1. Given the differential equation $\dot{x} = f(x,u)$. Linearize the differential equation about $x_0 = 0$ if the function $f(x,u)$ is given by

   a) $f(x,u) = -x + 3u$ (the function is linear),

   b) $f(x,u) = -x^3 + u$,

   c) $f(x,u) = -e^{-x} + 2u^2$

Solution:

For all cases, $f(x) \equiv f(x_0, u_0) + \frac{df}{dx} \bigg|_{x=x_0} (x-x_0)$

$+ \frac{df}{du} \bigg|_{u=u_0} (u-u_0)$,

where $(x_0, u_0)$ is chosen as an equilibrium point where

$\dot{x} = 0 \iff f(x_0, u_0) = 0$

a) For equilibrium at $x_0 = 0$, $u_0 = 0$. $f(x) = 0 + (1)(x-0) + 3(0-0) = -x + 3u$

so that the linearized ODE is $\dot{x} = -x + 3u$

b) For equilibrium at $x_0 = 0$, $u_0 = 0$.

$f(x) = 0 + (3x^2)\bigg|_{x=0} (x-0) + 1 \cdot (0-0) = u$

so that the linearized ODE is

$\dot{x} = u$

c) For equilibrium at $x_0 = 0$, $u_0 = \sqrt{e^{-0}/2} = 0.707$

$f(x) = 0 + (-e^{-0})\bigg|_{x=0} (x-0) + (4u)\bigg|_{u=0.707} (u-0.707)$

$= -x + 2.828 (u-0.707)$

so that the linearized ODE is

$\dot{x} = -x + 2.828 \dot{u}$

where $\dot{u} = u - 0.707$
2. Problem: Linearize the differential equation \( \dot{x} = f(x, u) \) about \( x_0 = 1 \) for each \( f(x) \) below:

A) \( f(x, u) = -x + 3u \)
B) \( f(x, u) = -x^2 + u \)
C) \( f(x, u) = -e^x + 2u^2 \)

Solution: The differential equation is linearized such that \( \hat{f}(x) = f(x_0, u_0) + \frac{\partial f}{\partial x} \Delta x_0 + \frac{\partial f}{\partial u} \Delta u_0 \), which is usually a valid approximation for \( (x, u) \) near \( (x_0, u_0) \). We choose a \( u_0 \) for each given \( x_0 \) such that \( \dot{x}_0 = f(x_0, u_0) = 0 \).

A) \( x_0 = 1 \)
\[ \dot{x}_0 = -1 + 3u_0 = 0 \rightarrow u_0 = \frac{1}{3} \]
Equilibrium is at \( (x_0, u_0) = (1, \frac{1}{3}) \), so our coordinates relative to equilibrium are \( \hat{x} = x - 1 \) and \( \hat{u} = u - \frac{1}{3} \)

\( \frac{\partial f}{\partial x} = -1 \) and \( \frac{\partial f}{\partial u} = 3 \), so our linearized equation is

\[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = -1(x - 1) + 3(u - \frac{1}{3}) \] or \[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = -\hat{x} + 3\hat{u} \]

The linearized equation has the same form as the original since the original was already linear.

B) \( x_0 = 1 \)
\[ \dot{x}_0 = -1 + 3u_0 = 0 \rightarrow u_0 = 1 \]
Equilibrium is at \( (x_0, u_0) = (1, 1) \), so our coordinates relative to equilibrium are \( \hat{x} = x - 1 \) and \( \hat{u} = u - 1 \)

\( \frac{\partial f}{\partial x} = -3x^2 \) and \( \frac{\partial f}{\partial u} = 1 \), so our linearized equation is

\[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = -3(x - 1)^2 + 1(u - 1) \] or \[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = -3\hat{x} + \hat{u} \]

C) \( x_0 = 1 \)
\[ \dot{x}_0 = -e^{-x} + 2u_0^2 = 0 \rightarrow u_0 = \frac{1}{\sqrt{2e}} \]
Equilibrium is at \( (x_0, u_0) = (1, \frac{1}{\sqrt{2e}}) \), so our coordinates relative to equilibrium are \( \hat{x} = x - 1 \) and \( \hat{u} = u - \frac{1}{\sqrt{2e}} \)

\( \frac{\partial f}{\partial x} = -e^{-x} \) and \( \frac{\partial f}{\partial u} = 4u_0 \), so our linearized equation is

\[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = -e^{-1}(x - 1) + 4(\frac{1}{\sqrt{2e}})(u - \frac{1}{\sqrt{2e}}) \] or \[ \hat{\dot{x}} = \hat{f}(\hat{x}, \hat{u}) = \frac{1}{e} \hat{x} + \frac{2\sqrt{2}}{e} \hat{u} \]
3. A tank of cross section area $A$ contains water that leaks out through a small opening of area $A_o$ at the bottom. The tank is filled with an input flowrate $q$. The differential equation for the water level $z$ in the tank is

$$\dot{z} = \frac{q}{\rho A} - \frac{A_o}{A} (2gz)^{1/2}$$

Linearize the differential equation about $z = 1$ for $A_o = 0.01 \text{ m}^2$ and $A = 1 \text{ m}^2$. For water $\rho = 1000 \text{ kg/m}^3$.

