}(s) = E_s^{outside}(s)$$

therefore the basic equation for circuit theory is

$$\begin{aligned} -E_s^{e}(s) + z^{i}(s)I_s(s) &= -\frac{\partial}{\partial s}\Phi(s) - j\omega A_s(s) \\ &= -\frac{j\omega}{\beta_o^2}\left[\frac{\partial}{\partial s}\nabla\cdot\vec{A}(s) + \beta_o^2 A_s(s)\right]. \end{aligned}$$

- **open and closed circuit expressions**

From the development above, the expression for the impressed electric field is

$$E_s^{e}(s) = \frac{\partial}{\partial s}\Phi(s) + j\omega A_s(s) + z^{i}(s)I_s(s).$$

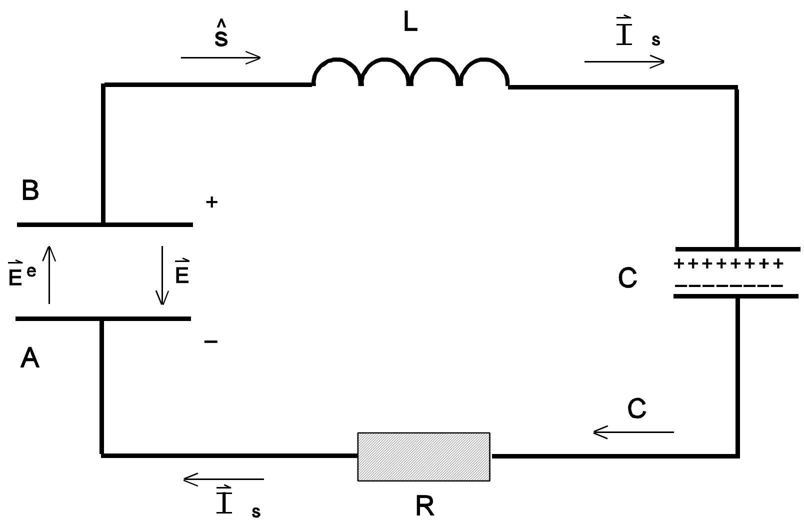

Figure 4. General circuit.

Integrating along a path *C* on the inner surface of the circuit from a point *s*1 to a point *s*2, which represent the ends of an open circuit, gives

$$\int_{s1}^{s2} E_S^e(s)\,ds = \int_{s1}^{s2} \frac{\partial}{\partial s}\Phi(s)\,ds + j\omega\int_{s1}^{s2} A_s(s)\,ds + \int_{s1}^{s2} z^i(s) I_s(s)\,ds$$

but, because the impressed electric field exists only in the source region

$$\int_{s1}^{s2} E_s^e(s)ds = \int_A^B E_s^e(s)\,ds = V_o^e$$

where V_o^e is the driving voltage. Now it can be seen that

$$\int_{s_1}^{s_2} \frac{\partial}{\partial s}\Phi(s)\,ds = \int_{s_1}^{s_2} d\Phi = \Phi(s_2) - \Phi(s_1)$$

and thus the equation for an open circuit can be expressed

$$V_0^e = \Phi(S_2) - \Phi(s_1) + j\omega\int_{s_1}^{s_2} A_s(s)\,ds + \int_{s_1}^{s_2} z^i(s) I_s(s)\,ds \,.$$

If the circuit is closed, then s_1=s_2 and $\Phi(s_2)$-$\Phi(s_1)$=0. In this case, the circuit equation becomes

$$V_0^e = \oint_C z^i(s) I_s(s)\,ds + j\omega\oint_C A_s(s)\,ds \,.$$

4.3 General equations for coupled circuits

Now the concepts developed above are extended to the case of two coupled circuits, each containing a generator, a resistor, a coil (inductor), and a capacitor. Circuit 1 will be referred to as the *primary circuit*, and circuit 2 will be referred to as the *secondary circuit*. This case is represented by a pair of coupled general circuit equations

$$V_{10}^e = \oint_{C_1} z_1^i(s_1) I_{1s}(s_1)\,ds_1 + j\omega\oint_{C_1}\left[\vec{A}_{11}(s_1) + \vec{A}_{12}(s_1)\right]\cdot d\vec{s}_1$$

$$V_{20}^e = \oint_{C_2} z_2^i(s_2) I_{2s}(s_2)\,ds_2 + j\omega\oint_{C_2}\left[\vec{A}_{21}(s_2) + \vec{A}_{22}(s_2)\right]\cdot d\vec{s}_2 \,.$$

