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Days: October 31 (M), November 2 (W), 4 (F) and 7 (M)  
Time: 9.00 AM

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1st Lecture Notes by  
Filippo de Monte - Monday, October 31, 2011

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1. NUMBERING SYSTEM IN HEAT CONDUCTION

The numbering system for heat conduction has the objective to describe a transient or steady state heat conduction problem by means of right a number, actually an alphanumeric string. This string is able to contain a great deal of information, nearly all the information contained in the governing equations of the problem of interest. The number system was first proposed by Beck and Litkouhi in 1985. A complete description is given in Ref. [1, Chapter 2].

The numbering system covers only regular geometries such as plates, cylinders and spheres as well as linear problems. It is also important in order to use the Green’s functions appendices given in [1].

1.1. Numbering System for rectangular (or Cartesian) coordinates

The X, Y and Z denote the x-, y- and z-coordinates, respectively. A 1D problem along x involve X. A 2D problem along x and y involves X and Y. A 3D problem along x, y and z involves X, Y and Z.

The type of boundary condition (BC) is indicated by 0, 1, 2 or 3, corresponding to zeroth kind, Dirichlet (1st kind), Neumann (2nd kind) or Robin (3rd kind) type. See table 2.1, p. 49, Ref. [1].

<table>
<thead>
<tr>
<th>Notation</th>
<th>Name of boundary condition</th>
<th>Description of boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Zeroth kind (natural)</td>
<td>No physical boundary</td>
</tr>
<tr>
<td>1</td>
<td>Dirichlet</td>
<td>Prescribed temperature, Eq. (2.3)</td>
</tr>
<tr>
<td>2</td>
<td>Neumann</td>
<td>Prescribed heat flux, Eq. (2.4)</td>
</tr>
<tr>
<td>3</td>
<td>Robin</td>
<td>Convective condition, Eq. (2.6)</td>
</tr>
<tr>
<td>4</td>
<td>Fourth kind (Carslaw)</td>
<td>Thin film, no convection, Eq. (2.7)</td>
</tr>
<tr>
<td>5</td>
<td>Fifth kind (Jaeger)</td>
<td>Thin film, convection, Eq. (2.8)</td>
</tr>
</tbody>
</table>

1st Example. X12 indicates a 1D rectangular slab of finite thickness having a boundary condition of Dirichlet type on the left side boundary (LSB), x = 0, and a boundary condition of Neumann type on the right side boundary (RSB), x = L, where L denotes its thickness.

2nd Example. X30 indicates a 1D rectangular slab semi-infinite along x having a boundary condition of Robin type at x = 0.

3rd Example. X13Y21 indicates a 2D rectangular slab of finite length L and width W having the boundary conditions below

- x = 0, boundary condition of Dirichlet type
- x = L, boundary condition of Robin type
- y = 0, boundary condition of Neumann type
- y = W, boundary condition of Dirichlet type.

4th Example. X13Y20 indicates a 2D rectangular slab of finite length L and semi-infinite along y having the boundary conditions below

- x = 0, boundary condition of Dirichlet type
- x = L, boundary condition of Robin type
- y = 0, boundary condition of Neumann type.
Of course, at this stage, we do not know if the boundary conditions of the examples listed before are homogeneous or non-homogeneous and, in the latter case, if they are time- and/or space-dependent. We can right say that, in the first two examples, we cannot have space-dependent boundary conditions otherwise we would have a 2D problem.

For this purpose, the symbol \( B \) denoting boundary condition was introduced. The homogeneous BC is simply indicated by \( B_0 \). The non-homogeneous BC is indicated by \( B \) followed by an alphanumeric string indicating the type of time- and space-variable function at the same BC.

For example, \( (t-x) \) denotes a function arbitrary in both time and space (along \( x \)). Hence, \( B(t-x) \) denotes a BC arbitrary in both time and space (along \( x \)). \( B(t1x1) \), shortly denoted by \( B1 \), indicates a constant and uniform BC.

