Recursive, Infinite Impulse Response (IIR) Digital Filters:

Filters defined by Laplace Domain transfer functions (analog devices) can be easily converted to Z domain transfer functions (digital, sampled data devices).

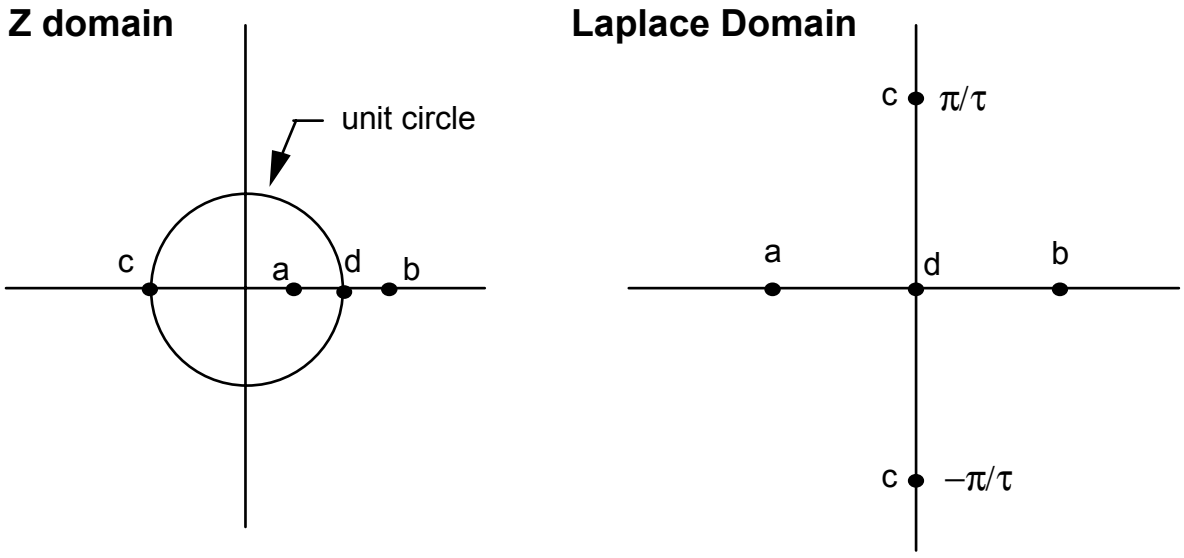
In the Laplace domain, a single time step or unit delay has the transfer function, $e^{-s\tau}$.

In the Z domain, a single time step or unit delay has the transfer function, z^{-1}.

Equating these two transfer functions provides a mapping between the Laplace domain and the Z domain, yields the transformation

$$z^{-1} = e^{-s\tau} \quad \Rightarrow \quad z = e^{s\tau} \tag{1}$$

where the time step or unit delay is given by the coefficient, τ.



Example #1: First Order Filter Design
Find the sampled data form of the first-order transfer function,