Solution:

1) The system's input is $q(t)$ and output is $z(t)$

2) Expressing the model as a function,

$$f(q, z, \dot{z}) = \dot{z} + \frac{A_o}{A} (2gz)^{1/2} - \frac{q}{\rho A} = 0$$

3) Defining the equilibrium operating point for output $z_0 = 1$

$$\dot{z}_0 = 0 \iff \frac{q_0}{\rho A} - \frac{A_o}{A} (2gz_0) = 0$$

$$q_0 = \rho A_0 (2gz_0)^{1/2} = 1000 \cdot 0.01 \cdot (2 \cdot 9.8)^{1/2} = 44.3 \text{ kg/sec}$$

so that the operating point $(q, z) = (q_0, z_0) = (44.3 \text{ kg/sec}, 1 \text{ m})$

4) Performing a Taylor series expansion,

$$f(q, z, \dot{z}) \approx f(q_0, z_0, 0) + \frac{df}{dq} \bigg|_{q=q_0} (q - q_0) + \frac{df}{d\dot{z}} \bigg|_{\dot{z}=z_0} (\dot{z} - \dot{z}_0) + \frac{df}{dz} \bigg|_{z=z_0} (z - z_0)$$

$$\frac{df}{dq} = \frac{d}{dq} [\dot{z} + \frac{A_o}{A} (2gz)^{1/2} - \frac{q}{\rho A}] = -\frac{1}{\rho A}$$

$$\frac{df}{d\dot{z}} = \frac{d}{d\dot{z}} [\dot{z} + \frac{A_o}{A} (2gz)^{1/2} - \frac{q}{\rho A}] = \frac{1}{2} \frac{A_o}{A} (2gz)^{-1/2} (2g)$$

$$\frac{df}{dz} = \frac{d}{dz} [\dot{z} + \frac{A_o}{A} (2gz)^{1/2} - \frac{q}{\rho A}] = \frac{1}{2} \frac{A_o}{A} (2gz)^{-1/2} (2g)$$
\[
\frac{df}{d\tilde{z}} = \frac{d}{d\tilde{z}} \left[ \tilde{z} + \frac{A_0}{A} (2g \tilde{z})^{\frac{1}{2}} - \frac{\tilde{g}}{pA} \right] = 1
\]

Using the given constants

\[
\frac{df}{d\tilde{z}} \bigg|_{(\tilde{z}_0, \tilde{g}_0)} = \frac{-1}{(1000 \text{ kg/m}^2)(1 \text{ m}^2)} = -10^{-3} m/kg
\]

\[
\frac{df}{d\bar{z}} \bigg|_{(\bar{z}_0, \bar{g}_0)} = \frac{\sqrt{5}}{2} \left( \frac{0.01 \text{ m}^3}{1 \text{ m}^2} \right) \sqrt{\frac{g}{l}} = 0.0211/\text{sec}
\]

So that the function becomes

\[
f(\tilde{z}, \bar{z}, \tilde{z}) \approx (0) + (-10^{-3})(\tilde{z} - \tilde{z}_0) + (0.0211)(\bar{z} - \bar{z}_0) + (1)(\tilde{z} - \tilde{z}_0) = 0
\]

5) Changing variables

\[
\tilde{\tilde{z}} = (\tilde{z} - \tilde{z}_0), \quad \tilde{\bar{z}} = (\bar{z} - \bar{z}_0), \quad \tilde{\tilde{z}} = (\tilde{z} - \tilde{z}_0)
\]

So that \( f(\tilde{z}, \bar{z}, \tilde{z}) \approx 0 + (-10^{-3})\tilde{\tilde{z}} + 0.0211\tilde{\bar{z}} + \tilde{\tilde{z}} = 0 \)