Also, \( B(t1x2y1) = B(x2) \) denotes a BC which is time-independent but variable linearly along \( x \) and uniform along \( y \). Finally, \( B(t1x5y2) = B(x5y2) \) indicates a step change along \( x \) and a linear variation along \( y \). See table 2.2, p. 52, Ref. [1].

### Table 2.2 Types of time- and space-variable function at boundary conditions

<table>
<thead>
<tr>
<th>Notation</th>
<th>Time-variable boundary function</th>
<th>Notation</th>
<th>Space-variable boundary function (two-dimensional)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>Arbitrary ( f(t) )</td>
<td>( Bx )</td>
<td>Arbitrary ( f(x) )</td>
</tr>
<tr>
<td>( B0 )</td>
<td>( f(t) = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B1 )</td>
<td>( f(t) = C )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B2 )</td>
<td>( f(t) = Ct ), ( p &gt; 1 )</td>
<td>( Bx2 )</td>
<td>( f(x) = Cx )</td>
</tr>
<tr>
<td>( B3 )</td>
<td>( f(t) = Cx^p )</td>
<td>( Bx3 )</td>
<td>( f(x) = Cx^p ), ( p &gt; 1 )</td>
</tr>
<tr>
<td>( B4 )</td>
<td>( f(t) = \exp(-at) )</td>
<td>( Bx4 )</td>
<td>( f(x) = \exp(-ax) )</td>
</tr>
<tr>
<td>( B5 )</td>
<td>Step changes in ( f(t) )</td>
<td>( Bx5 )</td>
<td>Step changes in ( f(x) )</td>
</tr>
<tr>
<td>( B6 )</td>
<td>( \sin(\omega t + E), \cos(\omega t + E) )</td>
<td>( Bx6 )</td>
<td>( \sin(\omega x + E), \cos(\omega x + E) )</td>
</tr>
</tbody>
</table>

Now, we need one more information, concerning the initial temperature of the body. For this purpose, we use the symbol \( T \). The \( T0 \) indicates that the body is initially at zero temperature. If the temperature is other than zero, we use \( T \) followed by an alphanumeric string indicating the type of the space-variable function at the initial time \( t = 0 \).

For example, \( (x-) \) denotes a function arbitrary in the \( x \)-direction. Hence, \( B(x-) \) = \( B(x-) \) denotes an initial condition for a 1D problem arbitrary along \( x \).

Then, \( (r-) = (x-y-z-) \) denotes a function arbitrary in the \( x-, y- \) and \( z- \)directions. Hence, \( B(r-) = B(x-y-z-) \) denotes an initial condition arbitrary in space for a 3D problem.

Also, \( T(r1) = T(x1y1z1) \), shortly denoted by \( T1 \), indicates a uniform initial temperature different from zero for a 3D problem. For a 1D problem, for example, we can have \( T(x2) \) which indicates a linear initial temperature. See table 2.3, p. 53, Ref. [1].

Examples of 1D transient problems: X21B10T0, X21B02T0, X21B00T2 and X21B12T2.

Examples of 2D transient problems: X13B01 Y21B(t-)1T0 (see Problem # 1) and X21B00 Y20B(x5)T2 (see Problem # 2).

Example of a 3D transient problem: X22B00 Y22B00 Z20B(x5y5) T0 G1 (see Problem # 5).
Then, the symbol C denotes a two-layer body with perfect thermal contact as well as C3 indicates an imperfect contact between the two regions. See table 2.4, p. 56, Ref. [1].

Finally, the symbol G denotes the volumetric heat source (or heat sink). The G0 = 0 indicates that there is no heat generation. For a 1D problem, the G(t1x1) = G1 indicates that the heat source is both constant and uniform as well as G(t2x2) = G2 indicates that the heat source/sink is linear in both time and space. See table 2.5, p. 57, Ref. [1].

Example of a 1D transient problem with volumetric heat generation: X2B10T0G2 (where \( g(x,t) = cxt \ W/m^3 \)).