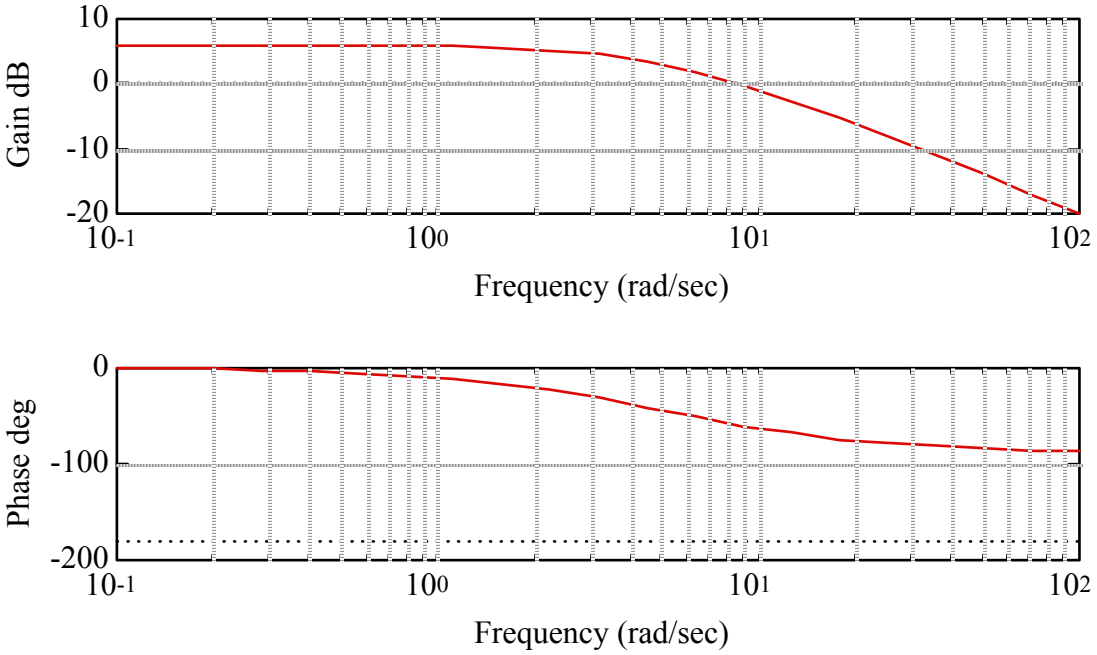
$$T(s) = 2/(0.2s+1) \tag{2}$$

This transfer function has a single pole at s = -5 and a single zero at s = -∞ that reduces all signals with frequencies above 5 radians/sec. and has a steady state gain,

$$T(\omega=0, z = 1) = 2.$$

The frequency response of this transfer function can be found with MatLab commands

```
num = [2]; den = [0.2 1]; bode(num,den);
```



To find the Z domain transfer function first specify the sampling rate. We will assume a sampling rate of 100 Hz., which is well above the "cutoff frequency",

$$\omega = 5 \text{ rad./sec.} = 0.796 \text{ Hz.}$$

and find the Z domain pole corresponding to the Laplace domain pole at s = -5,

$$z = e^{s\tau}$$

where $\tau = 1/100 = 0.01$
s = -5 rad/sec (real valued pole in this case)

This yields the Z domain pole at

$$z = e^{(-5*0.01)} = 0.951$$

The Laplace domain zero is at "-∞" so the Z domain zero is located at

$$z = e^{(-\infty*0.01)} = 0.0$$

The sampled data transfer function is now assembled with the Z domain transfer function,

$$T(z) = (Gz)/(z-0.951)$$

where G = unknown gain constant to be specified

Note that this transfer function has z-domain poles and zeros at the positions in the z-domain corresponding to the points calculated for the transformed Laplace domain poles and zeros. The variable z represents a positive unit time step into the future that cannot

be realized. We must modify our Z domain transfer function so that it is in units of delay "z^{-1}"by multiplying top and bottom through by enough factors of z^{-1} to yield only "delay" operators in the z-domain transfer function. In this case, we need only to multiply top and bottom by,

$$T(z) = G/(1-0.951*z^{-1})$$

where G = unknown gain constant to be specified

This transfer function has the zero frequency (z=1) magnitude at $\omega = 0$ of

$$T(\omega=0, z = 1) = 2 = G/(1-0.951) = G/0.0488$$

This magnitude must be matched to the S-domain magnitude at zero frequency where T(s) =T(0) = 2 so that the Z transform G parameter becomes

$$G = T(\omega=0)*0.0488 = 2*0.0488= 0.0975$$

and the final Z domain transfer function is,

$$T(z) = 0.0975/(1-0.951*z^{-1})$$

IIR Filter Implementation:

The derived z-domain filter must now be implemented with sampled data algebra. We start with the Z Transform,

$$Y(z)/X(z) = T(z) = G/(1-a*z^{-1})$$

and write the transfer functions in "delay operator" form on the input "X(z)" and the output "Y(z)".