Here $\vec{A}_{11}$ and $\vec{A}_{12}$ are the magnetic vector potentials at the surface of the primary circuit maintained by the currents I_{1s} and I_{2s} in the primary and secondary circuits, given by

$$\vec{A}_{11}(s_1) = \frac{\mu_o}{4\pi}\oint_{C_1'} I_{1s}(s_1')\frac{e^{-j\beta_o R_{11}}}{R_{11}(s_1, s_1')}\,d\vec{s}_1{}'$$

and

$$\vec{A}_{12}(s_1) = \frac{\mu_o}{4\pi}\oint_{C_2'} I_{2s}(s_2')\frac{e^{-j\beta_o R_{12}}}{R_{12}(s_1, s_2')}\,d\vec{s}_2{}'$$

and $\vec{A}_{21}$ and $\vec{A}_{22}$ are the vector potentials at the surface of the secondary circuit maintained by the currents I_{1s} and I_{2s} in the primary and secondary circuits, given by

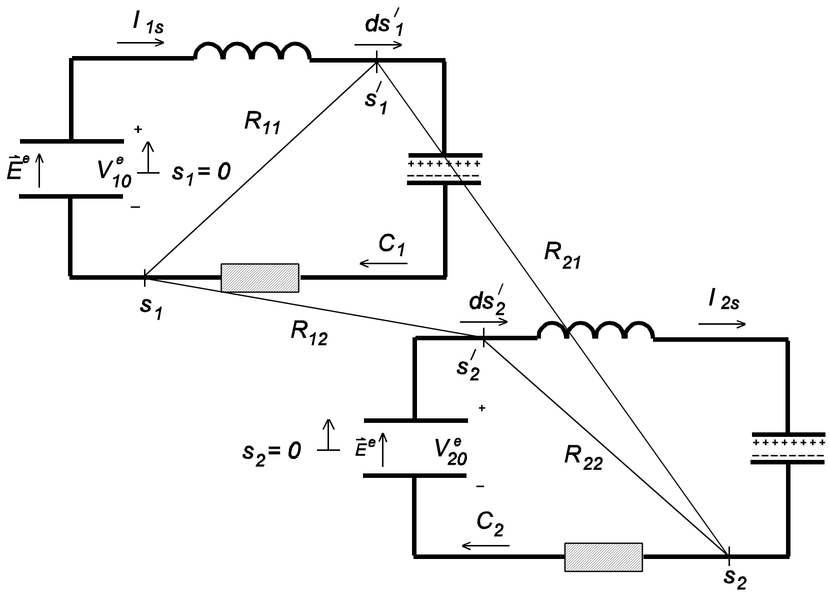

Figure 5. Generalized coupled circuits.

$$\vec{A}_{21}(s_2) = \frac{\mu_o}{4\pi}\oint_{C_1^{/}} I_{1s}(s_1^{/})\frac{e^{-j\beta_o R_{21}}}{R_{21}(s_2,\, s_1^{/})}\, d\vec{s}_1^{\,/}$$

and

$$\vec{A}_{22}(s_2) = \frac{\mu_o}{4\pi}\oint_{C_2^{/}} I_{2s}(s_2^{/})\frac{e^{-j\beta_o R_{22}}}{R_{22}(s_2,\, s_2^{/})}\, d\vec{s}_2^{\,/} .$$

Substituting these expressions for the various magnetic vector potentials into the general circuit equations above leads to

$$V_{10}^e = \oint_{C_1} z_1^i(s_1) I_{1s}(s_1)\, ds_1 + \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_1^{/}} I_{1s}(s_1^{/})\frac{e^{-j\beta_o R_{11}}}{R_{11}(s_1, s_1^{/})} d\vec{s}_1^{\,/}\right]$$

$$+ \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_2^{/}} I_{2s}(s_2^{/})\frac{e^{-j\beta_o R_{12}}}{R_{12}(s_1, s_2^{/})}\, d\vec{s}_2^{\,/}\right]$$

$$V_{20}^{e} = \oint_{C_2} z_2^{i}(s_2) I_{2s}(s_2)\, ds_2 + \frac{j\omega\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_1^{/}} I_{1s}(s_1^{/}) \frac{e^{-j\beta_o R_{21}}}{R_{21}(s_2, s_1^{/})} d\vec{s}_1^{\,/} \right]$$

$$+ \frac{j\omega\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_2^{/}} I_{2s}(s_2^{/}) \frac{e^{-j\beta_o R_{22}}}{R_{22}(s_2, s_2^{/})}\, d\vec{s}_2^{\,/} \right].$$