6) Now rewrite the model linearized equation in the standard form

\[
\tilde{\bar{z}} = -0.0211\tilde{\tilde{z}} + 0.001\tilde{\tilde{\tilde{z}}}
\]

where

\[
\tilde{\tilde{\tilde{z}}} = (\tilde{z} - 0.443), \quad \tilde{\tilde{z}} = (\bar{z} - 1), \quad \tilde{\bar{z}} = \tilde{z}
\]
Problem: An undamped, one degree-of-freedom vehicle with air-spring suspension with an external input force \( q(t) \) has the following second-order differential equation for vehicle displacement \( z(t) \):

\[
m \dddot{z} + A p_0 \left[ \frac{1}{1 - \frac{z}{d}} \right] - \frac{1}{1 + \frac{z}{d}}) = q(t)
\]

where \( m \) is the vehicle mass, \( A \) is the air-spring cross-sectional area, \( d \) is the maximum air-spring stroke, \( p_0 \) is the initial air-spring pressure charge, and \( g = 1.4 \) is the thermodynamic constant for air.

Linearize the differential equation about \( z = 0.05m \) for a suspension unit with \( A = 0.01m^2 \), \( d = 0.1m \), \( m = 1000 \text{ kg} \), and \( p_0 = 2 \text{ atm} \) (1 atm = 101.3 kPa = 101.3 kN/m²).

Solution: Follow the 6-step procedure in the lecture handout.

1) The input is \( q(t) \) and the output is \( z(t) \).

2) The nonlinear model of the system is

\[
f(q, z, z) = 0 = m \dddot{z} + A p_0 \left[ \frac{1}{1 - \frac{z}{d}} \right] - \frac{1}{1 + \frac{z}{d}}) = q(t)
\]

3) For the equilibrium position \( z_e = 0.05m \), the input \( q_e(t) = m \dddot{z} + A p_0 \left[ \frac{1}{1 - \frac{z}{d}} \right] - \frac{1}{1 + \frac{z}{d}}) = 0 \text{ N} \) (2 atm: 101,300 Pa/atm)

\[
(0.01m^2)(2 \text{ atm} \cdot 101,300 \text{ Pa/atm}) \left[ \frac{1}{1 - \frac{0.05}{0.1}} \right] - \frac{1}{1 + \frac{0.05}{0.1}} = 4198 \text{ N}
\]

So equilibrium is \( (q_e, z_e) = (4198 \text{ N}, 0.05m) \)

4) The Taylor expansion is

\[
f(q, z, z) \approx f(q_e, z_e, 0) + \frac{\partial f}{\partial q} \bigg|_{(q_e, z_e, 0)} (q - q_e) + \frac{\partial f}{\partial z} \bigg|_{(q_e, z_e, 0)} (z - z_e) + \frac{\partial f}{\partial z^2} \bigg|_{(q_e, z_e, 0)} (z - z_e)^2
\]

where \( \frac{\partial f}{\partial q} \bigg|_{(q_e, z_e, 0)} = -1 \frac{m}{z_e} \),

\[
\frac{\partial f}{\partial z} \bigg|_{(q_e, z_e, 0)} = A \cdot p_0 \left[ \frac{1}{1 - \frac{z}{d}} \right] - \frac{1}{1 + \frac{z}{d}} = (0.01 m^2)(2 \text{ atm} \cdot 101,300 \text{ Pa/atm}) \left[ \frac{(0.1m)(0.05m + 0.1m)}{0.05m + 0.1m} \right] - \frac{1}{1 + \frac{0.05}{0.1}} = 160,425 \frac{m}{z_e}
\]

and \( \frac{\partial f}{\partial z^2} \bigg|_{(q_e, z_e, 0)} = m = 1000 \text{ kg} \), so

\[
f(q, z, z) \approx 0 + -1 \frac{m}{z_e} (q - q_e) + 160,425 \frac{m}{z_e} (z - z_e) + 1000 \text{ kg} (z - z_e)
\]

5) Coordinates relative to equilibrium are \( \hat{q} = q - q_e \), \( \hat{z} = z - z_e \), and \( \hat{z} = z - z_e \), so

\[
\hat{f} (q, z, z) \approx (-1 \frac{m}{z_e}) \hat{q} + (160,425 \frac{m}{z_e}) \hat{z} + (1000 \text{ kg}) \hat{z}
\]

6) Rearrange into standard form:

\[
(1000 \text{ kg}) \dddot{\hat{z}} + (160,425 \frac{m}{z_e}) \hat{z} \approx (1 \frac{m}{z_e}) \hat{q} \text{ near } (q_e, z_e) = (4198 \text{ N}, 0.05m)
\]
2.22. Shown in Figure P2.22 is the block diagram of the servo-control system for one of the joints of a robot. This system is described in Section 2.12.