$$(1-a*z^{-1})Y(z) = G\ X(z)$$

Rearranging terms,

$$Y(z) - (a*z^{-1})*Y(z)= G\ X(z)$$

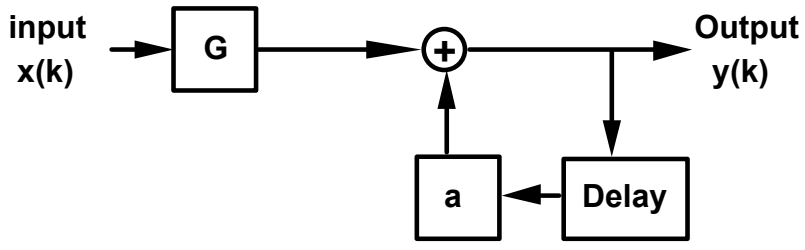
or

$$Y(z) = G\ X(z) + (a*z^{-1})*Y(z)$$

but z^{-1} is a unit delay, so in sampled data form,

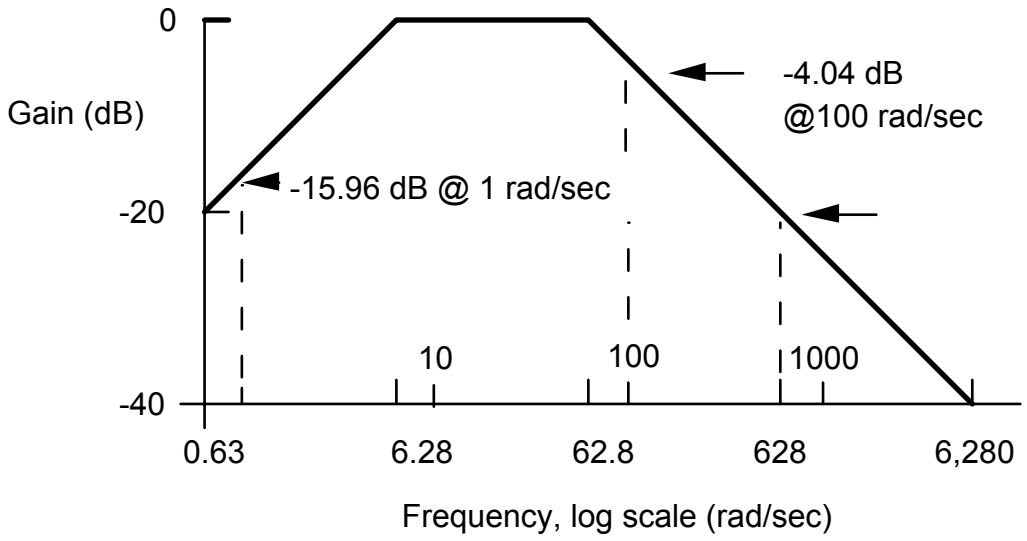
$$y(k) = G*x(k) + a*y(k-1)$$

This sampled data equation shown in block diagram form below.



Second-Order IIR Filter Design Example:

Derive the sampled data filter corresponding to the frequency response below.



For this example, pick a sampling interval corresponding to
1000 Hz. sampling frequency => $\tau = 0.001$ seconds

Now compute the Z-domain pole locations:

$$s = 62.8 \text{ rad/sec} \Rightarrow z = e^{s\tau} = e^{(-62.8*0.001)} = e^{(-0.0628)} = 0.939101$$

$$s = 6.28 \text{ rad/sec} \Rightarrow z = e^{s\tau} = e^{(-6.28*0.001)} = e^{(-0.00628)} = 0.993737$$

and compute the Z-domain zero locations:

$$s = -\infty \text{ rad/sec} \Rightarrow z = e^{s\tau} = e^{-\infty} = 0.0$$

$$s = 0.0 \text{ rad/sec} \Rightarrow z = e^{s\tau} = e^{(0.00)} = 1.00$$

Rewriting transfer function in Z domain,

$$T(z) = G\left[\frac{z(z-1)}{(z-0.993737)(z-0.939101)}\right] = G\left[\frac{z(z-1)}{z^2 - 1.93284z + 0.933219}\right]$$

$$= G\left[\frac{1-z^{-1}}{1-1.9328z^{-1}+0.933219z^{-2}}\right]$$

To compute gain, G, match the amplitude of T(z) at a single frequency. Here, set frequency, ω, to halfway between 6.28 rad/sec and 62.8 rad/sec on the log scale.

Amplitude matching frequency,

$$\omega = \text{alog}[\ \log(6.28) + 0.5(\ \log(6.28) - \log(62.8)\ )\ ] = 19.8569$$

so set $\omega = 20$ rad/sec where amplitude of T(z) = 0 dB = 1.0

and

$$G = \left\| \frac{1 - 1.9328z^{-1} + 0.933219z^{-2}}{1 - z^{-1}} \right\| = \left\| \frac{(z - 0.993737)(z - 0.939101)}{z(z-1)} \right\|$$

where: $z = e^{j\omega\tau} = e^{j(20*0.001)} = e^{j(0.020)}$

$$G = \left\| \frac{(z - 0.993737)(z - 0.939101)}{z(z-1)} \right\| = \frac{(0.0209)(0.0639)}{(1)(0.020)} = 0.067$$

our z-domain transfer function becomes:

$$T(z) = \frac{Y(z)}{X(z)} = 0.067\left[\frac{1 - z^{-1}}{1 - 1.9328z^{-1} + 0.933219z^{-2}} \right]$$

$$\left(1 - 1.9328z^{-1} + 0.933219z^{-2}\right)Y(z) = 0.067\left(1 - z^{-1}\right)X(z)$$

$$Y(z) = \left(1.9328z^{-1} - 0.933219z^{-2}\right)Y(z) + 0.067\left(1 - z^{-1}\right)X(z)$$

so that our filter equation becomes:

$$y(k) = 1.9328\ y(k-1) - 0.933219\ y(k-2) + 0.067\ [\ x(k) - x(k-1)\ ]$$

This filter is shown in block diagram form below.