Note that C_1 and C_2, lie along the inner periphery of the circuit while $C_1^{/}$ and $C_2^{/}$ lie along the centerline. When the circuit dimensions and V_{10}^{e}, V_{20}^{e} are specified, the equations above become a pair of coupled simultaneous integral equations for the unknown currents $I_{1s}(s_1)$ and $I_{2s}(s_2)$ in the primary and secondary circuits. These equations are in general too complicated to be solved exactly.

- **self and mutual impedances of electric circuits**

Reference currents I_{10} and I_{20} are chosen at the locations of the generators in the primary and secondary circuits, i.e.,

$$I_{10} = I_{1s}(s_1 = 0)$$

at the center of the primary circuit generator, and

$$I_{20} = I_{2s}(s_2 = 0)$$

at the center of the secondary circuit generator. Now let the currents be represented by

$$I_{1s}(s_1) = I_{10} f_1(s_1)$$

$$I_{2s}(s_2) = I_{20} f_2(s_2)$$

where $f_1(0) = f_2(0) = 1.0$, and f_1, f_2 are complex distribution functions. The general circuit equations formulated above may be expressed in terms of the reference currents as

$$V_{10}^{e} = I_{10} Z_{11} + I_{20} Z_{12}$$

$$V_{20}^{e} = I_{10} Z_{21} + I_{20} Z_{22}$$

where

Z_{11} = self-impedance of the primary circuit referenced to I_{10}
Z_{22} = self-impedance of the secondary circuit referenced to I_{20}
Z_{12} = mutual-impedance of the primary circuit referenced to I_{20}
Z_{21} = mutual-impedance of the secondary circuit referenced to I_{10}

and

$$Z_{11} = Z_1^i + Z_1^e$$

$$Z_{22} = Z_2^i + Z_2^e .$$

Here Z^i is referred to as the *internal self-impedance* of the primary and secondary circuits. This term depends primarily upon the internal impedance z^i per unit length of the conductors present in the circuits, and includes effects due to capacitance and resistance. Z^e is referred to as the *external self-impedance* of the primary and secondary circuits. This term depends entirely upon the interaction between currents in various parts of the circuit, and includes effects due to inductance. The various impedance terms are expressed as

$$Z_1^i = \oint_{C_1} z_1^i(s_1)\, f_1(s_1)\, ds_1$$

$$Z_2^i = \oint_{C_2} z_2^i(s_2)\, f_2(s_2)\, ds_2$$

$$Z_1^e = \frac{j\omega\mu_o}{4\pi} \oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_1'} f_1(s_1') \frac{e^{-j\beta_o R_{11}}}{R_{11}(s_1, s_1')} d\vec{s}_1{}' \right]$$

$$Z_2^e = \frac{j\omega\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_2'} f_2(s_2') \frac{e^{-j\beta_o R_{22}}}{R_{22}(s_2, s_2')} d\vec{s}_2{}' \right]$$

$$Z_{12} = \frac{j\omega\mu_o}{4\pi} \oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_2^/} f_2(s_2^/) \frac{e^{-j\beta_o R_{12}}}{R_{12}(s_1, s_2^/)} \, d\vec{s}_2^{\,/} \right]$$

$$Z_{21} = \frac{j\omega\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_1^/} f_1(s_1^/) \frac{e^{-j\beta_o R_{21}}}{R_{21}(s_2, s_1^/)} d\vec{s}_1^{\,/} \right].$$

It is noted that all of the circuit impedances depend in general on the current distribution functions $f_1(s_1)$ and $f_2(s_2)$.