(a) Find the plant transfer function $\theta_L(s)/E_a(s)$.

(b) Find the closed-loop system transfer function $\theta_L(s)/\theta_c(s)$.

(c) Find the transfer function from the system input $\theta_c(s)$ to the motor armature voltage $E_a(s)$ for the closed-loop system.

(d) Suppose that, in the closed-loop system, the signal $E_a(s)$ is known. Express the output $\theta_L(s)$ as the function of the signal $E_a(s)$.

**Solution:**

a) \[
\dot{\theta}_m = \left( E_a - 0.5 \dot{\theta}_m \right) \frac{1}{2s+21} \cdot 18 \cdot \frac{1}{2s+1} \quad (1)
\]

\[
\dot{\theta}_c = \dot{\theta}_m \cdot \frac{1}{5} \cdot \frac{1}{30} \quad (2)
\]

From (1) and (2)

\[
\frac{\dot{\theta}_c}{E_a} = \frac{0.15}{s^3 + 11s^2 + 7.55} \quad (3)
\]

b) Using the result of (a)

\[
\frac{\dot{\theta}_c}{E_a} = \frac{0.15}{s^3 + 11s^4 + 7.55} \quad (4)
\]

And from the block diagram

\[
E_a = (\theta_c - \theta_c) (K_p + K_0 s) \quad (5)
\]

From (1) and (5)

\[
\frac{\dot{\theta}_c}{\theta_c} = \frac{4.5 (K_p s + K_p)}{s^3 + 11s^2 + (4.5 K_o + 7.5) s + 4.5 K_p} \quad (6)
\]
c) Knowing that

\[ \frac{\theta_L}{E_a} = \frac{0.15}{s^3 + 115^2 + 7.55} \quad (1) \]

\[ \frac{\theta_L}{\theta_c} = \frac{4.5 (K_p s + K_p)}{s^3 + 115^2 + (4.5 K_p + 7.5) s + 4.5 K_p} \quad (2) \]

Thus

\[ \frac{E_a}{\theta_c} = \frac{30 (K_p s + K_p) (s^3 + 115^2 + 7.55)}{s^3 + 115^2 + (4.5 K_p + 7.5) s + 4.5 K_p} \]

d) \[ \frac{\theta_L}{E_a} = \frac{\theta_L}{\theta_c} \cdot \frac{\theta_c}{E_a} = \frac{0.15}{s^3 + 115^2 + 7.55} \]
2.29 Problem. The motion of the pendulum in the diagram is described by the differential equation

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{L} \theta = -\sin(t) \]

This nonlinear differential equation was linearized in section 2.4 as follows:

\[ \frac{d^2 \hat{\theta}}{dt^2} + \frac{g}{L} \hat{\theta} = 0 \quad \text{where} \quad \hat{\theta} = \theta - \theta_0 \quad \text{and the equilibrium} \theta_0 = 0 \]

A) Show that if the bob is held stationary at an initial angle \( \theta_i \) and then released, the resulting motion is sinusoidal in time.

B) Find the frequency in hertz of the sinusoidal motion as a function of the parameters.

C) Run the following MATLAB program and verify that the results match part (A):

\[
x = \text{Input}(\text{ODEs}) \quad x(0) = 0, x(1) = \pi
\]

Solution:

A) Take the Laplace Transform of the linearized differential equation:

\[ s^2 \theta - s \theta(0) - \frac{g}{L} \theta(0) + \frac{g}{L} \theta = 0 \quad \Rightarrow \quad (s^2 + \frac{g}{L}) \theta(s) = s \theta(0) \quad \Rightarrow \quad \theta(s) = \frac{s}{s^2 + \frac{g}{L}} \theta(0) \quad \theta(0) = \theta_i \]

\[ \theta(t) = \theta_i \cos \left( \sqrt{\frac{g}{L}} \cdot t \right) \]

This equation is sinusoidal with respect to time.

B) The equation of motion is of the form \( \theta(t) = A \cos(\omega t) \), so \( \omega = \sqrt{\frac{g}{L}} \). In Hz, \( f = \frac{\omega}{2\pi} \) \[ f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \]

C) The result is \( \theta(t) = \theta_i \left[ \frac{1}{2} e^{i\sqrt{\frac{g}{L}} \cdot t} + \frac{1}{2} e^{-i\sqrt{\frac{g}{L}} \cdot t} \right] = \theta_i \cos \left( \sqrt{\frac{g}{L}} \cdot t \right) \), which is the result obtained by hand.