1.2. Numbering System for cylindrical coordinates

Cylindrical coordinates are denoted by \( r, \phi \) and \( z \), where \( \phi \) is the ‘azimuth’ angle. In the problem notation devised by Beck et al. [1, Chapter 2], R denotes the \( r \)-radial coordinate as well as \( \Phi \) indicates the “angular” direction. Note that \( r \) and \( \phi \) are also called polar coordinates. We have Z for the z-direction.

For a full cylinder, we have only the outer radius. Hence, we have a zero kind boundary condition at \( r = 0 \). For an infinite body with a circular hole, we have only the inner radius. Hence, we have a zero kind boundary condition at \( r \to \infty \).
1st Example. R12 indicates a 1D hollow cylinder of finite thickness, \( b - a \), having a boundary condition of Dirichlet type on the inner radius, \( r = a \), and a boundary condition of Neumann type on the outer radius, \( r = b \).

2nd Example. R30 indicates a 1D an infinite body with a circular hole having a boundary condition of Robin type on the inner radius \( r = a \).

3rd Example. R02 indicates a 1D a full cylinder having a boundary condition of Neumann type on the outer radius \( r = b \).

4th Example. R13Z21 indicates a 2D hollow cylinder of finite height \( H \) having the boundary conditions below

- \( r = a \), boundary condition of Dirichlet type
- \( r = b \), boundary condition of Robin type
- \( z = 0 \), boundary condition of Neumann type
- \( x = H \), boundary condition of Dirichlet type.

5th Example. R13\( \Phi \)21 indicates a 2D semi-circular body finite in the radial direction having the boundary conditions below

- \( r = a \), boundary condition of Dirichlet type
- \( r = b \), boundary condition of Robin type
- \( \phi = 0 \), boundary condition of Neumann type
- \( \phi = \phi_0 \), boundary condition of Dirichlet type.

6th Example. R13\( \Phi \)00 indicates a 2D circular body (hollow cylinder) finite in the radial direction having the boundary conditions below

- \( r = a \), boundary condition of Dirichlet type
- \( r = b \), boundary condition of Robin type.

Examples of transient cylindrical problems

Examples of 1D transient problems: R13B01T0 and R01B2T0.
Examples of 2D transient problems: R02B0 Z20B(r5)T0 (see Problem # 3) and R20B(∅6) Φ00T1 (see Problem # 4), where T1 shortly indicates T(r1φ1).

1.3. Numbering System for spherical coordinates

Spherical coordinates (also called spherical polar coordinates) are denoted by r, θ and φ, where θ and φ are the ‘zenith’ (or polar) and ‘azimuth’ angles, respectively. In the problem notation devised by Beck et al. [1, Chapter 2], we use RS, Θ and Φ, respectively.

Similar considerations can be applied to this case.

References

PROBLEMS

Problem # 1 - Give the numbering system designation for the following 2D transient heat conduction problem in Cartesian coordinates

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L ; 0 < y < W ; t > 0)
\]

\[
T(0, y, t) = 0 \quad (0 < y < W ; t > 0)
\]

\[
-k \left( \frac{\partial T}{\partial x} \right)_{x=L} = h \left[ T(L, y, t) - T_w \right] \quad (0 < y < W ; t > 0)
\]

\[
-k \left( \frac{\partial T}{\partial y} \right)_{y=0} = q_0(t) \quad (0 < x < L ; t > 0)
\]

\[
T(x, W, t) = T_w \quad (0 < x < L ; t > 0)
\]

\[
T(x, y, 0) = 0 \quad (0 < x < L ; 0 < y < W)
\]

where \( T_w = 200^\circ C \), \( T_w = 100^\circ C \) and \( q_0(t) = 5t W / m^2 \). Also, draw a schematic of the problem.
Problem # 2 - Give the numbering system designation for the following 2D transient heat conduction problem in Cartesian coordinates