- **driving point impedance, coupling coefficient, and induced voltage**

Consider the case where only the primary circuit is driven by a generator excitation, i.e.,

$$V_{20}^e = 0$$

In this case the general circuit equations become

$$V_{10}^e = I_{10} Z_{11} + I_{20} Z_{12}$$

$$0 = I_{10} Z_{21} + I_{20} Z_{22} \, .$$

This leads to a pair of equations

$$I_{20} = -I_{10} \frac{Z_{21}}{Z_{22}}$$

$$V_{10}^e = I_{10} \left(Z_{11} - Z_{12} \frac{Z_{21}}{Z_{22}} \right)$$

which can be solved to yield

$$I_{10} = \frac{V_{10}^{e}}{Z_{11} - \dfrac{Z_{12}Z_{21}}{Z_{22}}}$$

and

$$I_{20} = \frac{-V_{10}^{e}}{\dfrac{Z_{11}Z_{22}}{Z_{21}} - Z_{12}}$$

which are the reference currents in the generator regions. From these, it can be seen that the driving point impedance of the primary circuit is

$$\left(Z_1\right)_{in} = \frac{V_{10}^{e}}{I_{10}} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22}} = Z_{11}\left(1 - \frac{Z_{12}Z_{21}}{Z_{11}Z_{22}}\right) .$$

For the case of a loosely coupled electric circuit having $\left(Z_{12}Z_{21}\right)/\left(Z_{11}Z_{22}\right) \ll 1$, Z_{12} and Z_{21} are both negligibly small, and therefore $(Z_1)_{in} \approx Z_{11}$. For this case the expressions

$$V_{10}^{e} = I_{10}Z_{11} + I_{20}Z_{12}$$

$$0 = I_{10}Z_{21} + I_{20}Z_{22} .$$

become

$$V_{10}^{e} + V_{12}^{i} = I_{10}Z_{11}$$

$$V_{21}^{i} = I_{20}Z_{22}$$

where $V_{12}^{i} = -I_{20}Z_{12}$ is the voltage induced in the primary circuit by the current in the secondary circuit, and $V_{21}^{i} = -I_{10}Z_{21}$ is the voltage induced in the secondary circuit by the current in the primary circuit. Because the circuits are considered to be loosely coupled

$$V_{10}^{e} \gg V_{12}^{i}$$

and therefore the general circuit equations become

$$V_{10}^{e} \cong I_{10} Z_{11}$$

$$V_{21}^{i} = -I_{10} Z_{21} = I_{20} Z_{22} .$$

- **near zone electric circuit**

For cases in which a circuit is operated such that the circuit dimensions are much less than a wavelength, then the phase of the current does not change appreciably as it travels around the circuit. The associated EM fields are such that the circuit is confined to the near- or induction-zones. Most electric circuits used at power and low radio frequencies may be modeled this way. In this type of *conventional* circuit

$$\beta_o R_{11} \ll 1, \quad \beta_o R_{12} \ll 1, \quad \beta_o R_{21} \ll 1, \quad \beta_o R_{22} \ll 1$$

and therefore the following approximation may be made

$$e^{-j\beta_o R_{ij}} = \left(1 - \frac{\beta_o^2 R_{ij}^2}{2!} + \frac{\beta_o^4 R_{ij}^4}{4!} + \ldots \right) - j\beta_o R_{ij} \left(1 - \frac{\beta_o^2 R_{ij}^2}{3!} + \frac{\beta_o^4 R_{ij}^4}{5!} + \ldots \right) \cong 1$$

for i=1,2 and j=1,2. Because the circuit dimensions are small compared to a wavelength

$$f_1(s_1) \cong f_2(s_2) \cong 1$$

and thus

$$f_i(s_i) \frac{e^{-j\beta_o R_{ij}(s_i, s_j^{/})}}{R_{ij}(s_i, s_j^{/})} \cong \frac{1}{R_{ij}(s_i, s_j^{/})}$$

for i=1,2 and j=1,2. Substituting the approximations stated above into the expressions for the various circuit impedances leads to

$$Z_1^{i} = \oint_{C_1} z_1^{i}(s_1)\, ds_1$$

$$Z_2^{i} = \oint_{C_2} z_2^{i}(s_2)\, ds_2$$

$$Z_1^e = jX_1^e = \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \oint_{C_1'} \frac{d\vec{s}_1'}{R_{11}(s_1, s_1')}$$

$$Z_2^e = jX_2^e = \frac{j\omega\mu_o}{4\pi}\oint_{C_2} d\vec{s}_2 \cdot \oint_{C_2'} \frac{d\vec{s}_2'}{R_{22}(s_2, s_2')}$$

$$Z_{12} = jX_{12} = \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \oint_{C_2'} \frac{d\vec{s}_2'}{R_{12}(s_1, s_2')}$$

$$Z_{21} = jX_{21} = \frac{j\omega\mu_o}{4\pi}\oint_{C_2} d\vec{s}_2 \cdot \oint_{C_1'} \frac{d\vec{s}_1'}{R_{21}(s_2, s_1')} .$$

Since the integrals above are frequency independent and functions only of the geometry of the circuit, the self and mutual inductances are defined as follows:

$X_1^e = \omega L_1^e$, where

$$L_1^e = \frac{\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \oint_{C_1'} \frac{d\vec{s}_1'}{R_{11}(s_1, s_1')}$$

is the *external self-inductance of the primary circuit*;

$X_2^e = \omega L_2^e$, where

$$L_2^e = \frac{\mu_o}{4\pi}\oint_{C_2} d\vec{s}_2 \cdot \oint_{C_2'} \frac{d\vec{s}_2'}{R_{22}(s_2, s_2')}$$

is the *external self-inductance of the secondary circuit*;

$$X_{12} = \omega L_{12}, \text{ where}$$

$$L_{12} = \frac{\mu_o}{4\pi} \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_2^{/}} \frac{d\vec{s}_2^{\,/}}{R_{12}(s_1, s_2^{/})}$$

is the *mutual inductance between the primary and secondary circuit*;

$$X_{21} = \omega L_{21}, \text{ where}$$

$$L_{21} = \frac{\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \oint_{C_1^{/}} \frac{d\vec{s}_1^{\,/}}{R_{21}(s_2, s_1^{/})}$$

is the *mutual inductance between the primary and secondary circuit*. It is seen by inspection that $L_{12} = L_{21}$.

- **radiating electric circuit**

At high, ultrahigh, and low microwave frequencies, the assumptions for conventional near-zone electric circuits are not valid since $\beta_o R \ll 1$ is not usually satisfied. In a *quasi-conventional* or *radiating* circuit, the less restrictive dimensional requirement of $\beta_o^2 R^2 \ll 1$ is assumed to be valid for the frequencies of interest, therefore

$$\beta_o^2 R_{11}^2 \ll 1, \quad \beta_o^2 R_{12}^2 \ll 1, \quad \beta_o^2 R_{21}^2 \ll 1, \quad \beta_o^2 R_{22}^2 \ll 1 .$$

In this case

$$e^{-j\beta_o R_{ij}} = \left(1 - \frac{\beta_o^2 R_{ij}^2}{2!} + \frac{\beta_o^4 R_{ij}^4}{4!} + \dots \right) - j\beta_o R_{ij} \left(1 - \frac{\beta_o^2 R_{ij}^2}{3!} + \frac{\beta_o^4 R_{ij}^4}{5!} + \dots \right)$$

$$\cong 1 - j\beta_o R_{ij} \left(1 - \frac{\beta_o^2 R_{ij}^2}{3!} \right)$$

for i=1,2 and j=1,2. In the approximation made above, the higher order term $\left(\beta_o^2 R_{ij}^2\right)/6$ is retained. For a quasi-conventional circuit

$$f_1(s_1) \cong f_2(s_2) \cong 1$$

is observed experimentally to remain approximately valid. Thus

$$f_i(s_i)\,\frac{e^{-j\beta_o R_{ij}(s_i,s_j^{/})}}{R_{ij}(s_i,s_j^{/})} \cong \frac{1}{R_{ij}(s_i,s_j)} - j\beta_o\left[1 - \frac{\beta_o^2 R_{ij}^2(s_i,s_j^{/})}{6}\right]$$

for i=1,2 and j=1,2. From this expression it is apparent why the higher order term $\left(\beta_o^2 R_{ij}^2\right)/6$ in the exponential series was retained. The leading term in the imaginary part of the exponential series integrates to zero in the impedance expressions, i.e.,

$$\oint_{C_i} -j\beta_o\, d\vec{s}_i^{\,/} = -j\beta_o \oint_{C_i} d\vec{s}_i^{\,/} = 0$$

for i=1,2. The impedance expressions for the quasi-conventional circuit are thus given by