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L ; \ y > 0 ; \ t > 0)
\]

\[
\frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \quad (y > 0 ; \ t > 0)
\]

\[
T(L, y, t) = 0 \quad (y > 0 ; \ t > 0)
\]

\[-k \frac{\partial T}{\partial y} \bigg|_{y=0} = \begin{cases} \frac{q_0}{2} & \text{for } 0 < x < L_0 \\ 0 & \text{for } L_0 < x < L \end{cases} \quad (0 < x < L ; \ t > 0)
\]

\[
T(x, \infty, t) = \text{finite} \quad (0 < x < L ; \ t > 0)
\]

\[
T(x, y, 0) = F(x) \quad (0 < x < L ; \ y > 0)
\]

where \( q_0 = 1 \text{ W/m}^2 \) and \( F(x) = 2x^\circ \text{C} \). Also, draw a schematic of the problem.

Problem # 3 - Give the numbering system designation for the following 2D transient heat conduction problem in cylindrical coordinates

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < r < b ; \ z > 0 ; \ t > 0)
\]

\[
T(0, z, t) = \text{finite} \quad (z > 0 ; \ t > 0)
\]

\[
\frac{\partial T}{\partial r} \bigg|_{r=b} = 0 \quad (z > 0 ; \ t > 0)
\]

\[-k \frac{\partial T}{\partial z} \bigg|_{z=0} = \begin{cases} \frac{q_0}{2} & \text{for } 0 < r < b_0 \\ 0 & \text{for } b_0 < r < b \end{cases} \quad (0 < r < b ; \ t > 0)
\]

\[
T(r, \infty, t) = \text{finite} \quad (0 < r < b ; \ t > 0)
\]

\[
T(r, z, 0) = 0 \quad (0 < r < b ; \ z > 0)
\]

where \( q_0 = 1 \text{ W/m}^2 \). Also, draw a schematic of the problem.
Problem # 4 - Give the numbering system designation for the following 2D transient heat conduction problem in cylindrical coordinates

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (a < r < \infty; 0 < \phi < 2\pi; t > 0)
\]

\[-k \left( \frac{\partial T}{\partial r} \right)_{r=0} = q_0 \sin(\pi\phi) \quad (0 < \phi < 2\pi; t > 0)\]

\[T(\infty, \phi, t) = \text{finite} \quad (0 < \phi < 2\pi; t > 0)\]

\[T(r, \phi, 0) = T_0 \quad (a < r < \infty; 0 < \phi < 2\pi)\]

where \(q_0 = 1 \text{ W/m}^2\) and \(T_0 = 100^\circ C\). Also, draw a schematic of the problem.

Problem # 5 - Give the numbering system designation for the following 3D transient heat conduction problem in Cartesian coordinates

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_0}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L; 0 < y < W; z > 0; t > 0)
\]

\[\left( \frac{\partial T}{\partial x} \right)_{x=0} = 0 \quad (0 < y < W; z > 0; t > 0)\]

\[\left( \frac{\partial T}{\partial x} \right)_{x=L} = 0 \quad (0 < y < W; z > 0; t > 0)\]

\[\left( \frac{\partial T}{\partial y} \right)_{y=0} = 0 \quad (0 < x < L; z > 0; t > 0)\]

\[\left( \frac{\partial T}{\partial y} \right)_{y=W} = 0 \quad (0 < x < L; z > 0; t > 0)\]

\[-k \left( \frac{\partial T}{\partial z} \right)_{z=0} = \begin{cases} q_0 & \text{for } 0 < x < L_0 \text{ & } 0 < y < W_0 \\ 0 & \text{for } L_0 < x < L \text{ & } W_0 < y < W \end{cases} \quad (0 < x < L; 0 < y < W; t > 0)\]

\[T(x, y, \infty, t) = \text{finite} \quad (0 < x < L; 0 < y < W; t > 0)\]

\[T(x, y, z, 0) = 0 \quad (0 < x < L; 0 < y < W; z > 0)\]

where \(q_0 = 1 \text{ W/m}^2\) and \(g_0 = 1 \text{ W/m}^3\). Also, draw a schematic of the problem.