$$Z_1^e = \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_1^{/}} \frac{d\vec{s}_1^{\,/}}{R_{11}(s_1,s_1^{/})} + \frac{j\beta_o}{6}\oint_{C_1^{/}} \beta_o^2 R_{11}^2(s_1,s_1^{/})\, d\vec{s}_1^{\,/}\right]$$

$$Z_2^e = \frac{j\omega\mu_o}{4\pi}\oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_2^{/}} \frac{d\vec{s}_2^{\,/}}{R_{22}(s_2,s_2^{/})} + \frac{j\beta_o}{6}\oint_{C_2^{/}} \beta_o^2 R_{22}^2(s_2,s_2^{/})\, d\vec{s}_2^{\,/}\right]$$

$$Z_{12} = \frac{j\omega\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \left[\oint_{C_2^{/}} \frac{d\vec{s}_2^{\,/}}{R_{12}(s_1,s_2^{/})} + \frac{j\beta_o}{6}\oint_{C_2^{/}} \beta_o^2 R_{12}^2(s_1,s_2^{/})\, d\vec{s}_2^{\,/}\right]$$

$$Z_{21} = \frac{j\omega\mu_o}{4\pi}\oint_{C_2} d\vec{s}_2 \cdot \left[\oint_{C_1^{/}} \frac{d\vec{s}_1^{\,/}}{R_{21}(s_2,s_1^{/})} + \frac{j\beta_o}{6}\oint_{C_1^{/}} \beta_o^2 R_{21}^2(s_2,s_1^{/})\, d\vec{s}_1^{\,/}\right]$$

and the expressions for the internal impedances of the primary and secondary circuits remain unchanged. It is seen that the impedance expressions above consist of both real and imaginary parts

$$Z_1^e = R_1^e + jX_1^e$$

$$Z_2^e = R_2^e + jX_2^e$$

and

$$Z_{12} = Z_{21} = R_{12} + jX_{12} .$$

Now, recalling that $v_p = \omega / \beta_o$

$$\frac{j\omega\mu_o}{4\pi}\frac{j\beta_o^3}{6} = -\frac{\mu_o v_p \beta_o^4}{24\pi} = -\frac{\eta_o\beta_o^4}{24\pi} = -\frac{120\pi\beta_o^4}{24\pi} = -5\beta_o^4$$

therefore

$$R_1^e = -5\beta_o^4 \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_1'} R_{11}^2(s_1, s_1')\, d\vec{s}_1'$$

$$X_1^e = \omega L_1^e, \text{ where}$$

$$L_1^e = \frac{\mu_o}{4\pi}\oint_{C_1} d\vec{s}_1 \cdot \oint_{C_1'} \frac{d\vec{s}_1'}{R_{11}(s_1, s_1')}$$

$$R_2^e = -5\beta_o^4 \oint_{C_2} d\vec{s}_2 \cdot \oint_{C_2'} R_{22}^2(s_2, s_2')\, d\vec{s}_2'$$

$$X_2^e = \omega L_2^e, \text{ where}$$

$$L_2^e = \frac{\mu_o}{4\pi} \oint_{C_2} d\vec{s}_2 \cdot \oint_{C_2'} \frac{d\vec{s}_2^{\,\prime}}{R_{22}(s_2, s_2')}$$

$$R_{12} = -5\beta_o^4 \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_2'} R_{12}^2(s_1, s_2')\, d\vec{s}_2^{\,\prime}$$

and

$$X_{12} = \omega L_{12}, \text{ where}$$

$$L_{12} = \frac{\mu_o}{4\pi} \oint_{C_1} d\vec{s}_1 \cdot \oint_{C_2'} \frac{d\vec{s}_2^{\,\prime}}{R_{12}(s_1, s_2')} .$$

It is noted that the inductances L_1^e, L_2^e and L_{12} for the quasi-conventional circuit are the same as those for the conventional near-zone circuit. In the quasi-conventional circuit, however, impedances Z_1^e, Z_2^e and Z_{12} become complex due to the presence of R_1^e, R_2^e and R_{12}. The resistive components R_1^e and R_2^e do not represent dissipation losses in the circuit (dissipation losses are included in the internal impedance terms Z_1^i and Z_2^i), but instead indicate a power loss from the circuit due to radiation of EM energy to space. R_1^e and R_2^e are therefore *radiation resistances* which describe the power loss from a quasi-conventional circuit due to EM radiation.

References

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